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Uniform Strict Practical Stability Criteria for Impulsive Functional Differential Equations

By Dilbaj Singh & Sanjay Kumar Srivastava

Lovely Professional University, Punjab, India

Abstract - Strict stability is the kind of stability that can give us some information about the rate of decay of the solution. There are some results about strict stability of functional differential equations. On other hand, in the study of stability, an interesting set of problems deal with bringing sets close to a certain state, rather than the equilibrium state. The desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. Many problems fall into this category. Such considerations led to the notion of practical stability which is neither weaker nor stronger than stability. In this paper, strict practical stability of Impulsive functional differential equations in which the state variables on the impulses are related to time delay is considered. By using Lyapunov functions and Razumikhin technique, some criteria for strict practical stability for functional differential equations, in which the state variables on the impulses are related to the time delay, are provided.

Keywords : *impulsive functional differential equation, strict stability, lyapunov function, razumikhin technique, time delay.*

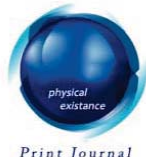
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6. Shen J. H., "Razumikhin techniques in impulsive functional differential equations", Nonlinear Analysis, vol. 36, pp. 119-130, 1999.

Uniform Strict Practical Stability Criteria for Impulsive Functional Differential Equations

Dilbaj Singh^α & Sanjay Kumar Srivastava^σ

Abstract - Strict stability is the kind of stability that can give us some information about the rate of decay of the solution. There are some results about strict stability of functional differential equations. On other hand, in the study of stability, an interesting set of problems deal with bringing sets close to a certain state, rather than the equilibrium state. The desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. Many problems fall into this category. Such considerations led to the notion of practical stability which is neither weaker nor stronger than stability. In this paper, strict practical stability of Impulsive functional differential equations in which the state variables on the impulses are related to time delay is considered. By using Lyapunov functions and Razumikhin technique, some criteria for strict practical stability for functional differential equations, in which the state variables on the impulses are related to the time delay, are provided.

Keywords : *impulsive functional differential equation, strict stability, lyapunov function, razumikhin technique, time delay.*

1. INTRODUCTION

The impulsive differential equations represent a more natural framework for mathematical modeling of many real world phenomena than ordinary differential equations. In recent years, significant progress has been made in the theory of impulsive differential Impulses can make unstable systems stable, so it has been widely used in many fields such as physics, chemistry, biology, population dynamics, industrial robotics and so on. The impulsive differential equations represent a more natural framework for mathematical modeling of many real world phenomena than ordinary differential equations. In recent years, significant progress has been made in the theory of impulsive differential equations [3-14]. In addition to that, functional differential equations have a wide application in our society. So it is important to study them. There are some results on impulsive functional differential equations [5,6,7,8,11,13,14].

We can easily see that in the previous works about impulsive functional differential equations the authors always suppose that the state variables on the impulses are only related to the present state. But in most cases, it is more applicable that the state variables on the impulses are also related to the former state. But there are rare results about impulsive functional differential equations in which state variable on the impulses are related to the time delay. In addition to that strict stability is analogous to

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Lyapunov's uniform asymptotic stability. It gives us some information about the rate of decay of the solutions. In [1], the authors have explored further the definitions of strict stability of differential equations and have gotten some results. In [13] authors have gotten some results about the strict stability of impulsive functional differential equations in which the state variables on the impulses are not related to the time delay. Moreover, in the study of Lyapunov stability, an interesting set of problems deal with bringing sets close to a certain state, rather than the equilibrium state. The desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. Many problems fall into this category including the travel of a space vehicle between two points, an aircraft or a missile which may oscillate around a mathematically unstable course yet its performance may be acceptable, the problem in a chemical process of keeping the temperature within certain bounds, etc. Such considerations led to the notion of practical stability which is neither weaker nor stronger than Lyapunov stability. In [11] authors have gotten some results about the practical stability of impulsive functional differential equations in Equations in terms of two measures. In [2] authors have gotten some results about strict practical stability of delay differential equations. In this paper, strict practical stability of impulsive functional differential equations in which the state variables on the impulses are related to the time delay is considered.

This paper is organized as follows. In Section II, we introduce some basic definitions and notations. In Section III, some criteria in the form of theorem for strict practical stability of impulsive functional differential equations is obtained in which state variables on the impulses are related to the time delay is investigated. Finally, concluding remarks are given in Section IV.

II. PRELIMINARIES

Consider the following Impulsive functional differential equation in which the state variables on the impulses are related to time delay.

$$\begin{aligned} x'(t) &= f(t, x_t), \quad t \geq t_0 \quad t \neq \tau_k \\ x(\tau_k) &= I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau)), \quad k=1, 2, 3, \dots, \end{aligned} \quad (1)$$

Where $x \in R^n$, $f \in C[R^+ \times D, R^n]$, $I_k, J_k \in C[R^n, R^n]$, D is an open set in $PC([- \tau, 0], R^n)$, where $\tau = \text{constant} > 0$,

$PC([- \tau, 0], R^n) = \{\phi : [- \tau, 0] \rightarrow R^n, \phi(t) \text{ is continuous everywhere except a finite number of points } \hat{t} \text{ at which } \phi(\hat{t}^+) \text{ and } \phi(\hat{t}^-) \text{ exist and } \phi(\hat{t}^+) = \phi(\hat{t}^-)\}$, $f(t, 0) = 0$, for all $t \in R$, $I_k(0) \equiv 0$, $J_k \equiv 0$, $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \dots < \tau_k < \dots, \tau_k \rightarrow \infty$, for $k \rightarrow \infty$ and $x(t^+) = \lim_{s \rightarrow t^+} x(s)$, $x(t^-) = \lim_{s \rightarrow t^-} x(s)$.

For each $t \geq t_0$, $x_t \in D$ is defined by $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$. For $\phi \in PC([- \tau, 0], R^n)$ $\|\phi\|_1$ is defined by $\|\phi\|_1 = \sup_{-\tau \leq s \leq 0} \|\phi\|$, $\|\phi\|_2$ is defined by $\|\phi\|_2 = \inf_{-\tau \leq s \leq 0} \|\phi\|$, where $\|\cdot\|$ denotes the norm of a vector R^n . We can see that $x(t) \equiv 0$ is a solution of (1) which we call the zero solution.

A function $x(t)$ is called a solution of (1) with the initial condition $x_\sigma = \phi$. Where $\sigma \geq t_0$ and $\phi \in PC([- \tau, 0], R^n)$, the initial value problems of equation (1) is

R_{ef.}

2. Lakshmikantham V. and Zhang Y., "Strict practical stability of delay differential equation", Applied Mathematics and Computation, vol. 118, pp. 275-285, 2001.

$$\begin{aligned}
 x'(t) &= f(t, x_t), \quad t \geq t_0 \quad t \neq \tau_k \\
 x(\tau_k) &= I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau)), \quad k=1, 2, 3, \dots, \quad x_\sigma = \varphi
 \end{aligned} \tag{2}$$

Throughout this paper we let the following hypothesis hold.

(H₁) For $t \in [\sigma - \tau, \sigma]$, the solution $x(t; \sigma, \varphi)$ coincides with the function $\varphi(t - \sigma)$.

(H₂) For each function $x(s) : [\sigma - \tau, \infty) \rightarrow R^n$, which is continuous everywhere except at the point $\{\tau_k\}$ at which $x(\tau_k^+), (\tau_k^-)$ exist and $x(\tau_k^+) = x(\tau_k^-)$, $f(t, x_t)$ is continuous for almost all $t \in [\sigma, \infty)$ and at the discontinuous points f is right continuous.

(H₃) $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in $PC([-\tau, 0], R^n)$.

(H₄) The functions $I_k, J_k, k=1, 2, \dots$, are such that if $x \in D, I_k \neq 0$ and $J_k \neq 0$, then $I_k(x) + J_k(x(t - \tau)) \in D$.

Under the hypothesis (H₁)–(H₂), there is a unique solution of problems (2) through-out (σ, φ) .

We are using the following notations:

$$S(\rho) = \{x \in R^n : \|x\| < \rho\},$$

$$K = \{a \in C[R^+, R^+] : a(t) \text{ is monotone strictly increasing and } a(0) = 0\},$$

We have following definitions.

Definition 2.1 : The trivial solution of (2) is said to be

- 1) Strictly practical stable, if for any $\sigma \geq t_0$ there exist (λ_1, A_1) , $\lambda_1 \leq A_1$, we have $\|\varphi\| \leq \lambda_1$ implies $\|x(t)\| \leq A_1$ for some $t_0 \in R^+$; and for every $\lambda_2 \leq \lambda_1$ there exist $A_2 \leq \lambda_2$, such that $\|\varphi\| > \lambda_2$ implies $\|x(t)\| > A_2$ for all $t \geq \sigma$.
- 2) Uniformly strictly practical stable if (1) holds for all $t_0 \in R^+$.

Definition 2.2 : The function $V(x) : [t_0, \infty) \times S(\rho) \rightarrow R^+$ belongs to class V_0 if:

- 1) The function V is continuous on each of the sets $[\tau_{k-1}, \tau_k) \times S(\rho)$ and for all $t \geq t_0$, $V(t, 0) \equiv 0$;
- 2) $V(t, x)$ is locally Lipschitzian in $x \in S(\rho)$;
- 3) For each $k = 1, 2, \dots$, there exist finite limits

$$\lim_{(t, y) \rightarrow (\tau_k^-, x)} V(t, y) = V(\tau_k^-, x)$$

With $V(\tau_k^+, x) = V(t, x)$ satisfied.

Definition 2.3 : Let $V \in V_0$, for $(t, x) \in [\tau_{k-1}, \tau_k) \times S(\rho)$, D^+V is defined as

$$D^+V(t, x(t)) = \lim_{\delta \rightarrow 0^+} \sup \frac{1}{\delta} \{V(t + \delta, x(t + \delta)) - V(t, x(t))\}$$

III. MAIN RESULT

Now we consider the uniformly strict practical stability of the impulsive functional differential equation in which the state variables on the impulses are related to the time delay. We have the following theorem about the uniform practical stability of the system (1).

Theorem 3.1 Assume that

- (i) There exist $0 < \lambda_1 \leq A_1$, $a_1, b_1 \in k$, $V_{\lambda_1}(t, x(t)) \in v_0$. Such that

$$b_1(\|x(t)\|) \leq V_{\lambda_1}(t, x(t)) \leq a_1(\|x(t)\|).$$

- (ii) For any solution of (1), $V_{\lambda_1}(t+s, x(t+s)) \leq V_{\lambda_1}(t, x(t))$ for $s \in [-\tau, 0]$ and $D^+V_{\lambda_1}(t, x(t)) \leq 0$.

- (iii) For all $k \in Z^+$, $x \in S(\rho)$.

$$V_{\lambda_1}(\tau_k, I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau))) \leq \frac{1+c_k}{2} [V_{\lambda_1}(\tau_k^-, x(\tau_k^-)) + V_{\lambda_1}(\tau_k^- - \tau, x(\tau_k^- - \tau))].$$

Where

$$c_k \geq 0 \text{ and } \sum_{k=1}^{\infty} c_k < \infty.$$

- (iv) For any $0 < \lambda_2 \leq \lambda_1$, there exist $a_2, b_2 \in k$, $V_{\lambda_2}(t, x(t)) \in v_0$. Such that

$$b_2(\|x(t)\|) \leq V_{\lambda_2}(t, x(t)) \leq a_2(\|x(t)\|).$$

- (v) For any solution of (1), $V_{\lambda_2}(t+s, x(t+s)) \geq V_{\lambda_2}(t, x(t))$ for $s \in [-\tau, 0]$ and

$$D^+V_{\lambda_2}(t, x(t)) \geq 0.$$

- (vi) For all $k \in Z^+$, $x \in S(\rho)$.

$$\begin{aligned} & V_{\lambda_2}(\tau_k, I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau))) \\ & \geq \frac{1-d_k}{2} [V_{\lambda_2}(\tau_k^-, x(\tau_k^-)) + V_{\lambda_2}(\tau_k^- - \tau, x(\tau_k^- - \tau))]. \end{aligned}$$

$$\text{Where } 0 \leq d_k < 1 \text{ and } \sum_{k=1}^{\infty} d_k < \infty.$$

Then the trivial solution of (1) is uniformly strictly practical stable.

Proof :

Since

$$\sum_{k=1}^{\infty} b_k < \infty, \sum_{k=1}^{\infty} c_k < \infty.$$

It follows that

$$\prod_{k=1}^{\infty} \frac{1+c_k}{2} = M \text{ and } \prod_{k=1}^{\infty} \frac{1-b_k}{2} = N.$$

Obviously

$$1 \leq M \leq \infty, 0 < N \leq 1.$$

Let $0 \leq A_1 < \rho$ and $\sigma \geq t_0$ be given and $\sigma \in [\tau_{k-1}, \tau_k]$ for some $k \in Z^+$.

Choose $0 < \lambda_1 \leq A_1$ such that $Ma_1(\lambda_1) < b_1(A_1)$.

We claim that $\|\varphi\| \leq \lambda_1$ implies $\|x(t)\| < A_1$ for $t \geq \sigma$.

Obviously for any $t \in [\sigma - \tau, \sigma]$, there exist a $\theta \in [-\tau, 0]$, Such that

$$\begin{aligned} V_{\lambda_1}(t, x(t)) &= V_{\lambda_1}(\sigma + \theta, x(\sigma + \theta)) \\ &\leq a_1(\|x(\sigma + \theta)\|) = a_1(\|x_\sigma(\theta)\|) = a_1(\|\varphi(\theta)\|) \leq a_1(\lambda_1). \end{aligned}$$

Then we claim that

$$V_{\lambda_1}(t, x(t)) \leq a_1(\lambda_1) \text{ for } \sigma \leq t < \tau_k. \quad (3)$$

In fact, if inequality (3) does not hold, then there exist a $\hat{t} \in (\sigma, \tau_k)$ such that $V_{\lambda_1}(\hat{t}, x(\hat{t})) > a_1(\lambda_1) \geq V_{\lambda_1}(\sigma, x(\sigma))$ which implies that there is a $\bar{t} \in (\sigma, \hat{t})$ such that

$$D^+V_{\lambda_1}(\bar{t}, x(\bar{t})) > 0 \quad (4)$$

and $V_{\lambda_1}(\bar{t} + s, x(\bar{t} + s)) \leq V_{\lambda_1}(\bar{t}, x(\bar{t}))$ for $s \in [-\tau, 0]$.

By condition (ii), which implies $D^+V_{\lambda_1}(\bar{t}, x(\bar{t})) \leq 0$. This contradict inequality (4) so inequality (3) holds.

In view of inequality (3) and condition (iii), we have

$$\begin{aligned} V_{\lambda_1}(\tau_k, x(\tau_k)) &= V_{\lambda_1}(\tau_k, I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau))) \\ &\leq \frac{1+c_k}{2}[V_{\lambda_1}(\tau_k^-, x(\tau_k^-)) + V_{\lambda_1}(\tau_k^- - \tau, x(\tau_k^- - \tau))] \leq (1+c_k)a_1(\lambda_1) \end{aligned}$$

Next we prove that

$$V_{\lambda_1}(t, x(t)) \leq (1+c_k)a_1(\lambda_1) \text{ for } \tau_k \leq t \leq \tau_{k+1}. \quad (5)$$

If inequality (5) does not hold, then there is a $\hat{s} \in (\tau_k, \tau_{k+1})$ such that

$$V_{\lambda_1}(\hat{s}, x(\hat{s})) > (1+c_k)a_1(\lambda_1) \geq V_{\lambda_1}(\tau_k, x(\tau_k))$$

Which implies that there is a $\bar{s} \in (\tau_k, \hat{s})$ such that

$$D^+V_{\lambda_1}(\bar{s}, x(\bar{s})) > 0 \quad (6)$$

and $V_{\lambda_1}(\bar{s} + s, x(\bar{s} + s)) \leq V_{\lambda_1}(\bar{s}, x(\bar{s}))$ for $s \in [-\tau, 0]$.

By condition (ii) which implies that $D^+V_{\lambda_1}(\bar{s}, x(\bar{s})) \leq 0$. This contradicts inequality (6), so inequality (5) holds.

In view of inequality (6) and the condition (iii), we have

$$\begin{aligned} V_{\lambda_1}(\tau_{k+1}, x(\tau_{k+1})) &= V_{\lambda_1}(\tau_{k+1}, I_{k+1}(x(\tau_{k+1}^-)) + J_{k+1}(x(\tau_{k+1}^- - \tau))) \\ &\leq \frac{1+c_{k+1}}{2}[V_{\lambda_1}(\tau_{k+1}^-, x(\tau_{k+1}^-)) + V_{\lambda_1}(\tau_{k+1}^- - \tau, x(\tau_{k+1}^- - \tau))] \\ &\leq (1+c_{k+1})(1+c_k)a_1(\lambda_1). \end{aligned}$$

This together with inequality (3) yields

$$V_{\lambda_1}(t, x(t)) \leq Ma_1(\lambda_1).$$

From this and condition (i) we have

$$b_1(\|x(t)\|) \leq V_{\lambda_1}(t, x(t)) \leq Ma_1(\lambda_1) \leq b_1(A_1) \quad \text{for } t \geq \sigma.$$

Thus, we have $\|x(t)\| < A_1$ for $t \geq \sigma$.

Now, let $0 < \lambda_2 \leq \lambda_1$, and Choose $0 < A_2 \leq \lambda_2$ such that $a_2(A_2) < Nb_2(\lambda_2)$.

Next we claim that $\|\varphi\| > \lambda_2$ implies $\|x(t)\| > A_2$ for $t \geq \sigma$.

If this holds then $A_2 < \|x(t)\| < A_1$ for $t \geq \sigma$.

Obviously for any $t \in [\sigma - \tau, \sigma]$, there exist a $\theta \in [-\tau, 0]$, Such that

$$V_{\lambda_2}(t, x(t)) = V_{\lambda_2}(\sigma + \theta, x(\sigma + \theta)) \geq b_2(\|x(\sigma + \theta)\|) = b_2(\|x_\sigma(\theta)\|) = b_2(\|\varphi(\theta)\|) \geq b_2(\lambda_2).$$

Then we claim that

$$V_{\lambda_2}(t, x(t)) \geq b_2(\lambda_2) \quad \text{for } \sigma \leq t < \tau_k. \quad (7)$$

In fact, if inequality (7) does not hold, then there exist a $\bar{t} \in (\sigma, \tau_k)$ such that

$$V_{\lambda_2}(\bar{t}, x(\bar{t})) < b_2(\lambda_2) \leq V_{\lambda_2}(\sigma, x(\sigma)) \quad (8)$$

which implies that there is a $t_1 \in (\sigma, \bar{t})$ such that

$$D^+V_{\lambda_2}(t_1, x(t_1)) < 0 \quad (9)$$

and

$$V_{\lambda_2}(t_1 + s, x(t_1 + s)) \geq V_{\lambda_2}(t_1, x(t_1)) \quad \text{for } s \in [-\tau, 0].$$

By condition (v), which implies $D^+V_{\lambda_2}(t_1, x(t_1)) \geq 0$. This contradict inequality (9) so inequality (8) holds.

In view of inequality (7) and condition (vi), we have

$$V_{\lambda_2}(\tau_k, x(\tau_k)) = V_{\lambda_2}(\tau_k, I_k(x(\tau_k^-)) + J_k(x(\tau_k^- - \tau))) \geq \frac{1-d_k}{2}[V_{\lambda_2}(\tau_k^-, x(\tau_k^-)) + V_{\lambda_2}(\tau_k^- - \tau, x(\tau_k^- - \tau))] \geq (1-d_k)b_2(\lambda_2)$$

Next we prove that

$$V_{\lambda_2}(t, x(t)) \geq (1-d_k)b_2(\lambda_2) \quad \text{for } \tau_k \leq t \leq \tau_{k+1}. \quad (10)$$

If inequality (10) does not hold, then there is a $\hat{r} \in (\tau_k, \tau_{k+1})$ such that

$$V_{\lambda_2}(\hat{r}, x(\hat{r})) < (1-d_k)b_2(\lambda_2) \leq V_{\lambda_2}(\tau_k, x(\tau_k))$$

Which implies that there is a $\bar{r} \in (\tau_k, \hat{r})$ such that

$$D^+V_{\lambda_2}(\bar{r}, x(\bar{r})) < 0 \quad (11)$$

And

$$V_{\lambda_1}(\bar{r} + s, x(\bar{r} + s)) \leq V_{\lambda_1}(\bar{r}, x(\bar{r})) \quad \text{for } s \in [-\tau, 0].$$

By condition (v) which implies that $D^+V_{\lambda_2}(\bar{r}, x(\bar{r})) \geq 0$. This contradicts inequality (11), so inequality (10) holds.

In view of inequality (11) and the condition (vi), we have

Which implies that there is a $\bar{r} \in (\tau_k, \hat{r})$ such that

$$D^+V_{\lambda_2}(\bar{r}, x(\bar{r})) < 0 \quad (11)$$

And $V_{\lambda_1}(\bar{s} + s, x(\bar{s} + s)) \leq V_{\lambda_1}(\bar{s}, x(\bar{s}))$ for $s \in [-\tau, 0]$.

By condition (v) which implies that $D^+V_{\lambda_2}(\bar{r}, x(\bar{r})) \geq 0$. This contradicts inequality (11), so inequality (10) holds.

In view of inequality (11) and the condition (vi), we have

$$\begin{aligned} V_{\lambda_2}(\tau_{k+1}, x(\tau_{k+1})) &= V_{\lambda_2}(\tau_{k+1}, I_{k+1}(x(\tau_{k+1}^-)) + J_{k+1}(x(\tau_{k+1}^- - \tau))) \\ &\geq \frac{1-d_{k+1}}{2} [V_{\lambda_2}(\tau_{k+1}^-, x(\tau_{k+1}^-)) + V_{\lambda_2}(\tau_{k+1}^- - \tau, x(\tau_{k+1}^- - \tau))] \\ &\geq (1-d_{k+1})(1-d_k)b_2(\lambda_2). \end{aligned}$$

By simple induction, we can prove that in general that for $k=0,1,2,\dots$

$$V_{\lambda_2}(t, x(t)) \geq (1-d_{k+m})\dots(1-d_k)b_2(\lambda_2) \quad \text{for } \tau_{k+m} \leq t < \tau_{k+m+1}.$$

This together with inequality (7) yields $V_{\lambda_2}(t, x(t)) \geq Nb_2(\lambda_2)$.

From this and condition (iv) we have

$$a_2(\|x(t)\|) \geq V_{\lambda_2}(t, x(t)) \geq Nb_2(\lambda_2) > a_2(A_2) \quad \text{for } t \geq \sigma.$$

Thus, we have $\|x(t)\| > A_2$ for $t \geq \sigma$.

Thus, the trivial solution of (1) is uniformly strictly practical stable.

IV. CONCLUSION

In this paper, the strict practical stability of impulsive functional differential equations in which the state variables on the impulses are related to the time delay is considered. By using Lyapunov functions and Razumikhin technique, we have obtained some results for the strict practical stability. Strict practical stability theorem for impulsive functional differential equation have been extended to impulsive functional differential equations in which the state variables on the impulses are related to the time delay.

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A Study on New Sequence of Functions Involving the Generalized Contour Integral

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Abstract - The aim of the present paper is an attempt to introduce a new sequence of functions $\{V_n^{(M,N;P,Q,R;\alpha)}(x;a,k,s)/n=0,1,2,\dots\}$ in equation (1.15), which involving with the generalized contour integral $I_{P,Q,R}^{M,N}[x]$. By using operational technique, some interesting generating relations and finite summation formulae are obtained in sections 2 and 3. These generating relations and finite summation formulae are unified in nature and act as key formulae from which, we can obtain as their special cases. For the sake of illustration we record here some special cases of our main formulas in section 4, which are also new and known.

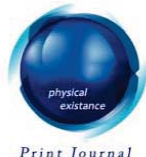
Keywords : generating function, generalized contour integral, operational techniques.

GJSFR-F Classification : MSC 2010: 11B34, 11R33



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A Study on New Sequence of Functions Involving the Generalized Contour Integral

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Abstract - The aim of the present paper is an attempt to introduce a new sequence of functions $\left\{V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) / n = 0, 1, 2, \dots\right\}$ in equation (1.15), which involving with the generalized contour integral $I_{P,Q;R}^{M,N}[x]$. By using operational technique, some interesting generating relations and finite summation formulae are obtained in sections 2 and 3. These generating relations and finite summation formulae are unified in nature and act as key formulae from which, we can obtain as their special cases. For the sake of illustration we record here some special cases of our main formulas in section 4, which are also new and known.

Keywords : generating function, generalized contour integral, operational techniques.

1. INTRODUCTION

Operational techniques have drawn the attention of several researchers in the study of sequences of functions and polynomials.

In 1956, Chak [1] defined a class of polynomials as,

$$G_{n,k}^{(\alpha)}(x) = x^{-\alpha-kn+n} e^x (x^k D)^n [x^\alpha e^{-x}] \quad (1.1)$$

Where k is constant and $n = 0, 1, 2, \dots, D \equiv \frac{d}{dx}$.

Gould and Hopper [5] introduced generalized Hermite polynomials in 1962 as,

$$H_n^r(x, a, p) = (-1)^n x^{-a} \exp(px^r) D^n [x^a \exp(-px^r)] \quad (1.2)$$

Chatterjea [4] studied as a class of polynomials for generalized Laguerre polynomials in 1964,

$$T_m^{(\alpha)}(x, p) = \frac{1}{n!} x^{-\alpha-n-1} \exp(px^r) (x^2 D)^n [x^{\alpha+1} \exp(-px^r)] \quad (1.3)$$

In 1968, Singh [16] studied the generalized Truesdell polynomials defined as,

$$T_n^{(\alpha)}(x, r, p) = x^{-\alpha} \exp(px^r) (xD)^n [x^\alpha \exp(-px^r)] \quad (1.4)$$

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Srivastava and Singh [19] introduced a general class of polynomials in 1971 as,

$$G_n^{(\alpha)}(x, r, p, k) = \frac{1}{n!} x^{-\alpha-kn} \exp(px^r) (x^{k+1}D)^n [x^\alpha \exp(-px^r)] \quad (1.5)$$

In 1971, the Rodrigues formula for generalized Laguerre polynomials is given by Mittal [8] as,

$$T_{kn}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} \exp(p_k(x)) D^n [x^{\alpha+n} \exp(-p_k(x))] \quad (1.6)$$

Where $p_k(x)$ is a polynomial in x of degree k .

Mittal [9] also proved following relation for (1.6) as,

$$T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!} x^{-\alpha-n} \exp(p_k(x)) T_s^n [x^\alpha \exp(-p_k(x))] \quad (1.7)$$

Where s is constant and $T_s \equiv x(s + xD)$.

Recently, Shukla and Prajapati [15] obtained several properties of (1.7).

Chandel [2] also studied a class of polynomials defined in 1973 as,

$$T_n^{\alpha,k}(x, r, p) = x^{-\alpha} \exp(px^r) (x^k D)^n [x^\alpha \exp(-px^r)] \quad (1.8)$$

In 1974, Chandel [3] established a generalization of polynomial system as,

$$G_n(h, g(x), k) = e^{-hg(x)} (x^k D)^n [e^{-hg(x)}] \quad (1.9)$$

Where $g(x)$ is a function of x , h and k are constants.

In the same year, Srivastava [14] discussed some operational formulas generalized function

$F_n^{(r,m)}(x, a, k, p)$ in the form,

$$F_n^{(r,m)}(x, a, k, p) = x^{-a} \exp(px^r) (x^k D)^n [x^{a+km} \exp(-px^r)] \quad (1.10)$$

In 1975, Joshi and Parjapat [6] introduced the a class of polynomial,

$$M_{vn}^{(\alpha)}(x, a, k) = \frac{x^{-\alpha-nk}}{n!} \exp(p_v(x)) [x^k (a + xD)]^n [x^\alpha \exp(-p_v(x))] \quad (1.11)$$

Subsequently in 1975, Patil and Thakare [11] have obtained several formulae and generating relation for

$$P_n^{(\alpha)}(x, r, s, a, k, \lambda) = x^{-\alpha} \exp(px^r) [x^k (\lambda + xD)]^n [x^{\alpha+sn} \exp(-px^r)] \quad (1.12)$$

R_{ef.}

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In 1979, Srivastava and Singh [17] studied a sequence of functions $V_n^{(\alpha)}(x; a, k, s)$ defined as

$$V_n^{(\alpha)}(x; a, k, s) = \frac{x^{-\alpha}}{n!} \exp\{p_k(x)\} \theta^n [x^\alpha \exp\{-p_k(x)\}] \quad (1.13)$$

By employing the operator $\theta \equiv x^a(s + xD)$, where s is constant and $p_k(x)$ is a polynomial in x of degree k .

J.C. Parjapati and N.K. Ahudia [12] introduced the sequence of function defined as,

$$V_n^{(\alpha, \beta, \delta)}(x; a, k, s) = \frac{1}{n!} x^{-\beta} W(\alpha, \delta; p_k(x)) (T_x^{a, s})^n [x^\beta W(\alpha, \delta; -p_k(x))] \quad (1.14)$$

A new sequence of function $\{V_n^{(M, N; P, Q; \alpha)}(x; a, k, s) / n = 0, 1, 2, \dots\}$ is introduced in this paper as,

$$V_n^{(M, N; P, Q; R; \alpha)}(x; a, k, s) = \frac{1}{n!} x^{-\alpha} I_{P, Q; R}^{M, N} [p_k(x)] (T_x^{a, s})^n [x^\alpha I_{P, Q; R}^{M, N} [-p_k(x)]] \quad (1.15)$$

Where $T_x^{a, s} \equiv x^a(s + xD)$, $D \equiv \frac{d}{dx}$, a and s are constants, $\beta \geq 0$, k is finite and non-negative integer, $p_k(x)$ is a polynomial in x of degree k and $I_{P, Q; R}^{M, N}[x]$ is I-function, defined and represented in the following manner [13]:

$$I_{P, Q; R}^{M, N} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{1, N} \\ (b_j, \beta_j)_{1, M} \end{matrix} \right. \begin{matrix} (a_{ji}, \alpha_{ji})_{N+1, P_i} \\ (b_{ji}, \beta_{ji})_{M+1, Q_i} \end{matrix} \right] = \frac{1}{2\pi i} \int_L \phi(\xi) z^\xi d\xi \quad (1.16)$$

$$\phi(\xi) = \frac{\prod_{j=1}^M \Gamma(b_j - \beta_j \xi) \prod_{j=1}^N \Gamma(1 - a_j + \alpha_j \xi)}{\sum_{i=1}^R \left[\prod_{j=M+1}^{Q_i} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \prod_{j=N+1}^{P_i} \Gamma(a_{ji} - \alpha_{ji} \xi) \right]} \quad (1.17)$$

and M, N, P_i, Q_i are integers satisfy $0 \leq N \leq P_i, 1 \leq M \leq Q_i (i = 1, \dots, R)$, R is finite, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are positive numbers and a_j, b_j, a_{ji}, b_{ji} are complex numbers. I-function which is a generalized form of the well known Fox's H-function [18]. In the sequel the I-function will be studied under the following conditions of existence:

$$(I) A_i > 0, |\arg z| < \frac{A_i \pi}{2} \quad (1.18)$$

$$(II) A_i \geq 0, |\arg z| \leq \frac{A_i \pi}{2} \text{ and } \operatorname{Re}(B+1) < 0 \quad (1.19)$$

$$\text{Where } A_i = \sum_{j=1}^N \alpha_j - \sum_{j=N+1}^{P_i} \alpha_{ji} + \sum_{j=1}^M \beta_j - \sum_{j=M+1}^{Q_i} \beta_{ji}, \forall i = (1, 2, \dots, R) \quad (1.20)$$

$$\text{And } B = \sum_{j=1}^M b_j + \sum_{j=M+1}^{Q_i} b_{ji} - \sum_{j=1}^N a_j - \sum_{j=N+1}^{P_i} a_{ji} + \frac{1}{2}(P_i - Q_i), \forall i = (1, 2, \dots, R) \quad (1.21)$$

Some generating relations and finite summation formulae of class of polynomials or sequence of function have been obtained by using the properties of the differential operators. $T_x^{a,s} \equiv x^a (s + xD)$, $T_x^{a,1} \equiv x^a (1 + xD)$, where $D \equiv \frac{d}{dx}$, is based on the work of Mittal [10], Patil and Thakare [11], Srivastava and Singh [17]. Some useful Operational Techniques are given below:

$$\exp(t T_x^{a,s}) (x^\beta f(x)) = x^\beta (1 - ax^a t)^{-\left(\frac{\beta+s}{a}\right)} f(x(1 - ax^a t)^{-1/a}) \quad (1.22)$$

$$\exp(t T_x^{a,s}) (x^{\alpha-an} f(x)) = x^\alpha (1 + at)^{-1+\left(\frac{\alpha+s}{a}\right)} f(x(1 + at)^{1/a}) \quad (1.23)$$

$$(T_x^{a,s})^n (xuv) = x \sum_{m=0}^{\infty} \binom{n}{m} (T_x^{a,s})^{n-m} (v) (T_x^{a,1})^m (u) \quad (1.24)$$

$$(1 + xD)(1 + a + xD)(1 + 2a + xD)(1 + 3a + xD) \dots (1 + (m-1)a + xD) x^{\beta-1} = a^m \left(\frac{\beta}{a}\right)_m x^{\beta-1} \quad (1.25)$$

$$(1 - at)^{-\frac{\alpha}{a}} = (1 - at)^{-\frac{\beta}{a}} \sum_{m=0}^{\infty} \left(\frac{\alpha - \beta}{a}\right)_m \frac{(at)^m}{m!} \quad (1.26)$$

II. GENERATING RELATIONS

$$\sum_{n=0}^{\infty} V_n^{(M,N;P,Q,R;\alpha)} (x; a, k, s) x^{-an} t^n = (1 - at)^{-\left(\frac{\alpha+s}{a}\right)} I_{P,Q;R}^{M,N} (p_k(x)) I_{P,Q;R}^{M,N} \left(-p_k(x(1 - at)^{-1/a})\right) \quad (2.1)$$

$$\sum_{n=0}^{\infty} V_n^{(M,N;P,Q,R;\alpha-an)} (x; a, k, s) x^{-an} t^n = (1 + at)^{-1+\left(\frac{\alpha+s}{a}\right)} I_{P,Q;R}^{M,N} (p_k(x)) I_{P,Q;R}^{M,N} \left(-p_k(x(1 + at)^{1/a})\right) \quad (2.2)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{m+n}{m} V_n^{(M,N;P,Q,R;\alpha)} (x; a, k, s) t^n \\ &= (1 - at)^{-\left(\frac{\alpha+s}{a}\right)} \frac{I_{P,Q;R}^{M,N} (p_k(x))}{I_{P,Q;R}^{M,N} (p_k(x(1 - at)^{-1/a}))} V_n^{(M,N;P,Q,R;\alpha)} (x(1 - at)^{-1/a}; a, k, s) \end{aligned} \quad (2.3)$$

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17. Srivastava, A. N. and Singh, S. N., (1979) Some generating relations connected with a function defined by a Generalized Rodrigues formula, Indian J. Pure Appl. Math., 10(10), 1312-1317.

Proof of (2.1)

From (1.15), we consider

$$\sum_{n=0}^{\infty} V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) t^n = x^{-\alpha} I_{P,Q;R}^{M,N}(p_k(x)) \exp(t T_x^{a,s}) [x^{\alpha} I_{P,Q;R}^{M,N}(-p_k(x))] \quad (2.4)$$

Using operational technique (1.22), above equation (2.4) reduces to:

$$\begin{aligned} & \sum_{n=0}^{\infty} V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) t^n \\ &= x^{-\alpha} I_{P,Q;R}^{M,N}(p_k(x)) x^{\alpha} (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} I_{P,Q;R}^{M,N}\left(-p_k\left(x(1 - ax^a t)^{-1/a}\right)\right) \\ &= (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} I_{P,Q;R}^{M,N}(p_k(x)) I_{P,Q;R}^{M,N}\left(-p_k\left(x(1 - ax^a t)^{-1/a}\right)\right) \end{aligned} \quad (2.5)$$

And replacing t by tx^{-a} , this gives (2.1).

Proof of (2.2)

Again from (1.15), we have

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(M,N;P,Q,R;\alpha-an)}(x; a, k, s) t^n = x^{-\alpha} I_{P,Q;R}^{M,N}(p_k(x)) \exp(t T_x^{a,s}) [x^{\alpha-an} I_{P,Q;R}^{M,N}(-p_k(x))] \quad (2.6)$$

applying the operational technique (1.23), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} x^{-an} V_n^{(M,N;P,Q,R;\alpha-an)}(x; a, k, s) t^n \\ &= x^{-\alpha} I_{P,Q;R}^{M,N}(p_k(x)) x^{\alpha} (1 + at)^{\frac{\alpha+s}{a}-1} I_{P,Q;R}^{M,N}\left[-p_k\left(x(1 + at)^{1/a}\right)\right] \\ &= (1 + at)^{\frac{\alpha+s}{a}-1} I_{P,Q;R}^{M,N}(p_k(x)) I_{P,Q;R}^{M,N}\left[-p_k\left(x(1 + at)^{1/a}\right)\right] \end{aligned} \quad (2.7)$$

This proves (2.2).

Proof of (2.3)

We can write (1.15) as,

$$(T_x^{a,s})^n [x^{\alpha} I_{P,Q;R}^{M,N}(-p_k(x))] = n! x^{\alpha} \frac{1}{I_{P,Q;R}^{M,N}(p_k(x))} V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) \quad (2.8)$$

Or

$$\exp\left(t(T_x^{a,s})\right) \left\{ (T_x^{a,s})^n [x^{\alpha} I_{P,Q;R}^{M,N}(-p_k(x))] \right\}$$

$$\begin{aligned}
&= n! \exp(t T_x^{a,\alpha}) \left[x^\alpha \frac{1}{I_{P,Q,R}^{M,N}(p_k(x))} V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) \right] \\
&\quad \sum_{m=0}^{\infty} \frac{t^m}{m!} (T_x^{a,s})^{m+n} \left[x^\alpha I_{P,Q,R}^{M,N}(-p_k(x)) \right] \\
&= n! \exp(T_x^{a,s}) \left[x^\alpha \frac{1}{I_{P,Q,R}^{M,N}(p_k(x))} V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) \right]
\end{aligned} \tag{2.9}$$

Using the operational technique (1.20), above equation can be written as:

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{t^m}{m!} (T_x^{a,s})^{m+n} \left[x^\alpha I_{P,Q,R}^{M,N}(-p_k(x)) \right] \\
&= n! x^\alpha (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \frac{1}{I_{P,Q,R}^{M,N}\left(p_k\left(x(1 - ax^a t)^{-1/a}\right)\right)} V_n^{(M,N;P,Q,R;\alpha)}\left(x(1 - ax^a t)^{-1/a}; a, k, s\right)
\end{aligned} \tag{2.10}$$

use of (2.8) gives

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{t^m (m+n)!}{m! n!} x^\alpha \frac{1}{I_{P,Q,R}^{M,N}(p_k(x))} V_{m+n}^{(M,N;P,Q,R;\alpha)}(x; a, k, s) \\
&= x^\alpha (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \frac{1}{I_{P,Q,R}^{M,N}\left[p_k\left(x(1 - ax^a t)^{-1/a}\right)\right]} V_n^{(M,N;P,Q,R;\alpha)}\left(x(1 - ax^a t)^{-1/a}; a, k, s\right)
\end{aligned} \tag{2.11}$$

Therefore

$$\begin{aligned}
&\sum_{m=0}^{\infty} \binom{m+n}{n} V_{m+n}^{(M,N;P,Q,R;\alpha)}(x; a, k, s) t^m \\
&= (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \frac{I_{P,Q,R}^{M,N}(p_k(x))}{I_{P,Q,R}^{M,N}\left(p_k\left(x(1 - ax^a t)^{-1/a}\right)\right)} V_n^{(M,N;P,Q,R;\alpha)}\left(x(1 - ax^a t)^{-1/a}; a, k, s\right)
\end{aligned} \tag{2.12}$$

And replacing t by tx^{-a} , this gives the result (2.3).

III. FINITE SUMMATION FORMULAE

$$V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(a \frac{x}{x}\right)^m \left(\frac{\alpha}{a}\right) V_{n-m}^{(M,N;P,Q,R;0)}(x; a, k, s) \tag{3.1}$$

$$V_n^{(M,N;P,Q,R;\alpha)}(x;a,k,s) = \sum_{m=0}^{\infty} \frac{1}{m!} (a \ x)^m \left(\frac{\alpha - \beta}{a} \right)_m V_{n-m}^{(M,N;P,Q,R;\beta)}(x;a,k,s) \quad (3.2)$$

Proof of (3.1)

From equation (1.15), we have:

$$V_n^{(M,N;P,Q,R;\alpha)}(x;a,k,s) = \frac{1}{n!} x^{-\alpha} I_{P,Q;R}^{M,N}(p_k(x)) (T_x^{(a,s)})^n [x x^{\alpha-1} I_{P,Q;R}^{M,N}(-p_k(x))] \quad (3.3)$$

Using the operational technique (1.24), we have:

$$\begin{aligned} & V_n^{(M,N;P,Q,R;\alpha)}(x;a,k,s) \\ &= \frac{1}{n!} x^{-\alpha} I_{P,Q;R}^{M,N}(p_k(x)) x \sum_{m=0}^n \binom{n}{m} (T_x^{(a,s)})^{n-m} [I_{P,Q;R}^{M,N}(-p_k(x))] (T_x^{a,1})^m (x^{\alpha-1}) \\ &= \frac{1}{n!} x^{-\alpha} I_{P,Q;R}^{M,N}(p_k(x)) x \sum_{m=0}^n \frac{n!}{m!(n-m)!} x^{a(n-m)} \times \\ & \quad \left[(s + xD)(s + a + xD)(s + 2a + xD) \dots (s + (n-m-1)a + xD) \right] \times \\ & \quad \left[I_{P,Q;R}^{M,N}(-p_k(x)) \right] x^{am} \left[(1 + xD)(1 + a + xD)(1 + 2a + xD) \dots (1 + (m-1)a + xD) \right] (x^{\alpha-1}) \end{aligned} \quad (3.4)$$

Using the result (1.24), we have:

$$= I_{P,Q;R}^{M,N}(p_k(x)) \sum_{m=0}^n \frac{1}{m!(n-m)!} x^a \prod_{i=0}^{n-m-1} (s + ia + xD) [I_{P,Q;R}^{M,N}(-p_k(x))] a^m \left(\frac{\alpha}{a} \right)_m \quad (3.5)$$

Put $\alpha = 0$ and replacing n by $n-m$ in (3.3), we get:

$$\begin{aligned} & V_{n-m}^{(M,N;P,Q,R;0)}(x;a,k,s) = \frac{1}{(n-m)!} I_{P,Q;R}^{M,N}(p_k(x)) (T_x^{a,s})^{n-m} [I_{P,Q;R}^{M,N}(-p_k(x))] \\ & \Rightarrow \frac{1}{(n-m)!} (T_x^{a,s})^{n-m} [I_{P,Q;R}^{M,N}(-p_k(x))] = \frac{1}{I_{P,Q;R}^{M,N}(p_k(x))} V_{n-m}^{(M,N;P,Q,R;0)}(x;a,k,s) \end{aligned} \quad (3.6)$$

This gives

$$\frac{1}{(n-m)!} \prod_{i=0}^{n-m-1} (s + ia + xD) [I_{P,Q;R}^{M,N}(-p_k(x))] = x^{a(m-n)} \frac{1}{I_{P,Q;R}^{M,N}(p_k(x))} V_{n-m}^{(M,N;P,Q,R;0)}(x;a,k,s) \quad (3.7)$$

From equation (3.5) and (3.7) we have the main the result.

Proof of (3.2)

Equation (1.15) can be written as:

$$\sum_{n=0}^{\infty} V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) t^n = x^{-\alpha} I_{P,Q;R}^{M,N}(p_k(x)) \exp(T_x^{(a,s)}) [x^{\beta} I_{P,Q;R}^{M,N}(-p_k(x))] \quad (3.8)$$

Applying the (1.21) in equation (3.8), we have:

$$\begin{aligned} & \sum_{n=0}^{\infty} V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) t^n \\ &= x^{-\alpha} I_{P,Q;R}^{M,N}(p_k(x)) x^{\alpha} (1-ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} I_{P,Q;R}^{M,N}\left(-p_k\left(x(1-ax^a t)^{-1/a}\right)\right) \\ &= (1-ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} I_{P,Q;R}^{M,N}(p_k(x)) I_{P,Q;R}^{M,N}\left(-p_k\left(x(1-ax^a t)^{-1/a}\right)\right) \end{aligned} \quad (3.9)$$

Applying the result from equation (1.26); equation (3.9) reduces to:

$$\begin{aligned} &= (1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)} \sum_{m=0}^{\infty} \left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a t)^m}{m!} I_{P,Q;R}^{M,N}(p_k(x)) I_{P,Q;R}^{M,N}\left(-p_k\left(x(1-ax^a t)^{-1/a}\right)\right) \\ &= \sum_{m=0}^{\infty} \left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a t)^m}{m!} x^{-\beta} I_{P,Q;R}^{M,N}(p_k(x)) \exp(T_x^{a,s}) [x^{\beta} I_{P,Q;R}^{M,N}(-p_k(x))] \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a)^m t^{n+m}}{m! n!} x^{-\beta} I_{P,Q;R}^{M,N}(p_k(x)) (T_x^{a,s})^n [x^{\beta} I_{P,Q;R}^{M,N}(-p_k(x))] \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a)^m t^n}{m! (n-m)!} x^{-\beta} I_{P,Q;R}^{M,N}(p_k(x)) (T_x^{a,s})^{n-m} [x^{\beta} I_{P,Q;R}^{M,N}(-p_k(x))] \end{aligned} \quad (3.10)$$

Now equating the coefficient of t^n , we get:

$$\begin{aligned} & V_n^{(M,N;P,Q,R;\alpha)}(x; a, k, s) \\ &= \sum_{m=0}^n \left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a)^m}{m! (n-m)!} x^{-\beta} I_{P,Q;R}^{M,N}(p_k(x)) (T_x^{a,s})^{n-m} [x^{\beta} I_{P,Q;R}^{M,N}(-p_k(x))] \end{aligned} \quad (3.11)$$

Using the equation (1.15) in (3.11) we have the result (3.2).

IV. SPECIAL CASES

- a) If we put $R=1$, I-function reduces to Fox's H-function [18, p. 10, Eqn. (2.1.1)], then the equation (2.1), (2.2) and (2.3) takes the following form:

$$\sum_{n=0}^{\infty} V_n^{(M,N;P,Q,1;\alpha)}(x; a, k, s) x^{-an} t^n$$

$$= (1-at)^{-1+\left(\frac{\alpha+s}{a}\right)} H_{P,Q}^{M,N}(p_k(x)) H_{P,Q}^{M,N}\left(-p_k\left(x(1-at)^{-1/a}\right)\right)$$
(4.1)

$$\sum_{n=0}^{\infty} V_n^{(M,N;P,Q,1;\alpha-an)}(x; a, k, s) x^{-an} t^n$$

$$= (1+at)^{-1+\left(\frac{\alpha+s}{a}\right)} H_{P,Q}^{M,N}(p_k(x)) H_{P,Q}^{M,N}\left(-p_k\left(x(1+at)^{1/a}\right)\right)$$
(4.2)

$$\sum_{n=0}^{\infty} \binom{m+n}{m} V_n^{(M,N;P,Q,1;\alpha)}(x; a, k, s) t^n$$

$$= (1-at)^{-1+\left(\frac{\alpha+s}{a}\right)} \frac{H_{P,Q}^{M,N}(p_k(x))}{H_{P,Q}^{M,N}\left(p_k\left(x(1-at)^{-1/a}\right)\right)} V_n^{(M,N;P,Q,1;\alpha)}\left(x(1-at)^{-1/a}; a, k, s\right)$$
(4.3)

- b) If we put $R=1; \alpha_j = \beta_j = 1$, then the I-function reduces to general type of G-function

$$[7] \text{ i.e. } I_{P,Q;R}^{M,N} \left[z \begin{matrix} (a_j, 1)_{1,N} \\ (b_j, 1)_{1,M} \end{matrix}, \begin{matrix} (a_j, 1)_{N+1,P} \\ (b_j, 1)_{M+1,Q} \end{matrix} \right] = G_{P,Q}^{M,N} \left[z \begin{matrix} (a_j, 1)_{1,p} \\ (b_j, 1)_{1,q} \end{matrix} \right], \text{ the equation (2.1), (2.2) and}$$

(2.3) takes the following form:

$$\sum_{n=0}^{\infty} V_n^{(M,N;P,Q,1;\alpha)}(x; a, k, s) x^{-an} t^n$$

$$= (1-at)^{-1+\left(\frac{\alpha+s}{a}\right)} G_{P,Q}^{M,N}(p_k(x)) G_{P,Q}^{M,N}\left(-p_k\left(x(1-at)^{-1/a}\right)\right)$$
(4.4)

$$\sum_{n=0}^{\infty} V_n^{(M,N;P,Q,1;\alpha-an)}(x; a, k, s) x^{-an} t^n$$

$$= (1+at)^{-1+\left(\frac{\alpha+s}{a}\right)} G_{P,Q}^{M,N}(p_k(x)) G_{P,Q}^{M,N}\left(-p_k\left(x(1+at)^{1/a}\right)\right)$$
(4.5)

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$$\sum_{n=0}^{\infty} \binom{m+n}{m} V_n^{(M,N;P,Q,1;\alpha)}(x; a, k, s) t^n$$

$$= (1-at)^{-\left(\frac{\alpha+s}{a}\right)} \frac{G_{P,Q}^{M,N}(p_k(x))}{G_{P,Q}^{M,N}(p_k(x(1-at)^{-1/a}))} V_n^{(M,N;P,Q,1;\alpha)}(x(1-at)^{-1/a}; a, k, s)$$

- c) If we put $R=1, N=P_i=P, M=1, Q_i=Q+1, b_1=0, \beta_1=1, a_j=1-a_j, b_{ji}=1-b_j, \beta_{ji}=\beta_j$, then I-function reduces to the generalized wright hypergeometric function [20, p.287],

$$\text{i.e. } I_{p,q+1;1}^{1,p} \left[z \left| \begin{matrix} (1-a_j, \alpha_j)_{1,p} \\ (0,1), (1-b_j, \beta_j)_{1,q} \end{matrix} \right. \right] = {}_p\psi_q \left[\begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} ; -z \right], \text{ the equation (2.1), (2.2) and}$$

(2.3) takes the following form:

$$\sum_{n=0}^{\infty} V_n^{(P,Q;\alpha)}(x; a, k, s) x^{-an} t^n$$

$$= (1-at)^{-\left(\frac{\alpha+s}{a}\right)} {}_p\psi_q(p_k(x)) {}_p\psi_q(-p_k(x(1-at)^{-1/a}))$$

$$\sum_{n=0}^{\infty} V_n^{(P,Q;\alpha-an)}(x; a, k, s) x^{-an} t^n$$

$$= (1+at)^{-1+\left(\frac{\alpha+s}{a}\right)} {}_p\psi_q(p_k(x)) {}_p\psi_q(-p_k(x(1+at)^{1/a}))$$

$$\sum_{n=0}^{\infty} \binom{m+n}{m} V_n^{(P,Q;\alpha)}(x; a, k, s) t^n$$

$$= (1-at)^{-\left(\frac{\alpha+s}{a}\right)} \frac{{}_p\psi_q(p_k(x))}{{}_p\psi_q(p_k(x(1-at)^{-1/a}))} V_n^{(P,Q;\alpha)}(x(1-at)^{-1/a}; a, k, s)$$

- d) The interesting special cases/relationships between (1.15) and class of polynomials (1.1)-(1.14) can also be obtained for appropriate values of parameters α, a, k, s and those of involving in I-function.

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Certain New Subclasses of Analytic Univalent Functions in the Unit Disk

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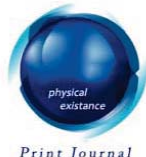
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Certain New Subclasses of Analytic Univalent Functions in the Unit Disk

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Abstract - Introduction of new subclasses $K(\omega, \alpha, \lambda)$ and $H(\omega, \alpha, \lambda)$ of analytic functions with negative co-efficients defined in the unit disk. We estimate some properties of functions in these classes and obtain Co-efficient bounds, Neighbourhood and Integral means inequality for the function $f(z)$ under these classes.

Keywords : convex, coefficient estimate, neighbourhood and integral means inequality.

1. INTRODUCTION AND DEFINITIONS

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Let S denote the subclass of A consisting of univalent functions $f(z)$ in E .

Let $A(\omega) \subset A$ denote the class of functions of the form

$$f(z) = (z - \omega) + \sum_{k=2}^{\infty} a_k (z - \omega)^k \quad (1.2)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$ and normalised with $f(\omega) = 0$ and $f'(\omega) - 1 = 0$ and ω is a fixed point in E .

Let $B(\omega, \alpha, \lambda)$ and $Q(\omega, \alpha, \lambda)$ denote the class of analytic function of order α in E defined as follows:

$$S(\omega) = \{f \in A(\omega) : f \text{ is univalent in } E\}$$

$$B(\omega, \alpha, \lambda) = \left\{ f \in S(\omega) : \operatorname{Re} \left(\frac{\lambda(z - \omega)^2 f''(z) + (z - \omega) f'(z)}{(1 - \lambda) f(z) + \lambda(z - \omega) f'(z)} \right) > \alpha, 0 \leq \lambda \leq 1, 0 \leq \alpha < 1, \right. \\ \left. z \in E \right\}$$

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$$Q(\omega, \alpha, \lambda) = \left\{ f \in S(\omega) : \operatorname{Re} \left(\frac{\lambda(z - \omega)^3 f'''(z) + (1 + 2\lambda)(z - \omega)^2 f''(z) + (z - \omega)f'(z)}{\lambda(z - \omega)^2 f''(z) + (z - \omega)f'(z)} \right) > \alpha, \right. \\ \left. 0 \leq \lambda \leq 1, 0 \leq \alpha < 1, z \in E \right\}$$

Let $T(\omega)$ denote subclass of $S(\omega)$ whose elements can be expressed in the form

$$f(z) = (z - \omega) - \sum_{k=2}^{\infty} a_k (z - \omega)^k. \quad (1.3)$$

Here we denote by $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$ respectively the subfamilies of $B(\omega, \alpha, \lambda)$ and $Q(\omega, \alpha, \lambda)$ obtained by taking intersection of $B(\omega, \alpha, \lambda)$ and $Q(\omega, \alpha, \lambda)$ with $T(\omega)$ that is,

$$H(\omega, \alpha, \lambda) = B(\omega, \alpha, \lambda) \cap T(\omega),$$

and

$$K(\omega, \alpha, \lambda) = Q(\omega, \alpha, \lambda) \cap T(\omega).$$

Also let $P(\omega) \subset P(\text{class of Cartheodory functions})$ denote the class of functions of the form

$$P_\omega(z) = 1 + \sum_{k=1}^{\infty} B_k (z - \omega)^k \quad (1.4)$$

that are regular in E and satisfy $P_\omega(\omega) = 1$, $\operatorname{Re} P_\omega(z) > 0$ for $z \in E$ and ω is a fixed point in E and

$$|B_k| \leq \frac{2}{(1+d)(1-d)^k}, \quad k \geq 1, \quad d = |\omega|.$$

II. CO-EFFICIENT ESTIMATES

For our main results we first derive the following:

Lemma2.1. A function $f(z) \in T(\omega)$ is in the class $H(\omega, \alpha, \lambda)$ if and only if

$$\sum_{k=2}^{\infty} (k - \alpha)(1 - d)^{k-1}(1 - \lambda + \lambda k)a_k \leq 1 - \alpha. \quad (2.1)$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z - \omega| = 1 - d < 1$ then we have

$$\left| \frac{\lambda(z - \omega)^2 f''(z) + (z - \omega)f'(z)}{(1 - \lambda)f(z) + \lambda(z - \omega)f'(z)} - 1 \right| = \left| \frac{-\sum_{k=2}^{\infty} (k - 1)a_k (z - \omega)^{k-1}(1 - \lambda + \lambda k)}{1 - \sum_{k=2}^{\infty} a_k (z - \omega)^{k-1}(1 - \lambda + \lambda k)} \right| \\ \leq \frac{\sum_{k=2}^{\infty} (k - 1)a_k (1 - d)^{k-1}(1 - \lambda + \lambda k)}{1 - \sum_{k=2}^{\infty} a_k (1 - d)^{k-1}(1 - \lambda + \lambda k)} \\ \leq 1 - \alpha.$$

This shows that the values of $\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)}$ lie in the circle centered at $\gamma = 1$ whose radius is $1 - \alpha$. Hence $f(z)$ is in the class $H(\omega, \alpha, \lambda)$. Then

$$\operatorname{Re} \left(\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)} \right) = \operatorname{Re} \left(\frac{1 - \sum_{k=2}^{\infty} k(1-\lambda+\lambda k)a_k(z-\omega)^{k-1}}{1 - \sum_{k=2}^{\infty} (1-\lambda+\lambda k)a_k(z-\omega)^{k-1}} \right) > \alpha. \quad (2.2)$$

for $z \in E$ and ω is a fixed point in E . Choose values of z on the real axis so that $\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)}$ is real. Upon clearing the denominator and letting $z \rightarrow 1^-$ through real values, we have

$$\alpha \left(1 - \sum_{k=2}^{\infty} a_k(1-d)^{k-1}(1-\lambda+\lambda k) \right) \leq 1 - \sum_{k=2}^{\infty} k(1-\lambda+\lambda k)a_k(1-d)^{k-1} \quad (2.3)$$

which is the required result.

Finally, we note that the assertion (2.1) of lemma 2.1 is sharp, with the external function

$$f(z) = (z-\omega) - \frac{1-\alpha}{(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}(z-\omega)^k. \quad (2.4)$$

Corollary 2.2. Let $f(z) \in T(\omega)$ be in the class $H(\omega, \alpha, \lambda)$. Then we have

$$a_k \leq \frac{1-\alpha}{(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}. \quad (2.5)$$

Equality in (2.5) holds true for the function $f(z)$ given by (2.4).

Lemma 2.3. A function $f(z) \in T(\omega)$ is in the class $K(\omega, \alpha, \lambda)$ if and only if

$$\sum_{k=2}^{\infty} k(k-\alpha)(1-d)^{k-1}(1-\lambda+\lambda k)a_k \leq 1-\alpha. \quad (2.6)$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z-\omega| = 1-d < 1$ then we have

$$\left| \frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)} - 1 \right|$$

$$= \left| \frac{-\sum_{k=2}^{\infty} k(k-1)a_k(z-\omega)^{k-1}(1-\lambda+\lambda k)}{1-\sum_{k=2}^{\infty} ka_k(z-\omega)^{k-1}(1-\lambda+\lambda k)} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty} k(k-1)a_k(1-d)^{k-1}(1-\lambda+\lambda k)}{1-\sum_{k=2}^{\infty} ka_k(1-d)^{k-1}(1-\lambda+\lambda k)} \leq 1-\alpha.$$

This shows that the values of $\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}$ lie in the circle centered at $\gamma = 1$ whose radius is $1 - \alpha$. Hence $f(z)$ is in the class $K(\omega, \alpha, \lambda)$. Then

$$Re \left(\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)} \right)$$

$$= Re \left(\frac{1 - \sum_{k=2}^{\infty} k^2(1-\lambda+\lambda k)a_k(z-\omega)^{k-1}}{1 - \sum_{k=2}^{\infty} k(1-\lambda+\lambda k)a_k(z-\omega)^{k-1}} \right) > \alpha \quad (2.7)$$

for $z \in E$ and ω is a fixed point in E . Choose values of z on the real axis so that $\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}$ is real. Upon clearing the denominator and letting $z \rightarrow 1^-$ through real values, we have

$$\alpha \left(1 - \sum_{k=2}^{\infty} ka_k(1-d)^{k-1}(1-\lambda+\lambda k) \right) \leq 1 - \sum_{k=2}^{\infty} k^2(1-\lambda+\lambda k)a_k(1-d)^{k-1} \quad (2.8)$$

which is the required result.

Finally, we note that the assertion (2.6) of Lemma 2.3 is sharp, with the external function

$$f(z) = (z-\omega) - \frac{1-\alpha}{k(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}(z-\omega)^k. \quad (2.9)$$

Corollary 2.4. Let $f(z) \in T(\omega)$ be in the class $K(\omega, \alpha, \lambda)$. Then we have

$$a_k \leq \frac{1-\alpha}{k(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}. \quad (2.10)$$

Equality in (2.10) holds true for the function $f(z)$ given by (2.9).

Theorem 2.5. Let $f(z) \in H(\omega, \alpha, \lambda)$ and $f(z) = (z-\omega) - a_2(z-\omega)^2 - \dots$ for $0 \leq \alpha < 1$, and ω is a fixed point in E . Then

$$|a_2| \leq \frac{-2(1-\alpha)}{(1+\lambda)(1-d^2)}, \quad (2.11)$$

$$|a_3| \leq \frac{-1}{1+2\lambda} \left[\frac{1-\alpha}{(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{(1-d^2)^2} \right],$$

$$|a_4| \leq \frac{-1}{1+3\lambda} \left[\frac{2(1-\alpha)}{3(1+d)(1-d)^3} + \frac{2(1-\alpha)^2}{(1-d)(1-d^2)^2} + \frac{4(1-\alpha)^3}{3(1-d^2)^3} \right].$$

Proof. Let us define

$$\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)} = \alpha + (1-\alpha)P_\omega(z). \quad (2.12)$$

That is,

$$\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z) = [(1-\lambda)f(z) + \lambda(z-\omega)f'(z)][\alpha + (1-\alpha) \sum_{k=1}^{\infty} B_k(z-\omega)^k]. \quad (2.13)$$

On comparing the coefficient in (2.13) the result follows.

We define the δ -neighbourhood of the function $f(z) \in T(\omega)$ by

$$N_\delta = \{g \in T(\omega) : g(z) = (z-\omega) - \sum_{k=2}^{\infty} b_k(z-\omega)^k, \sum_{k=2}^{\infty} k(1-d)^{k-1}|b_k| \leq \delta\}, \quad (2.14)$$

and in particular, for the identity function

$$e(z) = \left(1 - \frac{\omega}{z}\right)z, \quad (2.15)$$

we immediately have

$$N_\delta(e) = \{g \in T(\omega) : g(z) = (z-\omega) - \sum_{k=2}^{\infty} b_k(z-\omega)^k, \sum_{k=2}^{\infty} k(1-d)^{k-1}|b_k| \leq \delta\}. \quad (2.16)$$

Theorem 2.6. Let $f(z) \in K(\omega, \alpha, \lambda)$ and $f(z) = (z-\omega) - a_2(z-\omega)^2 - \dots$ for $0 \leq \alpha < 1$, and ω is a fixed point in E . Then

$$|a_2| \leq \frac{-(1-\alpha)}{(1+\lambda)(1-d^2)}, \quad (2.17)$$

$$|a_3| \leq \frac{-1}{1+2\lambda} \left[\frac{1-\alpha}{3(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{3(1-d^2)^2} \right],$$

$$|a_4| \leq \frac{-1}{1+3\lambda} \left[\frac{1-\alpha}{6(1+d)(1-d)^3} + \frac{(1-\alpha)^2}{2(1-d)(1-d^2)^2} + \frac{(1-\alpha)^3}{3(1-d^2)^3} \right].$$

Proof. Let us define

$$\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)} = \alpha + (1-\alpha)P_\omega(z). \quad (2.18)$$

That is,

$$\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z) = [\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)] \left[\alpha + (1-\alpha) \sum_{k=1}^{\infty} B_k(z-\omega)^k \right] \quad (2.19)$$

comparing the coefficients we get the required result.

From (2.14) and (2.15) we immediately have

$$N_\delta(e) = \{g \in T(\omega) : g(z) = (z-\omega) - \sum_{k=2}^{\infty} b_k(z-\omega)^k, \sum_{k=2}^{\infty} k^2(1-d)^{k-1}|b_k| \leq \delta\}. \quad (2.20)$$

Theorem 2.7. $H(\omega, \alpha, \lambda) \subset N_\delta(e)$ where $\delta = \frac{2(1-\alpha)}{(2-\alpha)(1-d)(1+\lambda)}$.

Proof. Let $f(z) \in H(\omega, \alpha, \lambda)$. Then in view of lemma 2.1, since $(k-\alpha)(1+\lambda)(1-d)^{k-1}$ is an increasing function of k ($k \geq 2$), we have

$$(1-d)(1+\lambda)(2-\alpha) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} (k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1} a_k \leq 1-\alpha, \quad (2.21)$$

which immediately yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{(2-\alpha)(1-d)(1+\lambda)}. \quad (2.22)$$

On the other hand, we also find from (2.3) that

$$\begin{aligned} (1-d) \sum_{k=2}^{\infty} k a_k (1-\lambda+\lambda k) - \alpha(1-d) \sum_{k=2}^{\infty} a_k (1-\lambda+\lambda k) &\leq \sum_{k=2}^{\infty} (k-\alpha) a_k (1-d)^{k-1} (1-\lambda+\lambda k) \\ &\leq 1-\alpha. \end{aligned} \quad (2.23)$$

From (2.22) and (2.23), we have

$$(1-d)(1+\lambda) \sum_{k=2}^{\infty} k a_k \leq (1-\alpha) + \alpha(1-d)(1+\lambda) \sum_{k=2}^{\infty} a_k \quad (2.24)$$

$$\begin{aligned}
&\leq (1-\alpha) + \frac{\alpha(1-\alpha)}{(2-\alpha)} \\
&\leq \frac{2(1-\alpha)}{2-\alpha}. \\
\sum_{k=2}^{\infty} ka_k &\leq \frac{2(1-\alpha)}{(2-\alpha)(1-d)(1+\lambda)}. \tag{2.25}
\end{aligned}$$

which proved the theorem.

Theorem 2.8. $K(\omega, \alpha, \lambda) \subset N_{\delta}(e)$ where $\delta = \frac{1-\alpha}{(2-\alpha)(1-d)(1+\lambda)}$.

Proof. Let $f(z) \in K(\omega, \alpha, \lambda)$. Then in view of lemma 2.3, since

$k(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}$ is an increasing function of $k(k \geq 2)$, we have

$$2(1-d)(1+\lambda)(2-\alpha) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} k(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1} a_k \leq 1-\alpha, \tag{2.26}$$

which immediately yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{2(2-\alpha)(1-d)(1+\lambda)}. \tag{2.27}$$

On the other hand, we also find from (2.8) that

$$\begin{aligned}
2(1-d)(1+\lambda) \sum_{k=2}^{\infty} ka_k - 2\alpha(1-d)(1+\lambda) \sum_{k=2}^{\infty} a_k &\leq \sum_{k=2}^{\infty} k(k-\alpha)(1-d)^{k-1} a_k (1-\lambda+\lambda k) \\
&\leq 1-\alpha. \tag{2.28}
\end{aligned}$$

From (2.27) and (2.28), we have

$$2(1-d)(1+\lambda) \sum_{k=2}^{\infty} ka_k \leq (1-\alpha) + 2\alpha(1-d)(1+\lambda) \sum_{k=2}^{\infty} a_k \tag{2.29}$$

$$\leq (1-\alpha) + \frac{\alpha(1-\alpha)}{(2-\alpha)} = \frac{2(1-\alpha)}{2-\alpha}.$$

$$\sum_{k=2}^{\infty} ka_k \leq \frac{1-\alpha}{(2-\alpha)(1-d)(1+\lambda)} \tag{2.30}$$

which proved the theorem.

III. INTEGRAL MEAN INEQUALITY

Lemma3.1. If f and g are analytic in E with $f < g$ then

$$\int_0^{2\pi} |g(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \quad (3.1)$$

where $\delta > 0, z = re^{i\theta}, \omega = de^{i\theta}$ and $0 < r + d < 1$.

Applying Lemma 3.1 and (1.2) we prove the following.

Theorem3.2. Let $\delta > 0$. If $f(z) \in H(\omega, \alpha, \lambda)$, then $z = re^{i\theta}, \omega = de^{i\theta}$ and $0 \leq d < r < 1$, we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\delta d\theta \quad (3.2)$$

where

$$f_2(z) = (z - \omega) - \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega)^2. \quad (3.3)$$

Proof. Let $f(z)$ defined by (1.3) and $f_2(z)$ be given by (3.3) we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} a_k (z - \omega)^{k-1} \right|^\delta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega) \right|^\delta d\theta. \quad (3.4)$$

By Lemma 3.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k (z - \omega)^{k-1} < 1 - \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega), \quad (3.5)$$

setting

$$1 - \sum_{k=2}^{\infty} a_k (z - \omega)^{k-1} = 1 - \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)} h(z). \quad (3.6)$$

From (3.6) and (2.1) we obtain

$$\begin{aligned} |h(z)| &= \left| \sum_{k=2}^{\infty} \frac{(2 - \alpha)(1 - d)(1 + \lambda)}{(1 - \alpha)} a_k (z - \omega)^{k-1} \right| \\ &\leq |z - \omega| \sum_{k=2}^{\infty} \frac{(k - \alpha)(1 - d)^{k-1}(1 + \lambda)}{(1 - \alpha)} a_k \leq |z - \omega|. \end{aligned} \quad (3.7)$$

This completes the proof of theorem 3.2.

Theorem 3.3. Let $\delta > 0$. If $f(z) \in K(\omega, \alpha, \lambda)$, then $z = re^{i\theta}$, $\omega = de^{i\theta}$ and $0 \leq d < r < 1$, we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\delta d\theta \quad (3.8)$$

where

$$f_2(z) = (z - \omega) - \frac{1 - \alpha}{2(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega)^2. \quad (3.9)$$

Proof. Let $f(z)$ defined by (1.3) and $f_2(z)$ be given by (3.9) we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} a_k(z - \omega)^{k-1} \right|^\delta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1 - \alpha}{2(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega) \right|^\delta d\theta. \quad (3.10)$$

By lemma 3.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k(z - \omega)^{k-1} < 1 - \frac{1 - \alpha}{2(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega), \quad (3.11)$$

setting

$$1 - \sum_{k=2}^{\infty} a_k(z - \omega)^{k-1} = 1 - \frac{1 - \alpha}{2(2 - \alpha)(1 - d)(1 + \lambda)} h(z). \quad (3.12)$$

From (3.12) and (2.6) we obtain

$$\begin{aligned} |h(z)| &= \left| \sum_{k=2}^{\infty} \frac{2(2 - \alpha)(1 - d)(1 + \lambda)}{(1 - \alpha)} a_k(z - \omega)^{k-1} \right| \\ &\leq |z - \omega| \sum_{k=2}^{\infty} \frac{2(k - \alpha)(1 - d)^{k-1}(1 + \lambda)}{(1 - \alpha)} a_k \\ &\leq |z - \omega|. \end{aligned} \quad (3.13)$$

This completes the proof of theorem 3.3.

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Some Results on a Lorentzian Para Sasakian Manifolds

By Mobin Ahmad, Archana Srivastava & Amit Prakash

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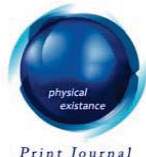
Keywords : *ip-sasakian manifold, generalized pseudo symmetric, generalized pseudo ricci symmetric, generalized ricci recurrent, semi pseudo symmetric.*

GJSFR-F Classification : *MSC 2010: 53C25, 22E43*



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Some Results on a Lorentzian Para Sasakian Manifolds

Mobin Ahmad^α, Archana Srivastava^σ & Amit Prakash^ρ

Abstract - The object of this paper is to study a type of non-flat differentiable manifold called Generalized Pseudo Symmetric, Generalized Pseudo Ricci symmetric, Generalized Ricci Recurrent, Semi pseudo symmetric and Semi pseudo Ricci symmetric manifold in a Lorentzian para-Sasakian manifold.

Keywords and phrases : *ip-sasakian manifold, generalized pseudo symmetric, generalized pseudo ricci symmetric, generalized ricci recurrent, semi pseudo symmetric.*

I. INTRODUCTION

A non -flat differentiable manifold (M^n, g) ($n > 3$), is called generalized pseudo symmetric $G(PS)_n$, if there exists a vector field P and 1- form A, B, C, D on M Such that

$$(D_X S)(Y, Z) = 2A(X)S(Y, Z) + B(R(X, Y), Z) + C(Y)S(X, Z) + D(Z)S(X, Y) + p(R(X, Z)Y). \quad (1.1)$$

In 1993, Chaki and Koley [6] introduced another type of non-flat differentiable manifold (M^n, g) ($n > 2$), satisfies the condition

$$(D_X S)(Y, Z) = 2A(X)S(Y, Z) + B(Y)S(X, Z) + C(Z)S(X, Y), \quad (1.2)$$

where A, B, C are three non-zero 1-form and D denotes the operator of covariant differentiation with respect to g . Such a manifold were called by them a generalized pseudo Ricci-symmetric manifold and an n - dimensional manifold of this kind were denoted by $G(PRS)_n$.

De, Guha and Kamilya [7] introduced and studied a type of differentiable manifold (M^n, g) ($n > 2$), whose Ricci tensors of type $(0, 2)$ satisfies the condition

$$(D_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z), \quad (1.3)$$

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where A and B are two non-zero 1-forms, P, Q are two vector fields such that

$$g(X, P) = A(X),$$

$$g(X, Q) = B(X),$$

such a manifold were called by them a generalized Ricci-recurrent manifold and an n -dimensional manifold of this kind were denoted by $(GR)_n$.

Tarafdar and Jawarneh [4] introduced a type of non-flat differentiable manifold (M^n, g) ($n > 3$), whose curvature tensor R satisfies the condition

$$(D_X R)(Y, Z)W = 2A(X)R(Y, Z)W + A(Y)R(X, Z)W + A(Z)R(Y, X)W \\ + A(W)R(Y, Z)X + A(W)R(Y, Z)X, \quad (1.4)$$

where A is a non-zero 1-form and $g(X, P) = A(X)$,

Such a manifold were called by them a Semi pseudo symmetric and an n -dimensional manifold of this kind is denoted by $(SPS)_n$.

Tarafdar and Jawarneh [5] introduced another type of non-flat Riemannian manifold (M^n, g) ($n > 3$), Whose Ricci-tensor of type $(0, 2)$ satisfies the condition

$$(D_X S)(Y, Z) = A(Y)S(X, Z) + A(Z)S(X, Y), \quad (1.5)$$

where A is a non-zero 1-form, such a manifold were called by them Semi pseudo Ricci-symmetric and an n -dimensional manifold of this kind is denoted by $(SPRS)_n$.

II. PRELIMINARIES

An n -dimensional differentiable manifold M^n is a Lorentzian para-Sasakian (LP -Sasakian) manifold if it admits a $(1, 1)$ -tensor field ϕ , a contravariant vector field ξ , a covariant vector field η and a Lorentzian metric g which satisfy ([2], [3])

$$\phi^2 X = X + \eta(X)\xi, \quad (2.1)$$

$$\eta(\xi) = -1, \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.3)$$

$$g(X, \xi) = \eta(X), \quad (2.4)$$

$$(D_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (2.5)$$

and

$$D_X \xi = \phi X, \quad (2.6)$$

for any vector fields X and Y , where D denotes covariant differentiation with respect to g ([1], [2], [3]).

In an LP -Sasakian manifold M^n with structure (ϕ, ξ, η, g) , it is easily seen that

$$(a) \phi \xi = 0 \quad (b) \eta(\phi X) = 0 \quad (c) \text{rank}(\phi) = n - 1. \quad (2.7)$$

R_{ref.}

1. Mihai, I and Rosca, R.: On Lorentzian P -Sasakian manifolds, Classical Analysis, World scientific Publ. Singapore (1992), 155-169.

Let us put

$$F(X, Y) = g(\phi X, Y). \quad (2.8)$$

Then the tensor field F is symmetric $(0, 2)$ tensor field i.e.,

$$F(X, Y) = F(Y, X), \quad (2.9)$$

and

$$F(X, Y) = (D_X \eta)(Y). \quad (2.10)$$

Also in an LP-Sasakian manifold, the following relation holds:

$$R(X, Y, Z, \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (2.11)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.12)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.13)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \quad (2.14)$$

and

$$S(X, \xi) = (n-1)\eta(X), \quad (2.15)$$

for any vector fields X, Y, Z , Where $R(X, Y)Z$ is the Riemannian curvature tensor, S is the Ricci tensor.

III. GENERALIZED PSEUDO SYMMETRIC LP- SASAKIAN MANIFOLD

Suppose that M^n is a generalized pseudo symmetric LP-sasakian manifold Putting $Z = \xi$ in (1.1), we get

$$\begin{aligned} (D_X S)(Y, \xi) &= 2A(X)S(Y, \xi) + B(R(X, Y)\xi) + C(Y)S(X, \xi) \\ &\quad + D(\xi)S(X, Y) + p(R(X, \xi)Y). \end{aligned} \quad (3.1)$$

Using (2.2), (2.12) and (2.15) in (3.1), we get

$$\begin{aligned} (D_X S)(Y, \xi) &= 2(n-1)A(X)\eta(Y) + \eta(Y)B(X) - \eta(X)B(Y) + (n-1)C(Y)\eta(X) \\ &\quad + D(\xi)S(X, Y) + \eta(Y)p(X) - g(X, Y)p(\xi). \end{aligned} \quad (3.2)$$

We know that

$$(D_X S)(Y, \xi) = (n-1)g(\phi X, Y) - S(Y, \phi X), \quad (3.3)$$

Therefore, from (3.2) and (3.3), we get

$$2(n-1)A(X)\eta(Y) + \eta(Y)B(X) - \eta(X)B(Y) + (n-1)C(Y)\eta(X) + D(\xi)S(X, Y)$$

$$+\eta(Y)p(X) - g(X,Y)p(\xi)=(n-1)g(\phi X, Y)-S(Y, \Phi X). \quad (3.4)$$

Therefore, putting $X=Y=\xi$ in (3.4) and using (2.1) and (2.15), we get

$$(n-1)[2A(\xi)+C(\xi)+D(\xi)]=0.$$

$$\text{Since } n \geq 3, \text{ we obtain} \quad 2A(\xi)+C(\xi)+D(\xi)=0. \quad (3.5)$$

So, the vanishing of the 1-form $2A+C+D$ over the vector field ξ is necessary in order that M be an LP-Sasakian manifold.

Now we will show that $2A+C+D=0$ holds for all vector fields on M .

Putting $Y=\xi$ in (1.1), we get

$$\begin{aligned} (D_X S)(\xi, Z) &= 2(n-1)A(X)\eta(Z)+\eta(Z)B(X)-g(X,Z)B(\xi)+C(\xi)S(X,Z) \\ &+ (n-1)D(Z)\eta(X)+\eta(Z)p(X)-\eta(X)p(Z). \end{aligned} \quad (3.6)$$

Also, We know that

$$(D_X S)(\xi, Z)=(n-1)g(\phi X, Z)-S(\phi X, Z). \quad (3.7)$$

From (3.6) and (3.7), We get

$$\begin{aligned} 2(n-1)A(X)\eta(Z)+\eta(Z)B(X)-g(X,Z)B(\xi)+C(\xi)S(X,Z)+(n-1)D(Z)\eta(X) \\ + \eta(Z)p(X)-\eta(X)p(Z) = (n-1)g(\phi X, Z)-S(\phi X, Z). \end{aligned} \quad (3.8)$$

Putting $Z=\xi$, in (3.8), we get

$$\begin{aligned} -2(n-1)A(X)-B(X)-\eta(X)B(\xi)+(n-1)C(\xi)\eta(X)+(n-1)D(\xi)\eta(X) \\ -p(X)-\eta(X)p(\xi)=0. \end{aligned} \quad (3.9)$$

Putting $X=\xi$ in (3.8), we get

$$2(n-1)A(\xi)\eta(X)+(n-1)C(\xi)\eta(X)-(n-1)D(X)+p(X)+\eta(X)p(\xi)=0. \quad (3.10)$$

Adding (3.9) and (3.10) and using (3.5), we get

$$-2(n-1)A(X)-\eta(X)B(\xi)-B(X)+(n-1)C(\xi)\eta(X)-(n-1)D(X)=0. \quad (3.11)$$

Now, Putting $X=\xi$ in (3.4), and replacing Z with X , we get

$$2(n-1)A(\xi)\eta(X)+B(\xi)\eta(X)+B(X)-(n-1)C(X)+(n-1)D(\xi)\eta(X)=0. \quad (3.12)$$

Adding (3.11) and (3.12) and using (3.5), we get

$$(n-1)[2A(X)+C(X)+D(X)]=0$$

Since $n \geq 3$, Hence we get

$$2A(X)+C(X)+D(X)=0, \text{ for all } X$$

This implies $2A+C+D=0$.

Hence, we can state the following theorem:

Theorem 3.1 .There exists no generalized pseudo symmetric LP - Sasakian manifold (M^n, g) , $n \geq 3$, if $2A+C+D$ is not every where zero.

IV. GENERALIZED PSEUDO RICCI SYMMETRIC LP - SASAKIAN MANIFOLD

Suppose that M^n is a generalized pseudo Ricci symmetric LP - Sasakian manifold then

$$(D_X S)(Y, Z) = 2A(X)S(Y, Z) + B(Y)S(X, Z) + C(Z)S(X, Y). \quad (4.1)$$

Putting $Z = \xi$, in (4.1) and using (2.15), we get

$$(D_X S)(Y, \xi) = 2(n-1)A(X)\eta(Y) + (n-1)B(Y)\eta(X) + C(\xi)S(X, Y). \quad (4.2)$$

Also, we know that

$$(D_X S)(Y, Z) = (n-1)g(\phi X, Y) - S(Y, \phi X). \quad (4.3)$$

From (4.2) and (4.3), we get

$$2(n-1)A(X)\eta(Y) + (n-1)B(Y)\eta(X) + C(\xi)S(X, Y) = (n-1)g(\phi X, Y) - S(Y, \phi X). \quad (4.4)$$

Putting $X = Y = \xi$ in (4.4), we get

$$(n-1)[2A(\xi) + B(\xi) + C(\xi)] = 0.$$

Which gives (since $n \geq 3$),

$$[2A(\xi) + B(\xi) + C(\xi)] = 0. \quad (4.5)$$

Putting $X = \xi$ in (4.4), we have

$$(n-1)\eta(Y)[2A(\xi) + C(\xi)] - (n-1)B(Y) = 0.$$

So, By virtue of (4.5) this yields

$$(n-1)[\eta(Y)B(\xi) + B(Y)] = 0,$$

$$\text{Which gives us (since } n \geq 3 \text{),} \quad B(Y) = -\eta(Y)B(\xi). \quad (4.6)$$

Similarly, Taking $Y = \xi$ in (4.4), we get

$$-2A(X) + \eta(X)[B(\xi) + C(\xi)] = 0.$$

Hence applying (4.5), we get

$$A(X) = -\eta(X)A(\xi). \quad (4.7)$$

Since $(\nabla_{\xi}S)(\xi, X) = 0$, then from (1.2), we obtain

$$(n-1)\eta(X)[2A(\xi) + B(\xi)] - (n-1)C(X) = 0. \quad (4.8)$$

So, by making the use of (4.5), the equation (4.8) reduces to

$$C(X) = -C(\xi)\eta(X) \quad (4.9)$$

Adding equation (4.6), (4.7) and (4.9), we get

$$2A(X) + B(X) + C(X) = -[2A(\xi) + B(\xi) + C(\xi)]\eta(X) \quad (4.10)$$

And then, from (4.5), it follows that

$$2A(X) + B(X) + C(X) = 0, \text{ for all } X,$$

$$\text{Thus } 2A + B + C = 0.$$

Hence, we can state the following theorem:

Theorem 4.1. There exists no generalized Pseudo Ricci-Symmetric LP-Sasakian manifold (M^n, g) , $n \geq 3$ if $2A + B + C$ is not everywhere zero.

V. GENERALIZED RICCI RECURRENT LP- SASAKIAN MANIFOLD ADMITTING CODAZZI TYPE RICCI TENSOR

We know that

$$(D_X S)(Y, Z) = D_X S(Y, Z) - S(D_X Y, Z) - S(Y, D_X Z). \quad (5.1)$$

Then, from (5.1) and (1.3), we get

$$A(X)S(Y, Z) + B(X)g(Y, Z) = D_X S(Y, Z) - S(Y, D_X Z). \quad (5.2)$$

Putting $Z = \xi$ in above relation, we get

$$(n-1)A(X)\eta(Y) + B(X)\eta(Y) = (n-1)g(\emptyset X, Y) - S(Y, \emptyset X). \quad (5.3)$$

Putting $Y = \xi$ in (5.3), we have

$$(n-1)A(X) + B(X) = 0. \quad (5.4)$$

Here we assume that a generalized Ricci recurrent manifold admits Codazzi type Ricci tensor

$$\text{i.e. } (D_X S)(Y, Z) = (D_Y S)(X, Z). \quad (5.5)$$

Then by virtue of (1.3), it follows from (5.5) that

$$A(X)S(Y, Z) + B(X)g(Y, Z) = A(Y)S(X, Z) + B(Y)g(X, Z). \quad (5.6)$$

Putting $X = \xi$ in above, we get

$$A(\xi)S(Y, Z) + B(\xi)g(Y, Z) = [(n-1)A(Y) + B(Y)]\eta(Z). \quad (5.7)$$

In view of (5.4), (5.7) yields

$$S(Y, Z) = \lambda g(Y, Z), \text{ where } \lambda = -\frac{B(\xi)}{A(\xi)}.$$

i.e. M^n is an Einstein manifold.

Hence, we can state the following theorem:

Theorem 5.1. If a generalized Ricci recurrent LP - Sasakian manifold admits a Codazzi type Ricci tensor, then it is an Einstein manifold with constant $\lambda = -\frac{B(\xi)}{A(\xi)}$.

VI. SEMI PSEUDO SYMMETRIC LP - SASAKIAN MANIFOLD (SPS)_N (M^n, g), $n > 3$

Suppose that M^n is a Semi pseudo symmetric LP-Sasakian manifold then from (1.4), we obtain

$$(D_X S)(Y, \xi) = (2n - 1)A(X)\eta(Y) + (n - 2)A(Y)\eta(X) + A(\xi)S(X, Y). \text{ Since } g(X, P) = A(X) \Rightarrow A(\xi) = g(\xi, P) = \eta(P) \quad (6.1)$$

$$\text{This implies } (D_X S)(Y, \xi) = (2n - 1)A(X)\eta(Y) + (n - 2)\eta(X)A(Y) + \eta(P)S(X, Y). \quad (6.2)$$

We also know that

$$(D_X S)(Y, \xi) = (n - 1)g(\phi X, Y) - S(Y, \phi X). \quad (6.3)$$

From equation (6.2) and (6.3), we get

$$(2n - 1)A(X)\eta(Y) + (n - 2)\eta(X)A(Y) + \eta(P)S(X, Y) = (n - 1)g(\phi X, Y) - S(Y, \phi X). \quad (6.4)$$

Putting $X = \xi$ in (6.4), we get

$$(3n - 2)\eta(P)\eta(Y) - (n - 2)A(Y) = 0. \quad (6.5)$$

Putting $Y = \xi$ in (6.5), we get

$$\eta(P) = 0 \quad (6.6)$$

Hence, from (6.5) and (6.6), we get

$$A(Y) = 0.$$

Which is inadmissible by the definition of (SPS)_n.

Hence, we can state the following theorem:

Theorem 6.1. A (SPS)_n ($n > 3$), cannot be an LP sasakian manifold.

VII. SEMI PSEUDO RICCI SYMMETRIC $(SPRS)_n$ $(M^n, g), n > 3$ LP -SASAKIAN MANIFOLD

In this section we assume that a $(SPRS)_n$ is an LP sasakian manifold then from (1.5), we obtain

$$(D_X S)(Y, \xi) = (n-1)A(Y)\eta(X) + A(\xi)S(X, Y). \quad (7.1)$$

Also we know that

$$(D_X S)(Y, \xi) = (n-1)g(\phi X, Y) - S(Y, \phi X). \quad (7.2)$$

From (7.1) and (7.2), we get

$$(n-1)A(Y)\eta(X) + A(\xi)S(X, Y) = (n-1)g(\phi X, Y) - S(\phi X, Y). \quad (7.3)$$

Putting $X = \xi$ in (7.3), we get

$$A(Y) = A(\xi)\eta(Y). \quad (7.4)$$

Putting $Y = \xi$ in (7.4), we get

$$A(\xi) = 0. \quad (7.5)$$

Using (7.5) in (7.4), we get

$$A(Y) = 0.$$

Which is inadmissible by the definition of $(SPRS)_n$.

Hence, we can state the following theorem .:

Theorem 7.1. A $(SPRS)_n$ $(n > 3)$ can not be an LP sasakian manifold.

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Multi-Presheaves and Multi-Sheaves Constructions

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Abstract - In this paper, we modify the classical theory of sheaves, in such a way that it includes applications where more than one restriction map between sections are needed, we also show how the class of multi-presheaves on a topological space may be made into a category, with suitable notion of morphisms based on a generalization of natural transformation. Several examples are given.

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GJSFR-F Classification : 18B30, 54B40, 55N30



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Multi-Presheaves and Multi-Sheaves Constructions

E. M. Solouma ^α & Abdullah Aljouiee ^σ

Abstract - In this paper, we modify the classical theory of sheaves, in such a way that it includes applications where more than one restriction map between sections are needed, we also show how the class of multi-presheaves on a topological space may be made into a category, with suitable notion of morphisms based on a generalization of natural transformation. Several examples are given.

Keywords : multivalued presheaves, stalks, multivalued sheaves, germs.

1. INTRODUCTION

In many occasions, we may be interested in algebraic structures defined over local neighborhoods. For example, a theory of cohomology of a topological space often concerns with sets of maps from a local neighborhood to some abelian groups, which possesses a natural $\mathbb{Z}$ -module structure. Another example is line bundles (either real or complex): since $\mathbb{R}$ or $\mathbb{C}$ are themselves rings, the set of sections over a local neighborhood forms an $\mathbb{R}$ or $\mathbb{C}$ -module. To analyze this local algebraic information, mathematicians came up with the notion of sheaves, which accommodate local and global data in a natural way. However, there are many fashions of introducing sheaves; Tennison [7] and Bredon [1] have done it in two very different styles in their separate books, though both of which bear the name "Sheaf Theory".

In this paper, we introduce some modifications on the set up in [3], putting extra structure on the set of restriction maps to developing a genuine "Multivalued" sheaf theory. One of the main results we will prove is the fact, just as in the classical case, there is a canonical way of associating to any multi-presheaf a "minimal" multi-sheaf. As this construction may be viewed as that of a right adjoint to inclusion, it follows that the category of multi-sheaves is a reflective subcategory of the category of multi-presheaves. This paper is organized as follows: In section 2, we introduce the notion of multi-presheaf over a local monoid defined in [8], and we give some examples. In section 3, we make use of stalks of

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multi-presheaves which are used as a main ingredient of construction and study of multi-sheaves. In successive Section 5, we introduce and study the notion of multi-sheaf, which is just a multi-presheaf satisfying some additional properties.

II. MULTI-PRESHEAVES

Throughout this section, we fix a topological space X with a topology τ and $\mathcal{N}_x$, the family of the open neighborhoods of the point $x \in X$ directed by inclusion, i.e., $U \leq V$ if $U \supseteq V$. In order to introduce the notion of multi-presheaf and multi-sheaf, we will need the notion of *local monoid* defined in [8]. This notion will also be needed later, when we define and study stalks of multi-presheaves and their use in the framework of sheafification.

Let us fix a set I such that, for any $a \leq b$ in a direct index set $\mathcal{A}$, there exists a non-empty set $I_{ab} \subseteq I$. We call $\mathcal{I} = \bigcup \{I_{ab}; I_{ab} \subseteq I\}$ a local monoid (index by $\mathcal{A}$), if the following properties are satisfied:

1. There exists some $e \in I$ such that $e \in \bigcap_{b \geq a} I_{ab}$.
2. It is endowed with some associated composition

$$I_{ab} \times I_{bc} \rightarrow I_{ac} : (i, j) \mapsto ij$$

with the property that for any $a \leq b \leq c$ in $\mathcal{A}$ and any $i \in I_{ab}$ resp. $j \in I_{bc}$, we have $ie = i$ resp. $ej = j$. In particular, this implies $I_{ab} \subseteq I_{ac}$ resp. $I_{bc} \subseteq I_{ac}$ and I_{aa} is a monoid with unit e for every $a \in \mathcal{A}$.

3. Whenever $a \leq b, c$ in $\mathcal{A}$ and $i \in I_{ab}$ resp. $j \in I_{bc}$, then there exists some $d \in \mathcal{A}$ with $d \geq b, c$ and some $k \in I_{bd}$ resp. $\ell \in I_{cd}$ with the property that $ik = j\ell$ in I_{ad} .

Note that once some $d \in \mathcal{A}$ and corresponding elements k and ℓ with the feature required in the last property has been found, This property clearly remains valid for all $d' \geq d$.

Definition 2.1. Consider a topological space X , with topology τ directed by inclusion. Consider a fixed set I , such that for any $U \supseteq V$ in τ , a non-empty set $I_{UV} \subseteq I$. The union $\mathfrak{I} = \bigcup \{I_{UV}; I_{UV} \subseteq I\}$ is called a global monoid if the following properties are satisfied:

G₁: There exists some elements $e \in \bigcap_{U \supseteq V} I_{UV}$.

G₂: It is endowed with some associated composition

$$I_{UV} \times I_{VW} \rightarrow I_{UW} : (i, j) \mapsto ij,$$

R_{ref.}

[8] A. Verschoren and N. A. Souttan, *Inductive limit of multi-spectra*, J. of Comm. in Algebra, **27**, No. 4, (1999), 1895-1902.

for any $U \supseteq V \supseteq W$ open subsets of X , and any $i \in I_{UV}$ then $ie = ei = i$.

G₃: For any $U \supseteq V, W$ in τ and $i \in I_{UV}$ resp. $j \in I_{VW}$, there exists $W' \in \tau$ with $W' \subseteq V, W$ and some $k \in I_{VW'}$ resp. $\ell \in I_{WW'}$ with the property that $ik = j\ell$ in $I_{UW'}$.

For each $x \in X$, the family $\mathfrak{I}_x = \bigcup \{I_{UV}, x \in V \subseteq U\}$ is called local monoid.

Definition 2.2. A ($\mathfrak{I}$ -indexed) multi-presheaf F of sets (abelian groups) on the space X consists the following data:

1. For each open subset U of X , there is given a set (abelian groups) $F(U)$.
2. For each pair of open subsets $U \supseteq V$ of X , there is given a non-empty set $F_{UV} = \{F_{UV}^i, i \in I_{UV}\}$, the so called restriction maps (homeomorphisms).

These data should satisfy the following conditions:

P1. For any open subset U of X , we have $id_{F(U)} = F_{UU}^e$, the identity map on F_{UU} .

P2. For any open subsets $W \subseteq V \subseteq U$ of X , and any F_{UV}^i resp. F_{VW}^j for some $i \in I_{UV}$ resp. $j \in I_{VW}$, the composition $F_{VW}^j F_{UV}^i$ belongs to F_{UW} .

P3. Whenever $x \in V \subseteq U$, any $s \in F(U)$ and any $i, j \in I_{UV}$, there exists $W \in \mathcal{N}_x$ with $W \subseteq V$ and indices $k, \ell \in I_{VW}$ such that $ik = j\ell$ and

$$F_{VW}^k F_{UV}^i(s) = F_{VW}^\ell F_{UV}^j(s).$$

P4. For any open neighborhoods $W \subseteq V \subseteq U$ of $x \in X$ and any $i \in I_{UV}$ resp. $j \in I_{VW}$, there exists some open set $W' \in \mathcal{N}_x$ with $W' \subseteq W$ such that

$$F_{VW'}^j F_{UV}^i = F_{UW'}^{ij}.$$

Note that once an open neighborhood $W' \in \mathcal{N}_x$ as in (**P4**) has been found, the same conclusion remains valid for any open neighborhood $W'' \subseteq W'$ of x . Indeed, it is sufficient to note that $F_{VW'}^j F_{UV}^i = F_{UW'}^{ij}$ which implies that

$$F_{VW''}^j F_{UV}^i = F_{W'W''}^e F_{VW'}^j F_{UV}^i = F_{W'W''}^e F_{UW'}^{ij} = F_{UW''}^{ij}.$$

Let us also point out that, in the terminology of [7], any multi-presheaf may be viewed as a contravariant multifunctor from the category of open subsets of X to the category of sets.

Example 1. Let X be the closed interval $[0, 1] \subseteq R$, endowed with a relative topology τ on X with respect to usual topology on R . Assume that τ is directed

by inclusion, i.e., $U \leq V$ if $U \supseteq V$. We will construct a multi-presheaf F on X , indexed by a constant global monoid $\mathfrak{J} = N$, i.e., $I_{UV} = N$ for all $U \supseteq V$ with the composition

$$N \times N \rightarrow N : (i, j) \mapsto \max(i, j).$$

For any strictly positive integer $m \in N$, let $f_m : X \rightarrow R$ be defined by

$$f_m(x) = \left(1 - \frac{1}{m}\right)x, \forall x \in X.$$

If U is an open subset of X , denote f_m^U the restriction $f_m|_U$ of f_m on U , i.e., $f_m^U : U \rightarrow R : x \mapsto \left(1 - \frac{1}{m}\right)x$. Let $F(U) = \{f_m^U, m \in N\}$ and whenever $V \subseteq U$ open subsets of X , we put $F_{UV} = \{F_{UV}^i, i \in N\}$ where $F_{UV}^i : F(U) \rightarrow F(V)$ is given by

$$F_{UV}^i(f_m^U) = \begin{cases} f_{m-1}^V & \text{if } m \text{ is even and } \frac{m}{2} \leq i, \\ f_m^V & \text{otherwise.} \end{cases}$$

It is clear that F_{UU}^0 is the identity on $F(U)$ for any open subset U of X and straightforward verification shows that for any $W \subseteq V \subseteq U$ open subsets of X , and $i, j \in N$, we have $F_{VW}^j F_{UV}^i = F_{UW}^{\ell^*}$ where $\ell^* = \max(i, j)$. From this one easily deduces that F is a multi-presheaf, Indeed.

Example 2. Let X be the unite circle C^1 endowed with a relative topology τ on X , with respect to usual topology on R^2 . Let τ be directed by inclusion, i.e., $U \leq V$ if $U \supseteq V$. Let H be the helix in R^3 parameterized by

$$R \rightarrow H : t \mapsto (t, \sin(t), \cos(t)),$$

both endowed with their standard topology, let

$$p : H \rightarrow X : (t, \sin(t), \cos(t)) \mapsto (\sin(t), \cos(t)),$$

be the canonical projection. We construct a multi-presheaf F^H of abelian groups on X , indexed by a constant global monoid $\mathfrak{J} = Z$, i.e., $I_{UV} = Z$ for all $U \supseteq V$, with composition

$$Z \times Z \rightarrow Z : (i, j) \mapsto i + j.$$

For any open subset U on X , denote $F^H(U)$ for $\left\{ \text{Continuous functions } s : U \rightarrow H, p \circ s = id_U \right\}$, and whenever $V \subseteq U$, consider $F_{UV} = \{F_{UV}^i, i \in Z\}$ where the

$R_{\text{ef.}}$

[7] B. R. Tennison, *Sheaf Theory*, Cambridge Univ. Press, London, New York, Melbourne, 1975.

Example 2. Let X be the unite circle C^1 endowed with a relative topology τ on X , with respect to usual topology on R^2 . Let τ be directed by inclusion, i.e., $U \leq V$ if $U \supseteq V$. Let H be the helix in R^3 parameterized by

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$$Z \times Z \rightarrow Z : (i, j) \mapsto i + j.$$

For any open subset U on X , denote $F^H(U)$ for $\left\{ \text{Continuous functions } s : U \rightarrow H, p \circ s = id_U \right\}$, and whenever $V \subseteq U$, consider $F_{UV} = \{F_{UV}^i, i \in Z\}$ where the homomorphism $F_{UV}^i : F^H(U) \rightarrow F^H(V)$ is given by $F_{UV}^i(s) = s|_V + (0, 0, 2\pi i)$. It is clear that F_{UU}^0 is identity on $F^H(U)$ for any U on X . Whenever $W \subseteq V \subseteq U$ and any $i, j \in Z$ we have $F_{VW}^j F_{UV}^i = F_{UW}^{i+j}$, it easy to verify that F^H is actually a multi-presheaf, indexed by constant global monoid $\mathfrak{I} = Z$.

III. STALKS OF MULTI-PRESHEAVES

Just as ordinary presheaves give rise to stalks, which are used as a main ingredient of construction and study of sheaves. In this section we show how the multi-presheaves we introduce above allow for the construction of stalks. The actual construction of this stalks is of course the idea of inductive limit of multi-spectra,[8], and its properties and thus depends heavily upon these results.

Construction 1. Let F be a multi-presheaf on X and $x \in X$. Consider the set $F(U)$ of all multi-sections of F on U , where U is running over all open subsets of X such that $U \in \mathcal{N}_x$. Then, as mentioned before, the system $\{F(U), F_{UV}, \mathcal{N}_x\}$ is a multi-spectrum over a local monoid $\mathfrak{I}_x = \bigcup \{I_{UV}, x \in V \subseteq U\}$. Let $P = \coprod_{U \in \mathcal{N}_x} F(U)$ the disjoint union of sets $F(U)$. We define a relation " $\sim$ " on P as follows. Let $s \in F(U)$ and $t \in F(V)$, be two multi-section of F on U, V

respectively, then $s \sim t$ if there exists $W \in \mathcal{N}_x$, $W \subseteq U \cap V$ and $i \in I_{UW} \cap I_{VW}$ such that $F_{UW}^i(s) = F_{VW}^i(t)$. This is actually an equivalence relation.

Definition 3.1. The multi-stalk (abbreviated: M^* -stalk), F_x at $x \in X$ is the quotient of $P = \coprod_{U \in \mathcal{N}_x} F(U)$ under the equivalence relation " $\sim$ ". For any $s \in F(U)$, let s_x denote the equivalence class contain $s \in F(U)$ in F_x , which is called multi-germ (abbreviated: M^* -germ) of multi-section s at x , we thus obtain the map $F_{U,x}^e : F(U) \rightarrow F_x : s \mapsto s_x$. For any $U \in \mathcal{N}_x$, we put $I_U = \bigcup \{I_{UV}, U \leq V, V \in \mathcal{N}_x\}$, then any $i \in I_U$ where $i \in I_{UV}$ yields an obvious map $F_{U,x}^i : F(U) \rightarrow F_x$ defined

$$\begin{array}{ccc} F(U) & \xrightarrow{F_{U,x}^i} & F_x \\ F_{UV}^i \downarrow & \nearrow F_{V,x}^e & \\ F(V) & & \end{array}$$

or equivalently, $F_{U,x}^i(s) = [F_{UV}^i(s)]$ where $U \supseteq V \ni x$ is chosen such that $i \in I_{UV}$.

It easy to say that the M^* -stalk F_x of F at x is none other than the inductive limit of multi-spectrum $\{F(U), F_{UV}, \mathcal{N}_x\}$. We write

$$F_x = \varinjlim_{x \in U} F(U).$$

This comes equipped with maps $F(U) \rightarrow F_x : s \mapsto s_x$, whenever an open $U \ni x$.

IV. MORPHISMS OF MULTI-PRESHEAVES

In this section, we give the definition of morphisms between multi-presheaves and we discuss the composition of such morphisms of multi-presheaves and their stalks.

Definition 4.1. Let F, G be a multi-presheaves of sets (abelian groups) over X . A morphism of multi-presheaves $f : F \rightarrow G$ is the family $\{f(U)\}$ of maps (homomorphisms) $f(U) : F(U) \rightarrow G(U)$, one for each $U \in \tau$ such that whenever $U \supseteq V$ and $i \in I_{UV}$, the following diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{F_{UV}^i} & F(V) \\ f(U) \downarrow & & \downarrow f(V) \\ G(U) & \xrightarrow{G_{UV}^i} & G(V) \end{array}$$

is commutative. The composition of such morphisms of multi-presheaves is defined in the same way: If $f : F \rightarrow G$ and $g : G \rightarrow H$ are two morphisms of multi-presheaves, then the composition $gf : F \rightarrow H$ is defined by $(gf)(U) = g(U) \circ f(U)$, for each open subset U of X . The identity morphism $id_F : F \rightarrow F$ of multi-sheaf is defined by $id_F(U) = id_{F(U)}$ for any open subset U of X .

Example 3. Let $X = N$ endowed with the topology τ_0 , consisting of open sets of the form $U_\infty = \phi$ and $U_n = \{n, n+1, \dots\}$, where $n \in N$. Assume that τ_0 is directed by inclusion, i.e., $U_n \leq U_m$ if $U_n \supseteq U_m$, for any $n \leq m$. Let $\mathfrak{I} = N$ be the constant global monoid, i.e., $I_{U_n U_m} = N$ for all $U_n \supseteq U_m$ with the mapping $N \times N \rightarrow N : (i, j) \mapsto ij$. If U_n is an open subset of X , consider $G(U_\infty) = \{1\}$ and $G(U_n) = \{2^{nz}, z \in Z\}$, whenever $U_n \supseteq U_m$ open subsets of X . Let $G_{nm} = \{G_{nm}^i, i \in N\}$ where the homomorphisms $G_{nm}^i : G(U_n) \rightarrow G(U_m)$ are defined by $G_{nm}^i(2^{nz}) = 2^{(iz)m}$. In case of $m = \infty$, let $G_{nm}^i(2^{nz}) = 1$ (constant homomorphism) for all $i \in N$. It is clear that G_{nn}^1 is the identity on $G(U_n)$ for any open subset U_n of X . Moreover, if $n \leq m \leq p$, then

$$\begin{aligned} G_{mp}^j G_{nm}^i(2^{nz}) &= G_{mp}^j(2^{(iz)m}) \\ &= 2^{(ij)zp} = G_{np}^{ij}(2^{nz}) \end{aligned}$$

i.e., $G_{mp}^j G_{nm}^i = G_{np}^{ij}$. It is easy to see that the conditions **(P3)** and **(P3)** of Definition 2.2 are verified. Thus the system $G = \{G(U_n), G_{nm}, \tau_0\}$ is a multi-presheaf of abelian groups.

Definition 4.2. Let F, G be multi-presheaves of sets (abelian groups) indexed by a local monoid $\mathfrak{I}_x = \bigcup \{I_{UV}, x \in V \subseteq U\}$. A morphism $f : F \rightarrow G$ of multi-presheaves of sets (abelian groups) on a space X , induces for each point $x \in X$, a map (homomorphism) of M^* -stalks $f_x : F_x \rightarrow G_x$ such that for any $s \in F(U)$, $i \in I_{UV}$, we have $f_x(s_x^i) = f(U)(s)_x^i$, where for simplicity s_x^i , $f(U)(s)_x^i$ means $F_{U,x}^i$, $(f(U)(s))_x^i$ respectively.

It is not difficult to show that f well is defined. For, let $s_x^i = t_x^i$, where $s \in F(U)$ and $t \in F(V)$, then there exists $W \in \mathcal{N}_x$, $W \subseteq U \cap V$ and $i \in I_{UW} \cap I_{VW}$

such that $F_{UW}^i(s) = F_{VW}^i(t)$, then $f(W)F_{UW}^i(s) = f(W)F_{VW}^i(t)$. Since f is a morphism of multi-presheaves then the following diagrams

$$\begin{array}{ccc} F(U) & \xrightarrow{F_{UW}^i} & F(W) \\ f(U) \downarrow & & \downarrow f(W) \\ G(U) & \xrightarrow{G_{UW}^i} & G(W) \end{array} \quad \begin{array}{ccc} F(V) & \xrightarrow{F_{VW}^i} & F(W) \\ f(V) \downarrow & & \downarrow f(W) \\ G(V) & \xrightarrow{G_{VW}^i} & G(W) \end{array}$$

commute. Therefore,

$$G_{UW}^i(f(U)(s)) = f(W)F_{UW}^i(s) = f(W)F_{VW}^i(t) = G_{VW}^i(f(V)(t)),$$

then $f(U)(s)_x^i = f(V)(t)_x^i$. Which proves that f_x is well defined.

Theorem 4.1. Let F, G, H be multi-presheaves of sets (abelian groups) on a space X , then for any $x \in X$

1. $f : F \rightarrow G$ and $g : G \rightarrow H$ are morphisms of multi-presheaves, then $(gf)_x = g_x f_x$.
2. $(id_F)_x = id_{F_x}$.

Proof. Consider M^* -germ $(s)_x^i$ of multi-section $s \in F(U)$ for some $U \in \mathcal{N}_x$ and some $i \in I_{UV}$, then

1.
$$\begin{aligned} (g_x f_x)(s)_x^i &= g_x(f_x(s)_x^i) = g_x(f(U)(s)_x^i) \\ &= g(U)f(U)(s)_x^i = (gf)(U)(s)_x^i = (gf)_x(s)_x^i \end{aligned}$$
2.
$$(id_F)_x(s)_x^i = id_{F(U)}(U)(s)_x^i = s_x^i = id_{F_x}(s)_x^i.$$

V. MULTI-SHEAVES

In this section we introduce and study the notion of multi-sheaf, which is just that a multi-presheaf satisfying some additional properties guaranteeing a nice, well-behaving local-global structure. The axioms for multi-sheaves, just as from the ordinary sheaf case, [7].

R_{ef.}

[7] B. R. Tennison, *Sheaf Theory*, Cambridge Univ. Press, London, New York, Melbourne, 1975.

Definition 5.1. Suppose $\mathfrak{I}_x = \bigcup \{I_{UV}, I_{UV} \subseteq I\}$ is a fixed global monoid indexed by the open subsets of the topological space X . An ($\mathfrak{I}$ -indexed) multi-presheaf F of sets on X is called multi-sheaf if it satisfies the following conditions:

- S1.** For any open subset $U \subseteq X$, any $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ open covering of U and any two multi-sections $s, t \in F(U)$. If for any $\alpha \in \mathcal{A}$ there exists some $i \in I_{UU_\alpha}$ such that $F_{UU_\alpha}^i(s) = F_{UU_\alpha}^i(t)$, then $s = t$.
- S2.** For any open subset $U \subseteq X$, any $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ open covering of U and any family $(s_\alpha)_{\alpha \in \mathcal{A}}$ of multi-sections $s_\alpha \in F(U_\alpha)$. If for any $(\alpha, \beta) \in \mathcal{A} \times \mathcal{A}$, there exists some $i \in I_{U_\alpha U_\alpha \cap U_\beta} \cap I_{U_\beta U_\alpha \cap U_\beta}$ such that

$$F_{U_\alpha U_\alpha \cap U_\beta}^i(s_\alpha) = F_{U_\beta U_\alpha \cap U_\beta}^i(s_\beta),$$

then there exists some $s \in F(U)$ and some $j \in I_{UU_\alpha}$ such that $F_{UU_\alpha}^j(s) = s_\alpha$, for all $\alpha \in \mathcal{A}$.

- S3.** For any open subset $U \subseteq X$, any $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ open covering of U and any two multi-sections $s, t \in F(U)$. If for any $\alpha \in \mathcal{A}$ there exists some $i, j \in I_{UU_\alpha}$ with the property that $F_{UU_\alpha}^i(s) = F_{UU_\alpha}^j(t)$, then there exists $k, \ell \in I_{UU}$ such that $F_{UU}^k(s) = F_{UU}^\ell(t)$, then we call F a strong multi-sheaf.

Note that the condition (**S3**) asserts that the result of the glueing procedure inferred in (**S2**) is essentially unique. Also we can define in the obvious way the notion of abelian multi-sheaf (multi-sheaf of abelian groups). In this case the condition (**S1**) can be simplified by putting $t = 0$.

Definition 5.2. (1) The global monoid $\mathfrak{I} = \bigcup \{I_{UV}, I_{UV} \subseteq I\}$ is called divisible if for any $V \subseteq U$ open subsets of X and any pair of indices $i, j \in I_{UV}$, there exists some $k \in I_{UU}$ with $i = kj$ or $j = ki$.

- (2) The global monoid $\mathfrak{I}$ is called commutative if for any pair of open subsets $V \subseteq U$ and any $i \in I_{UV}$ resp. $k \in I_{VV}$, there exists some $k' \in I_{UU}$ such that $ik = k'i \in I_{UV}$.

- (3) The global monoid $\mathfrak{I}$ is called torsion free if for any pair of open subsets $V \subseteq U$ and any $i \in I_{UV}$ resp. any $k', k'' \in I_{UU}$ such that $k''i = k'i$ then $k' = k''$.

Note that the constant global monoid $\mathfrak{J}$, where $\mathfrak{J}$ is additive monoid N_0 or any commutative group, then $\mathfrak{J}$ satisfies the pervious properties.

Theorem 5.1. Assume that the global monoid $\mathfrak{J}$ satisfies the following properties:

1. $\mathfrak{J}$ is divisible
2. $\mathfrak{J}$ is commutative and torsion free.

Then any ($\mathfrak{J}$ -indexed) multi-sheaf F on X is a strong multi-sheaf.

Proof. Assume that $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is open covering of $U \subseteq X$. Choose $x \in U$ and some $\alpha \in \mathcal{A}$ with $x \in U_\alpha$, and suppose that for some $i, j \in I_{UU_\alpha}$, and some $s, t \in F(U)$, we have $F_{UU_\alpha}^i(s) = F_{UU_\alpha}^j(t)$ for all $\alpha \in \mathcal{A}$. Let us first suppose that $\mathfrak{J}$ is divisible. Pick $k \in I_{UU}$ such that, for example, $i = kj$. Up to shrinking the open neighborhood U_α of x , we may assume that

$$F_{UU_\alpha}^j(t) = F_{UU_\alpha}^i(s) = F_{UU_\alpha}^j F_{UU}^k(s).$$

Since U_α covers U , it follows from (S1) that $t \in F_{UU}^k(s)$, which proves our claim. Next, let us assume that condition (2) holds. Choose some open neighborhood $U'_\alpha \subseteq U_\alpha$ of x with the property that there exists some $k, \ell \in I_{U_\alpha U'_\alpha}$ with $ik = j\ell \in I_{UU'_\alpha}$. shrink U'_α such that $F_{U_\alpha U'_\alpha}^k F_{UU_\alpha}^i = F_{UU'_\alpha}^{ik}$ resp. $F_{U_\alpha U'_\alpha}^\ell F_{UU_\alpha}^j = F_{UU'_\alpha}^{j\ell}$. Since $\mathfrak{J}$ is commutative, there exist $k', \ell' \in I_{UU}$ such that $ik = k'i$ resp. $j\ell = \ell'j$, within $I_{U_\alpha U'_\alpha}$, note that $i, j \in I_{U_\alpha U'_\alpha}$. Since $\mathfrak{J}$ is assumed to be torsion free, k' and ℓ' are actually independent of the particular choice of x and U_α . Again shrinking the open neighborhood U'_α of x , we may assume that $F_{UU'_\alpha}^i F_{UU}^{k'} = F_{UU'_\alpha}^{k'i}$ resp. $F_{UU'_\alpha}^j F_{UU}^{\ell'} = F_{UU'_\alpha}^{j\ell'}$. We now obtain:

$$\begin{aligned} F_{UU'_\alpha}^i F_{UU}^{k'}(t) &= F_{UU'_\alpha}^{k'i}(t) = F_{UU'_\alpha}^{ik}(t) = F_{UU'_\alpha}^{j\ell'}(t) \\ &= F_{U_\alpha U'_\alpha}^\ell F_{UU_\alpha}^j(t) = F_{U_\alpha U'_\alpha}^\ell F_{UU_\alpha}^i(s) \\ &= F_{UU'_\alpha}^{i\ell}(s) = F_{UU'_\alpha}^{\ell'i}(s) = F_{UU'_\alpha}^i F_{UU}^{\ell'}(s). \end{aligned}$$

Since U'_α is a covering of U , applying (S1) now clearly yields that $F_{UU}^{k'}(t) = F_{UU}^{\ell'}(s)$, which proves our claim.

Example 4. Consider a multi-presheaf F^H of abelian groups on the space C^1 , given in Example 2. Assume that $(U_\alpha)_{\alpha \in \mathcal{A}}$ is an open covering of an open set $U \in C^1$, let for all $\alpha \in \mathcal{A}$, $s, t \in F^H(U)$, we have $F_{UU_\alpha}^i(s) = F_{UU_\alpha}^i(t)$ for some $i \in I_{UU_\alpha}$, then $s|_{U_\alpha} + (0, 0, 2\pi i) = t|_{U_\alpha} + (0, 0, 2\pi i)$ thus $s|_{U_\alpha} = t|_{U_\alpha}$ which means that $s(x) = t(x)$ for all $x \in U_\alpha$. Since $(U_\alpha)_{\alpha \in \mathcal{A}}$ is an open covering of U , then $s(x) = t(x)$ for all $x \in U$, this implies that $s = t$. In order to verify the condition (S2), for any family of multi-sections $(s_\alpha)_{\alpha \in \mathcal{A}}$ with $s_\alpha \in F(U_\alpha)$ and some $i \in I_{U_\alpha U_\alpha \cap U_\beta} \cap I_{U_\beta U_\alpha \cap U_\beta}$, we have for any $(\alpha, \beta) \in \mathcal{A} \times \mathcal{A}$

$$F_{U_\alpha U_\alpha \cap U_\beta}^i(s_\alpha) = F_{U_\beta U_\alpha \cap U_\beta}^i(s_\beta),$$

then we have $s_\alpha|_{U_\alpha \cap U_\beta} + (0, 0, 2\pi i) = s_\beta|_{U_\alpha \cap U_\beta} + (0, 0, 2\pi i)$ which yields to $s_\alpha(x) = s_\beta(x)$ for all $x \in U_\alpha \cap U_\beta$, since $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ we define $s \in F(U)$ by $s(x) = s_\alpha(x)$ for all $x \in U$, then we have $s|_{U_\alpha}(x) = s_\alpha(x)$ which yields to $s|_{U_\alpha} + (0, 0, 2\pi i) = s_\alpha$, then $F_{UU_\alpha}^j(s) = s_\alpha$, for some $j \in I_{UU_\alpha}$.

Theorem 5.2. If X is a topological space and F is a multi-sheaf on X . Then for any pair of multi-section $s, t \in F(U)$, we have $s = t$ if and only if for all $x \in U$, there is $U_x \in \mathcal{N}_x$ and $i \in I_{UU_x}$ such that $s_x^i = t_x^i$.

Proof. Let $s = t$ this implies that for any $W_x \in \mathcal{N}_x$ with $W_x \subseteq U$ and for some $i \in I_{UW_x}$, we have $F_{UW_x}^i(s) = F_{UW_x}^i(t)$, which means that $s_x^i = t_x^i$ for all $x \in U$. Conversely, if $s, t \in F(U)$ such that for all $x \in U$, there is $U_x \in \mathcal{N}_x$ and $i \in I_{UU_x}$ such that $s_x^i = t_x^i$. It follows that we may find some open neighborhood $V_x \subseteq U_x$ of x such that $i \in I_{UV_x}$ with $F_{UV_x}^i(s) \sim F_{UV_x}^i(t)$, and up to making V_x even smaller and changing i if necessary, we may assume that we actually have $F_{UV_x}^i(s) = F_{UV_x}^i(t)$. Since $(V_x)_{x \in U}$ is obviously an open cover for U , if we make x vary through U , it follows from the multi-sheaf axioms that $s = t$.

Theorem 5.3. A morphism $f : F \rightarrow G$ of multi-sheaves on a topological space X satisfies the condition $f_x : F_x \rightarrow G_x$ is injective for all $x \in X$ if and only if the map $f(U) : F(U) \rightarrow G(U)$ is injective for all open subset U of X .

Proof. Assume that $f(U) : F(U) \rightarrow G(U)$ is injective, and suppose that $t \in F_x$ is such that $f_x(t) = 0$, then there exists an open set U , and $s \in F(U)$ such that s

has germ t at x , for some $i \in I_{UV}$, so that $f(U)(s)$ has germ 0 at x . Then there exists $V \subseteq U$, $V \in \mathcal{N}_x$ and some $i \in I_{UV}$ such that $G_{UV}^i(f(U)(s)) = 0$, but f is a morphism of multi-sheaf, therefore

$$G_{UV}^i(f(U)(s)) = 0 = f(V)(F_{UV}^i(s)).$$

But $f(V)$ is injective by hypothesis, so $F_{UV}^i(s) = 0$ and thus $t = 0$.

Conversely, assume that $f_x : F_x \rightarrow G_x$ is injective and suppose $s \in F(U)$ is such that $f(U)(s) = 0 \in G(U)$, therefore $G_{U,x}^i(f(U)(s)) = 0$, for some $i \in I_{UV}$, then $f_x(F_{U,x}^i(s)) = 0$, since f_x is injective by hypothesis. Then there exists $U_x \in \mathcal{N}_x$ with $U_x \subseteq U$ and some $i \in I_{UU_x}$ such that $F_{UU_x}^i(s) = 0$. Since $(U_x)_{x \in U}$ is an open covering for U , by a multi-sheaf condition (S2), $s = 0$ on U . Thus $f(U)$ is injective.

Definition 5.3. Let X be a topological space. A sheaf space over X is a pair $\mathbf{E} = (E, p)$, where E is a topological space and the projection p is a continuous onto map $E \rightarrow X$ which has an additional structure, that is, p is a local homeomorphism, i.e., any $e \in E$ has a neighborhood $\mathcal{N}$ such that $p|_{\mathcal{N}}$ maps $\mathcal{N}$ homeomorphically onto a neighborhood of $p(e)$.

Definition 5.4. Let $\mathbf{E} = (E, p)$ and $\mathbf{E}^1 = (E^1, p^1)$ are two sheaf spaces, then a morphism $\mathbf{f} : \mathbf{E} \rightarrow \mathbf{E}^1$ is by definition a map $f : E \rightarrow E^1$, which is continuous and makes the following diagram commutative:

$$\begin{array}{ccc} E & \xrightarrow{p} & X \\ f \downarrow & \nearrow p^1 & \\ E^1 & & \end{array}$$

Construction 2. Consider a sheaf space $\mathbf{E} = (E, p)$. We can construct a multi-sheaf of sets $\Gamma E = \tilde{E}$ (the multi-sheaf of sections of E) in such a way that a morphism $\mathbf{f} : \mathbf{E} \rightarrow \mathbf{E}^1$ of sheaf spaces gives rise to a morphism $\tilde{f} : \tilde{E} \rightarrow \tilde{E}^1$ of multi-sheaves. Now, let $\mathcal{I} = \bigcup \{I_{UV}, U \supseteq V\}$ be a local monoid indexed by open subsets of X . We say that $\mathcal{I}$ acts on the sheaf space $\mathbf{E}$, if there is given for any pair of open subsets $V \subseteq U$ in X a map

$$E|_U \times I_{UV} \rightarrow E|_V : (m, i) \mapsto m^i,$$

where $E|_U = p^{-1}(U)$, with the property that for any open subsets $W \subseteq V \subseteq U$ of X , the induced diagram

$$\begin{array}{ccc} E|_U \times I_{UV} \times I_{VW} & \longrightarrow & E|_V \times I_{VW} \\ \downarrow & & \downarrow \\ E|_U \times I_{UW} & \longrightarrow & E|_W \end{array}$$

Commutates, where the left vertical map is induced by the composition $I_{UV} \times I_{VW} \rightarrow I_{UW}$. This means that for any $m \in E|_U$ and any $i \in I_{UV}$ resp. $j \in I_{VW}$, we have $m^{ij} = (m^i)^j$. If $\mathcal{I}$ acts on a sheaf space $\mathbf{E}$, then it easy to verify that we may associate to it a multi-sheaf $\tilde{E}$ on X , by letting $\tilde{E}(U) = \Gamma(U, E) = \{\text{continuous maps } s : U \rightarrow E, p \circ s = id_U\}$, for any open subset $U \subseteq X$ and putting $\tilde{E}_{UV} = \{\tilde{E}_{UV}^i, i \in I_{UV}\}$, for any $U \supseteq V$ and $i \in I_{UV}$, where $\tilde{E}_{UV}^i : \tilde{E}(U) \rightarrow \tilde{E}(V)$ is defined by $s \rightarrow i(s|_V)$, where $i(s)(y) = s(y)^i$. One can easily show that $\{\tilde{E}(U), \tilde{E}_{UV}, \tau\}$ is a multi-sheaf of sets.

Construction 3. For each multi-presheaf F of sets on X over a local monoid $\mathcal{I} = \bigcup\{I_{UV}, U \supseteq V\}$, we can construct a sheaf space LF in such a way that a morphism $f : F \rightarrow G$ of multi-presheaves gives rise to a morphism $Lf : LF \rightarrow LG$ of sheaf spaces. Consider $LF = \coprod_{x \in X} F_x$ (the disjoint union of the stalks F_x) with $p : LF \rightarrow X$ the natural projection. We topologize LF as follows: Let U be an open set in X , and $s \in F(U)$. Define a map

$$\tilde{s}^e : U \rightarrow LF : x \mapsto s_x; s_x \in F_{U,x}^e(s).$$

Moreover, for any $i \in I_{UV}$ there is a map

$$\tilde{s}^i : V \rightarrow LF : x \mapsto F_{V,x}^i(s) = F_{V,x}^e(F_{UV}^i(s)).$$

From this, it also follows that we have a commutative diagram

$$\begin{array}{ccc} s \in F(U) & \longrightarrow & \Gamma(U, LF) \ni F_{U,x}^e(s) \\ F_{UV}^i \downarrow & & \downarrow \Gamma(F_{UV}^i) \\ F_{UV}^i(s) \in F(V) & \longrightarrow & \Gamma(V, LF) \ni F_{U,x}^i(s) \end{array}$$

Therefor, we obtain a multi-sheaf ΓLF of sets called the associated multi-sheaf (for simplicity we call $\Gamma LF = \underline{a}F$).

Theorem 5.4. F is a multi-sheaf if and only if $F \rightarrow \underline{a}F$ is an isomorphism of multi-sheaves.

Proof. First, we shall prove that if $F \rightarrow \underline{a}F$ is an isomorphism of multi-presheaves and $\underline{a}F$ is a multi-sheaf, then F is a multi-sheaf. Let U be an open set of X and $\{U_x\}_{x \in X}$ be an open cover of U , i.e., $U = \bigcup_{x \in X} U_x$, let s and t be two sections in $F(U)$ and suppose that $F_{UU_x}^i(s) = F_{UU_x}^i(t)$. Since $F \rightarrow \underline{a}F$ is a morphism of multi-presheaves, then the following diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{n_F(U)} & \underline{a}F(U) \\ F_{UU_x}^i \downarrow & & \downarrow \underline{a}F_{UU_x}^i \\ F(U_x) & \xrightarrow{G_{UV}^i} & \underline{a}F(U_x) \end{array}$$

is commutative. Then $\underline{a}F_{UU_x}^i(n_F(U)(s)) = \underline{a}F_{UU_x}^i(n_F(U)(t))$ for some $i \in I_{UU_x}$, but $\underline{a}F$ is a multi-sheaf, therefore, from the condition (S1) of Definition 5.1, we have

$$n_F(U)(s) = n_F(U)(t),$$

since n_F is an isomorphism, that is $n_F(U)$ is bijective, it follows that $s = t$. For the glueing condition (S2), let U be an open set of X and let $U = \bigcup_{x \in X} U_x$ be an open covering of U . Consider $(s^x)_{x \in X}$ the family of sections of F with $\forall x \in X$, $s^x \in F(U_x)$ such that

$$F_{U_x U_x \cap U_y}^i(s^x) = F_{U_y U_x \cap U_y}^i(s^y)$$

for some $i \in I_{U_x U_x \cap U_y} \cap I_{U_y U_x \cap U_y}$. Therefore

$$n_F(U_x \cap U_y) F_{U_x U_x \cap U_y}^i(s^x) = n_F(U_x \cap U_y) F_{U_y U_x \cap U_y}^i(s^y),$$

since n_F is morphism of multi-presheaves, therefore

$$\underline{a}F_{UU_x}^i(n_F(U)(s^x)) = \underline{a}F_{UU_x}^i(n_F(U)(s^y))$$

but $\underline{a}F$ is a multi-sheaf, thus there exists $\tilde{s} \in \underline{a}F(U)$ such that $\forall x \in U$, and for some $i \in I_{UU_x}$

$$\underline{a}F_{UU_x}^i(\tilde{s}) = n_F(U_x)(s^x)$$

$$F_{U,x}^i(s) = F_{U,x}^e(s^x)$$

$$F_{U_x, x}^e(F_{UU_x}^i(s)) = F_{U_x, x}^e(s^x)$$

In other words, for all $x \in U$ there exists $s \in F(U)$ such that $F_{U_x, x}^i(s) = s^x$ which prove that F satisfies the glueing condition (S2) of Definition 5.1. Now we want to prove that if F a multi-sheaf then $F(U) \rightarrow \underline{a}F(U)$ is bijective.

(i) the map is injective for

$$\tilde{s} = \tilde{t} \Leftrightarrow \forall x \in U \ F_{U_x, x}^i(s) = F_{U_x, x}^i(t) \text{ for some } i \in I_{UU_x}$$

$$\Leftrightarrow \forall x \in U \ F_{U_x, x}^e(F_{UU_x}^i(s)) = F_{U_x, x}^e(F_{UU_x}^i(t))$$

$$\Leftrightarrow U \text{ has open cover } U_x \text{ such that}$$

$$F_{UU_x}^i(s) = F_{UU_x}^i(t), \text{ for some restriction maps } F_{UU_x}^i$$

$$\Leftrightarrow s = t \text{ (since } F \text{ is multi-sheaf).}$$

(ii) Let $t \in \underline{a}F(U)$ and $x \in U_x \subseteq U$, if $\tilde{s}^x \in F(U_x)$ neighborhood $\tilde{s}^x(U_x) \subseteq t(U)$ for some open $U_x \subseteq U$ this mean that $s^x \in F(U_x)$ satisfies the glueing condition of Definition 5.1, for, let $s^y \in F(U_y)$, take $V = U_x \cap U_y$. Let $i \in I_{U_x V} \cap I_{U_y V}$ and $\tilde{s}^x : V \rightarrow LF$. Therefore

$$F_{V, z}^i(s^z) = F_{U_x, x}^i(s^x)F_{U_y, y}^i(s^y).$$

Thus $F_{U_x, x}^i(s^x)F_{U_y, y}^i(s^y)$, then $F_{U_x, V}^i(s^x)F_{U_y, V}^i(s^y)$ for some $i \in I_{U_x V} \cap I_{U_y V}$, but F is multi-sheaf then there exists $s \in F(U)$ such that for all $x \in U$, $F_{UU_x}^i(s) = s^x$. Therefore, there is $s \in F(U)$ such that $t \in F_{U_x, x}^e(s^x)$ which proves that the map is injective.

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Semirelib Cutvertex Graph

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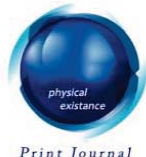
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GJSFR-F Classification : 10C 75, 10C 10



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Semirelib Cutvertex Graph

Venkanagouda M Goudar ^α & K B Manjunatha prasad ^σ

Abstract - In this communications, the concept of semirelib cutvertex graph of a planar graph is introduced. We present a characterization of those graphs whose semirelib cutvertex graphs are planar, outer planar, eulerian, hamiltonian with crossing number one.

Keywords : blocks, edge degree, inner vertex number, line graph, regions.

1. INTRODUCTION

The concept of block edge cut vertex graph was introduced by Venkanagouda M Goudar [6]. For the graph $G(p,q)$, if $B = u_1, u_2, \dots, u_r : r \geq 2$ is a block of G , then we say that the vertex u_i and the block B are incident with each other. If two blocks B_1 and B_2 are incident with a common cutvertex, then they are adjacent blocks. The line graph $n(G)$ was defined in [5]. For the graph G the line graph $n(G)$ is the graph whose vertex set is the union of the set of edges and the set of cutvertices of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent or the corresponding members of G are incident.

All undefined terminology will conform with that in Harary[2]. All graphs considered here are finite, undirected, planar and without loops or multiple edges.

The semirelib cutvertex graph of a planar graph G denoted by $R_c(G)$ is the graph whose vertex set is the union of set of edges, set of blocks, set of cutvertices and set of regions of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent, the edges lie on the blocks, the edges lie on the region and edges are incident to the cutvertex.

The edge degree of an edge uv is the sum of the degree of the vertices of u and v . For the planar graph G , the inner vertex number $i(G)$ of a graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be minimally nonouterplanar if $i(G)=1$ as was given by Kulli [4].

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II. PRELIMINARY NOTES

We need the following results to prove further results.

Theorem 2.1 (2) . If G is a (p, q) graph whose vertices have degree d_i then the line graph $L(G)$ has q vertices and q_L edges, where $q_L = -q + \frac{1}{2} \sum d_i^2$ edges.

Theorem 2.2 (2) . The line graph $L(G)$ of a graph is planar if and only if G is planar, $\Delta(G) \leq 4$ and if $\deg v = 4$, for a vertex v of G , then v is a cutvertex.

Theorem 2.3 (4) . A graph is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$.

III. MAIN RESULTS

We start with few preliminary results.

Lemma 3.1 For any planar graph G , $L(G) \subseteq R_c(G)$.

Lemma 3.2 If a graph G contains a block as K_2 , then it becomes a pendent vertex in $R_c(G)$.

In the following theorem we obtain the number of vertices and edges of a semirelib cutvertex graph of a plane graph.

Theorem 3.3 For any planar graph G with r regions and b blocks, the semirelib cutvertex graph $R_c(G)$ whose vertices have degree d_i , has $(q + r + b + c)$ vertices and $\frac{1}{2} \sum d_i^2 + \sum q_j + \sum c_k$ edges where c_k be the number of edges incident to the cutvertex c_k .

Proof. By the definition of $R_c(G)$, the number of vertices is the union of edges, regions, blocks and the number of cutvertices of G . Hence $R_c(G)$ has $(q + r + b + c)$ vertices. Further by the Theorem 2.1, number of edges in $L(G)$ is $q_L = -q + \frac{1}{2} \sum d_i^2$. Thus the number of edges in $R_c(G)$ is the sum of the number of edges in $L(G)$, the number of edges bounded by the regions which is q , the number of edges lies on the blocks is $\sum q_j$ and the number of edges incident to the cut vertices which is $\sum c_k$. Hence $E[R_c(G)] = -q + \frac{1}{2} \sum d_i^2 + q + \sum q_j + \sum c_k = \frac{1}{2} \sum d_i^2 + \sum q_j + \sum c_k$.

Theorem 3.4 For any edge $e_i \in G$ which is either an internal edge not incident to a cutvertex or a pendent edge with edge degree n , then degree of the corresponding vertex in $R_c(G)$ is $n+1$.

Proof. Suppose a pendent edge $e_i \in E(G)$ have edge degree n . Let v'_i be the corresponding vertex in $R_c(G)$. Clearly there are $n - 1$ edges adjacent to e_i . Also, by the definition of semirelib cutvertex graph, e_i lies on only one region and also on one block. Hence in $R_c(G)$, $\deg(v'_i) = n - 1 + 1 + 1 = n + 1$. Further, if e_i is an internal edge which is not incident to a cutvertex and v'_i be the corresponding vertex in $R_c(G)$ then there are $n - 2$ edges adjacent to v'_i . Also e_i lies on only one block and lies in two regions. Hence $\deg v'_i = n - 2 + 1 + 2 = n + 1$.

Theorem 3.5 *For any edge $e_i \in G$ which is an internal edge incident to a cutvertex with edge degree n , then degree of the corresponding vertex in $R_c(G)$ is $n + 2$.*

Proof. Suppose a pendent edge $e_i \in E(G)$ have edge degree n . Let v'_i be the corresponding vertex in $R_c(G)$. Clearly there are $n - 1$ edges adjacent to e_i . Since e_i is an internal edge which is incident to a cutvertex then there are $n - 2$ edges adjacent to v'_i . Also e_i lies on only one block, lies in two regions and incident to a cutvertex. Hence $\deg v'_i = n - 2 + 1 + 2 + 1 = n + 2$.

Theorem 3.6 *For any edge $e_i \in G$ which is an internal edge incident to two cutvertices with edge degree n , then degree of the corresponding vertex in $R_c(G)$ is $n + 3$.*

Proof. Suppose a pendent edge $e_i \in E(G)$ have edge degree n . Let v'_i be the corresponding vertex in $R_c(G)$. Clearly there are $n - 1$ edges adjacent to e_i . Since e_i is an internal edge which is incident to a cutvertex then there are $n - 2$ edges adjacent to v_i . Also e_i lies on only one block, lies in two regions and incident to two cutvertices. Hence $\deg v'_i = n - 2 + 1 + 2 + 2 = n + 3$.

Theorem 3.7 *For any graph G with each block contains at least three edges, then $R_c(G)$ is nonseparable.*

Proof. Let $e_1, e_2, \dots, e_n \in E(G)$, $r_1, r_2, \dots, r_t$ be the regions, $b_1, b_2, \dots, b_m$ be the blocks which are not K_2 and $c_1, c_2, \dots, c_k$. By the definition of line graph, the corresponding vertices $v'_1, v'_2, \dots, v'_n \in R_c(G)$ form a subgraph without isolated vertex. Since each block contains at least three edges and each region of G has at least three edges. By the definition of $R_c(G)$, each vertex is of degree at least three. Also the cutvertices adjacent to $v'_1, v'_2, \dots, v'_n$. Hence $R_c(G)$ is nonseparable.

In the following theorem we obtain the condition for the planarity on semirelib cutvertex graph of a graph.

Theorem 3.8 *For any planar graph G , the $R_c(G)$ is planar if and only if G is a tree such that $\Delta(G) \leq 3$.*

Proof. Suppose $R_c(G)$ is planar. Assume that $\exists v_i \in G$ such that $\deg v_i \geq 4$. Suppose $\deg v_i = 4$ and e_1, e_2, e_3, e_4 are the edges incident to v_i . Let v'_1, v'_2, v'_3, v'_4 be the vertices of $R_c(G)$ corresponding to e_1, e_2, e_3, e_4 . By the definition of line graph, v'_1, v'_2, v'_3, v'_4 form K_4 as an induced subgraph. In $R_c(G)$, the region vertex r_i is adjacent with all vertices of $L(G)$ to form K_5 as an induced subgraph. Further the corresponding block vertices $b_1, b_2, b_3, \dots, b_{n-1}$ of blocks $B_1, B_2, B_3, \dots, B_n$ in G are adjacent to vertices of K_5 , which forms graph homeomorphic to K_5 . By the Theorem 2.3, it is non planar, a contradiction.

Conversely, Suppose $\deg v \leq 3$ and let e_1, e_2, e_3 be the edges of G incident to v . Let v'_1, v'_2, v'_3 be the vertices of $R_c(G)$ corresponding to e_1, e_2, e_3 . By the definition of line graph v'_1, v'_2, v'_3 form K_3 as a subgraph. By the definition of $R_c(G)$, the region vertex r_i is adjacent to v'_1, v'_2, v'_3 to form K_4 as a subgraph. Further, by the lemma 2.2, the blocks vertices b_1, b_2, b_3 of T forms a pendent vertices of $R_c(G)$. The cutvertex c_i is adjacent to v'_1, v'_2, v'_3 to form a graph which is homeomorphic to K_4 . Hence $R_c(G)$ is planar.

In the following theorem we obtain the condition for the outer planarity on semirelib cutvertex graph of a graph.

Theorem 3.9 *For any planar graph G , $R_c(G)$ is outer planar if and only if G is a path P_n .*

Proof. Suppose $R_c(G)$ is outer planar. Assume that G is a tree with at least one vertex v such that $\deg v = 3$. Let e_1, e_2, e_3 be the edges of G incident to v . Let v'_1, v'_2, v'_3 be the vertices of $R_c(G)$ corresponding to e_1, e_2, e_3 . By the definition of line graph v'_1, v'_2, v'_3 form K_3 as a subgraph. In $R_c(G)$, the region vertex r_i is adjacent to v'_1, v'_2, v'_3 to form K_4 as induced subgraph. Further by the lemma 3.2, $b_1 = e_1, b_2 = e_2, \dots, b_{n-1} = e_{n-1}$ becomes $n-1$ pendant vertices in $R_c(G)$. Also the cutvertex c_i is adjacent to v'_1, v'_2, v'_3 to form a graph homeomorphic to K_4 . Clearly $i[R_c(G)] \geq 1$, which is non-outer planar, a contradiction. Hence G must be a path.

Conversely, Suppose G is a path. Let $e_1, e_2, e_3, \dots, e_{n-1} \in E(G)$. Let $v'_1, v'_2, \dots, v'_{n-1}$ be the vertices of $R_c(G)$ corresponding to $e_1, e_2, \dots, e_{n-2}$. By the definition of line graph, $L[P_n] = P_{n-1}$. Further by the lemma 3.2, $b_1 = e_1, b_2 = e_2, \dots, b_{n-1} = e_{n-1}$ becomes $n-1$ pendant vertices and it becomes a caterpillar. Further the region vertex r_i is adjacent to all the vertices of $L[P_n]$. Also the cutvertices $c_1, c_2, e_3, \dots, c_{n-2}$ are adjacent to $v'_1, v'_2, \dots, v'_{n-1}$ to form the graph such that each block is K_3 . Clearly $R_c(G)$ is outer planar.

In the following theorem we obtain the condition for the minimally non outer planar on semirelib cutvertex graph of a graph.

Theorem 3.10 *For any planar graph G , $R_c(G)$ is not minimally nonouter planar.*

Proof. Follows from the above theorem.

In the following theorem we obtain the condition for the non eulerian on semirelib cutvertex graph of a graph.

Theorem 3.11 *For any planar graph G , $R_c(G)$ is always non Eulerian.*

Proof. We consider the following cases.

Case1. Assume that G is a tree. In a tree each edge is a block and hence $b_1 = e_1, b_2 = e_2, \dots, b_{n-1} = e_{n-1} \forall e_{n-1} \in E(G)$ and $\forall b_{n-1} \in V[R_c(G)]$. In $R_c(G)$, the block vertex b_i is always a pendent vertex, which is non Eulerian.

Case2. Assume that G is a graph which contains at least one pendent vertex. Suppose $e_i \in G$ is a pendent edge. In $e_i = b_i \in V[R_c(G)]$ becomes a pendent vertex, which is non Eulerian.

Case3. Assume that G is K_2 -free graph. We have the following sub cases of case3.

Sub case1. Suppose G itself is a block with even number of edges. Clearly each edge of G is of even degree. By the definition of $R_c(G)$, both the region vertices and blocks have even degree. By the Theorem 2.3 $e_i = b_i \in V[R_c(G)]$ is of odd degree, which is non Eulerian. Further if G is a block with odd number of edges, then by the Theorem 3.3, each $e_i = b_i \in V[R_c(G)]$ is of even degree. Also the block vertex b_i and region vertex r_i are adjacent to these vertices. Clearly degree of b_i and r_i is odd, which is non Eulerian.

Sub case2. Suppose G is a graph such that it contains at least one cutvertex. If each edge is even degree then by the sub case 1, it is non Eulerian. Assume that G contains at least one edge with odd edge degree. Clearly for any $e_j \in E(G)$, degree of $e_j \in V[R_c(G)]$ is odd, which is non Eulerian. Hence for any graph G $R_c(G)$ is always non Eulerian.

In the following theorem we obtain the condition for the hamiltonian on semirelib cutvertex graph of a graph.

Theorem 3.12 *For any graph G , $R_c(G)$ is hamiltonian if and only if G is a graph such that each block contains at least three edges.*

Proof. Suppose $R_c(G)$ is hamiltonian. Assume that G has at least one block which is K_2 . Let $e_i \in E(G)$ such that $e_i = b_i \in V[R_c(G)]$. By the Lemma 3.2, b_i becomes a pendent vertex. Clearly $R_c(G)$ is non hamiltonian, a contradiction.

Conversely, suppose G is K_2 - free graph. Let $e_1, e_2, \dots, e_{n-1} \in E(G)$ and $v'_1, v'_2, \dots, v'_{n-1}$ be the corresponding vertices of $R_c(G)$. Let $b_1, b_2, \dots, b_i$ be the blocks, $r_1, r_2, \dots, r_k$ be the regions and $c_1, c_2, \dots, c_k$ be the cutvertices of G such that $e_1, e_2, \dots, e_l \in V(b_1)$, $e_{l+1}, e_{l+2}, \dots, e_m \in V(b_2)$, $\dots$ $e_{m+1}, e_{m+2}, \dots, e_{n-1} \in V(b_i)$. By the Theorem 3.3, $V[R_c(G)] = e_1, e_2, \dots, e_{n-1} \cup b_1, b_2, \dots, b_i \cup r_1, r_2, \dots, r_k \cup c_1, c_2, \dots, c_k$. By theorem 3.7, $R_c(G)$ is non separable. By the definition, $b_1 e_1, c_1, e_2, \dots, e_{l-1} r_1 b_2, c_2, \dots, r_2 e_m b_3, \dots, e_{k+1}, c_k, e_{k+2}, \dots, e_{n-1} b_k r_k e_l b_1$ form a cycle which contains all the vertices of $R_c(G)$. Hence $R_c(G)$ is hamiltonian.

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Notes



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Bicomplex Riesz-Fischer Theorem

By K. S. Charak, R. Kumar & D. Rochon

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Abstract - This paper continues the study of infinite dimensional bicomplex Hilbert spaces introduced in previous articles on the topic. Besides obtaining a Best Approximation Theorem, the main purpose of this paper is to obtain a bicomplex analogue of the Riesz-Fischer Theorem. There are many statements of the Riesz-Fischer (R-F) Theorem in the literature, some are equivalent, some are consequences of the original versions. The one referred to in this paper is the R-F Theorem which establishes that the spaces l^2 is the canonical model space.

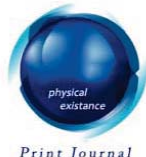
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Bicomplex Riesz-Fischer Theorem

K. S. Charak ^α, R. Kumar ^σ & D. Rochon ^ρ

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Keywords : bicomplex numbers, bicomplex algebra, generalized Hilbert spaces, Riesz-Fischer theorem.

I. INTRODUCTION

Hilbert spaces over the field of complex numbers are indispensable for mathematical structure of quantum mechanics [15] which in turn play a great role in molecular, atomic and subatomic phenomena. The work towards the generalization of quantum mechanics to bicomplex number system have been recently a topic in different quantum mechanical models [2, 3, 4, 19, 20]. More specifically, in [7, 8] the authors made an in depth study of bicomplex Hilbert spaces and operators acting on them. After obtaining reasonable results responsible for investigations on finite and infinite dimensional bicomplex Hilbert spaces and applications to quantum mechanics [9, 10, 11, 12], they in [8] asked for extension of Riesz-Fischer Theorem and Spectral Theorem on infinite dimensional Hilbert spaces. Recently, the bicomplex analogue of the Spectral Decomposition Theorem was proven using bicomplex eigenvalues [6]. In this paper, we obtain a bicomplex analogue of the Riesz-Fischer Theorem [13, 14] on infinite dimensional Hilbert spaces. Our proof of R-F Theorem is essentially different from its complex Hilbert space analogue in the sense that we do not make use of the so called Parseval's identity as done in general Hilbert spaces over $\mathbb{R}$ or $\mathbb{C}$. To support our results, we prove A Best Approximation Theorem and we show that the bicomplex analogue of l^2 , the space of all (real, complex or bicomplex) sequences $\{w_l\}$ such that $\sum_{l=1}^{\infty} |w_l|^2 < \infty$, is a bicomplex Hilbert space. As for the standard quantum mechanics, this specific result is fundamental to understand the space where live the wave functions of the bicomplex Quantum Harmonic Oscillator [9, 19, 20].

II. PRELIMINARIES

This section first summarizes a number of known results on the algebra of bicomplex numbers, which will be needed in this paper. Much more details as well as proofs can be found in [16, 18, 19, 20]. Basic definitions related to bicomplex modules and scalar products are also formulated as in [7, 19], but here we make no restrictions to finite dimensions following definitions of [8].

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a) *Bicomplex Numbers*i. *Definition*

The set $\mathbb{M}(2)$ of *bicomplex numbers* is defined as

$$\mathbb{M}(2) := \{w = z_1 + z_2 \mathbf{i}_2 \mid z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)\}, \quad (2.1)$$

where $\mathbf{i}_1$ and $\mathbf{i}_2$ are independent imaginary units such that $\mathbf{i}_1^2 = -1 = \mathbf{i}_2^2$. The product of $\mathbf{i}_1$ and $\mathbf{i}_2$ defines a hyperbolic unit $\mathbf{j}$ such that $\mathbf{j}^2 = 1$. The product of all units is commutative and satisfies

$$\mathbf{i}_1 \mathbf{i}_2 = \mathbf{j}, \quad \mathbf{i}_1 \mathbf{j} = -\mathbf{i}_2, \quad \mathbf{i}_2 \mathbf{j} = -\mathbf{i}_1.$$

With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{M}(2)$ makes up a commutative ring. They are a particular case of the so-called *Multicomplex Numbers* (denoted $\mathbb{M}(n)$) [16, 17] and [23]. In fact, bicomplex numbers

$$\mathbb{M}(2) \cong \text{Cl}_{\mathbb{C}}(1, 0) \cong \text{Cl}_{\mathbb{C}}(0, 1)$$

are unique among the complex Clifford algebras (see [1, 5] and [21]) in the sense that this set form a commutative, but not division algebra.

Three important subsets of $\mathbb{M}(2)$ can be specified as

$$\mathbb{C}(\mathbf{i}_k) := \{x + y \mathbf{i}_k \mid x, y \in \mathbb{R}\}, \quad k = 1, 2;$$

$$\mathbb{D} := \{x + y \mathbf{j} \mid x, y \in \mathbb{R}\}.$$

Each of the sets $\mathbb{C}(\mathbf{i}_k)$ is isomorphic to the field of complex numbers, while $\mathbb{D}$ is the set of so-called *hyperbolic numbers*, also called duplex numbers (see, e.g. [22], [18]).

ii. *Conjugation and Moduli*

Three kinds of conjugation can be defined on bicomplex numbers. With w specified as in (2.1) and the bar ($\bar{}$) denoting complex conjugation in $\mathbb{C}(\mathbf{i}_1)$, we define

$$w^{\dagger_1} := \bar{z}_1 + \bar{z}_2 \mathbf{i}_2, \quad w^{\dagger_2} := z_1 - z_2 \mathbf{i}_2, \quad w^{\dagger_3} := \bar{z}_1 - \bar{z}_2 \mathbf{i}_2.$$

It is easy to check that each conjugation has the following properties:

$$(s + t)^{\dagger_k} = s^{\dagger_k} + t^{\dagger_k}, \quad (s^{\dagger_k})^{\dagger_k} = s, \quad (s \cdot t)^{\dagger_k} = s^{\dagger_k} \cdot t^{\dagger_k}.$$

Here $s, t \in \mathbb{M}(2)$ and $k = 1, 2, 3$.

With each kind of conjugation, one can define a specific bicomplex modulus as

$$|w|_{\mathbf{i}_1}^2 := w \cdot w^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i}_1),$$

$$|w|_{\mathbf{i}_2}^2 := w \cdot w^{\dagger_1} = (|z_1|^2 - |z_2|^2) + 2 \operatorname{Re}(z_1 \bar{z}_2) \mathbf{i}_2 \in \mathbb{C}(\mathbf{i}_2),$$

$$|w|_{\mathbf{j}}^2 := w \cdot w^{\dagger_3} = (|z_1|^2 + |z_2|^2) - 2 \operatorname{Im}(z_1 \bar{z}_2) \mathbf{j} \in \mathbb{D}.$$

It can be shown that $|s \cdot t|_k^2 = |s|_k^2 \cdot |t|_k^2$, where $k = \mathbf{i}_1, \mathbf{i}_2$ or $\mathbf{j}$.

In this paper we will often use the Euclidean $\mathbb{R}^4$ -norm defined as

$$|w| := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\operatorname{Re}(|w|_{\mathbf{j}}^2)}.$$

Clearly, this norm maps $\mathbb{M}(2)$ into $\mathbb{R}$. We have $|w| \geq 0$, and $|w| = 0$ if and only if $w = 0$. Moreover [18], for all $s, t \in \mathbb{M}(2)$,

$$|s + t| \leq |s| + |t|, \quad |s \cdot t| \leq \sqrt{2} |s| \cdot |t|.$$

R_{ef.}

[16] G. Balev Price, An Introduction to Multicomplex Spaces and Functions, Marcel Dekker, 1991.

iii. *Idempotent Basis*

The operations of the bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers $\mathbf{e}_1$ and $\mathbf{e}_2$ defined as

$$\mathbf{e}_1 := \frac{1+\mathbf{j}}{2}, \quad \mathbf{e}_2 := \frac{1-\mathbf{j}}{2}.$$

In fact $\mathbf{e}_1$ and $\mathbf{e}_2$ are hyperbolic numbers. They make up the so-called *idempotent basis* of the bicomplex numbers. One easily checks that ($k = 1, 2$)

$$\mathbf{e}_1^2 = \mathbf{e}_1, \quad \mathbf{e}_2^2 = \mathbf{e}_2, \quad \mathbf{e}_1 + \mathbf{e}_2 = 1, \quad \mathbf{e}_k^{\dagger_3} = \mathbf{e}_k, \quad \mathbf{e}_1 \mathbf{e}_2 = 0. \quad (2.2)$$

Any bicomplex number w can be written uniquely as

$$w = z_1 + z_2 \mathbf{i}_2 = z_{\hat{1}} \mathbf{e}_1 + z_{\hat{2}} \mathbf{e}_2, \quad (2.3)$$

where

$$z_{\hat{1}} = z_1 - z_2 \mathbf{i}_1 \quad \text{and} \quad z_{\hat{2}} = z_1 + z_2 \mathbf{i}_1$$

both belong to $\mathbb{C}(\mathbf{i}_1)$. Note that

$$|w| = \frac{1}{\sqrt{2}} \sqrt{|z_{\hat{1}}|^2 + |z_{\hat{2}}|^2}.$$

The caret notation ($\hat{1}$ and $\hat{2}$) will be used systematically in connection with idempotent decompositions, with the purpose of easily distinguishing different types of indices. As a consequence of (2.2) and (2.3), one can check that if $\sqrt[n]{z_{\hat{1}}}$ is an n th root of $z_{\hat{1}}$ and $\sqrt[n]{z_{\hat{2}}}$ is an n th root of $z_{\hat{2}}$, then $\sqrt[n]{z_{\hat{1}}} \mathbf{e}_1 + \sqrt[n]{z_{\hat{2}}} \mathbf{e}_2$ is an n th root of w .

The uniqueness of the idempotent decomposition allows the introduction of two projection operators as

$$P_1 : w \in \mathbb{M}(2) \mapsto z_{\hat{1}} \in \mathbb{C}(\mathbf{i}_1),$$

$$P_2 : w \in \mathbb{M}(2) \mapsto z_{\hat{2}} \in \mathbb{C}(\mathbf{i}_1).$$

The P_k ($k = 1, 2$) satisfies

$$[P_k]^2 = P_k, \quad P_1 \mathbf{e}_1 + P_2 \mathbf{e}_2 = \mathbf{Id},$$

and, for $s, t \in \mathbb{M}(2)$,

$$P_k(s+t) = P_k(s) + P_k(t), \quad P_k(s \cdot t) = P_k(s) \cdot P_k(t).$$

The product of two bicomplex numbers w and w' can be written in the idempotent basis as

$$w \cdot w' = (z_{\hat{1}} \mathbf{e}_1 + z_{\hat{2}} \mathbf{e}_2) \cdot (z'_{\hat{1}} \mathbf{e}_1 + z'_{\hat{2}} \mathbf{e}_2) = z_{\hat{1}} z'_{\hat{1}} \mathbf{e}_1 + z_{\hat{2}} z'_{\hat{2}} \mathbf{e}_2.$$

Since 1 is uniquely decomposed as $\mathbf{e}_1 + \mathbf{e}_2$, we can see that $w \cdot w' = 1$ if and only if $z_{\hat{1}} z'_{\hat{1}} = 1 = z_{\hat{2}} z'_{\hat{2}}$. Thus w has an inverse if and only if $z_{\hat{1}} \neq 0 \neq z_{\hat{2}}$, and the inverse w^{-1} is then equal to $(z_{\hat{1}})^{-1} \mathbf{e}_1 + (z_{\hat{2}})^{-1} \mathbf{e}_2$. A nonzero w that does not have an inverse has the property that either $z_{\hat{1}} = 0$ or $z_{\hat{2}} = 0$, and such a w is a divisor of zero. Zero divisors make up the so-called *null cone* $\mathcal{NC}$. That terminology comes from the fact that when w is written as in (2.1), zero divisors are such that $z_1^2 + z_2^2 = 0$.

Any hyperbolic number can be written in the idempotent basis as $x_{\hat{1}} \mathbf{e}_1 + x_{\hat{2}} \mathbf{e}_2$, with $x_{\hat{1}}$ and $x_{\hat{2}}$ in $\mathbb{R}$. We define the set $\mathbb{D}_+$ of positive hyperbolic numbers as

$$\mathbb{D}_+ := \{x_{\hat{1}} \mathbf{e}_1 + x_{\hat{2}} \mathbf{e}_2 \mid x_{\hat{1}}, x_{\hat{2}} \geq 0\}.$$

Since $w^{\dagger_3} = \bar{z}_{\hat{1}} \mathbf{e}_1 + \bar{z}_{\hat{2}} \mathbf{e}_2$, it is clear that $w \cdot w^{\dagger_3} \in \mathbb{D}_+$ for any w in $\mathbb{M}(2)$.

The set of bicomplex numbers is a commutative ring. Just like vector spaces are defined over fields, modules are defined over rings. A module M defined over the ring of bicomplex numbers is called an $\mathbb{M}(2)$ -module [19, 7, 8].

Let M be an $\mathbb{M}(2)$ -module. For $k = 1, 2$, we define V_k as the set of all elements of the form $\mathbf{e}_k|\psi\rangle$, with $|\psi\rangle \in M$. Succinctly, $V_1 := \mathbf{e}_1 M$ and $V_2 := \mathbf{e}_2 M$. In fact, V_k is a vector space over $\mathbb{C}(\mathbf{i}_1)$ and any element $|v_k\rangle \in V_k$ satisfies $|v_k\rangle = \mathbf{e}_k|v_k\rangle$ for $k = 1, 2$. For arbitrary $\mathbb{M}(2)$ -modules, vector spaces V_1 and V_2 bear no structural similarities. For more specific modules, however, they may share structure. It was shown in [7] that if M is a finite-dimensional free $\mathbb{M}(2)$ -module, then V_1 and V_2 have the same dimension.

For any $|\psi\rangle \in M$, there exist a unique decomposition

$$|\psi\rangle = |v_1\rangle + |v_2\rangle, \quad (2.4)$$

where $v_k \in V_k$, $k = 1, 2$.

It will be useful to rewrite (2.4) as

$$|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle,$$

where

$$|\psi_1\rangle := \mathbf{e}_1|\psi\rangle \quad \text{and} \quad |\psi_2\rangle := \mathbf{e}_2|\psi\rangle.$$

In fact, the $\mathbb{M}(2)$ -module M can be viewed as a vector space M' over $\mathbb{C}(\mathbf{i}_1)$, and $M' = V_1 \oplus V_2$. From a set-theoretical point of view, M and M' are identical. In this sense we can say, perhaps improperly, that the **module** M can be decomposed into the direct sum of two vector spaces over $\mathbb{C}(\mathbf{i}_1)$, i.e. $M = V_1 \oplus V_2$.

iv. Bicomplex Scalar Product

A *bicomplex scalar product* maps two arbitrary kets $|\psi\rangle$ and $|\phi\rangle$ into a bicomplex number $(|\psi\rangle, |\phi\rangle)$, so that the following always holds ($s \in \mathbb{M}(2)$):

1. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle)$;
2. $(|\psi\rangle, s|\phi\rangle) = s(|\psi\rangle, |\phi\rangle)$;
3. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^{\dagger 3}$;
4. $(|\psi\rangle, |\psi\rangle) = 0 \Leftrightarrow |\psi\rangle = 0$.

The bicomplex scalar product was defined in [19] where, as in this paper, the physicists' convention is used for the order of elements in the product.

Property 3 implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$, while properties 2 and 3 together imply that $(s|\psi\rangle, |\phi\rangle) = s^{\dagger 3}(|\psi\rangle, |\phi\rangle)$. However, in this work we will also require the bicomplex scalar product $(\cdot, \cdot)$ to be *hyperbolic positive*, i.e.

$$(|\psi\rangle, |\psi\rangle) \in \mathbb{D}_+, \quad \forall |\psi\rangle \in M.$$

This is a necessary condition if we want to recover the standard quantum mechanics from the bicomplex one (see [9]).

Definition 2.1. Let M be a $\mathbb{T}$ -module and let $(\cdot, \cdot)$ be a bicomplex scalar product defined on M . The space $\{M, (\cdot, \cdot)\}$ is called a $\mathbb{M}(2)$ -inner product space, or bicomplex pre-Hilbert space. When no confusion arises, $\{M, (\cdot, \cdot)\}$ will simply be denoted by M .

In this work, we will sometimes use the Dirac notation

$$(|\psi\rangle, |\phi\rangle) = \langle\psi|\phi\rangle$$

R_{ef.}

[19] D. Rochon and S. Tremblay, *Bicomplex quantum mechanics: I. The generalized Schrödinger equation*, Adv. Appl. Clifford Algebr **14**, 231–248 (2004).

for the scalar product. The one-to-one correspondence between *bra* $\langle \cdot |$ and *ket* $|\cdot\rangle$ can be established from the Bicomplex Riesz Representation Theorem [8, Th. 3.7]. As in [12], subindices will be used inside the ket notation. In fact, this is simply a convenient way to deal with the Dirac notation in V_1 and V_2 . Note that the following projection of a bicomplex scalar product:

$$(\cdot, \cdot)_{\widehat{k}} := P_k((\cdot, \cdot)) : M \times M \longrightarrow \mathbb{C}(\mathbf{i}_1)$$

is a **standard scalar product** on V_k , for $k = 1, 2$. One easily show [8] that

$$\begin{aligned} (|\psi\rangle, |\phi\rangle) &= \mathbf{e}_1 P_1((|\psi_1\rangle, |\phi_1\rangle)) + \mathbf{e}_2 P_2((|\psi_2\rangle, |\phi_2\rangle)) \\ &= \mathbf{e}_1 (|\psi_1\rangle, |\phi_1\rangle)_{\widehat{1}} + \mathbf{e}_2 (|\psi_2\rangle, |\phi_2\rangle)_{\widehat{2}}. \\ &= \mathbf{e}_1 \langle \psi_1 | \phi_1 \rangle_{\widehat{1}} + \mathbf{e}_2 \langle \psi_2 | \phi_2 \rangle_{\widehat{2}}. \end{aligned} \quad (2.5)$$

We point out that a bicomplex scalar product is **completely characterized** by the two standard scalar products $(\cdot, \cdot)_{\widehat{k}}$ on V_k . In fact, if $(\cdot, \cdot)_{\widehat{k}}$ is an arbitrary scalar product on V_k , for $k = 1, 2$, then $(\cdot, \cdot)$ defined as in (2.5) is a bicomplex scalar product on M .

From this scalar product, we can define a **norm** on the vector space M' :

$$\begin{aligned} ||\phi|| &:= \frac{1}{\sqrt{2}} \sqrt{(|\phi_1\rangle, |\phi_1\rangle)_{\widehat{1}} + (|\phi_2\rangle, |\phi_2\rangle)_{\widehat{2}}} \\ &= \frac{1}{\sqrt{2}} \sqrt{||\phi_1||_1^2 + ||\phi_2||_2^2}. \end{aligned} \quad (2.6)$$

Here we wrote

$$||\phi_k||_k = \sqrt{(|\phi_k\rangle, |\phi_k\rangle)_{\widehat{k}}},$$

where $|\cdot|_k$ is the natural scalar-product-induced norm on V_k . Moreover,

$$||\phi|| = \frac{1}{\sqrt{2}} \sqrt{(|\phi_1\rangle, |\phi_1\rangle)_{\widehat{1}} + (|\phi_2\rangle, |\phi_2\rangle)_{\widehat{2}}} = |\sqrt{(|\phi\rangle, |\phi\rangle)}|.$$

Definition 2.2. Let M be an $\mathbb{M}(2)$ -module and let M' be the associated vector space. We say that $||\cdot|| : M \longrightarrow \mathbb{R}$ is a **$\mathbb{M}(2)$ -norm** on M if the following holds:

1. $||\cdot|| : M' \longrightarrow \mathbb{R}$ is a norm;
2. $||w \cdot |\psi\rangle|| \leq \sqrt{2} |w| \cdot |||\psi\rangle||$, $\forall w \in \mathbb{T}$, $\forall |\psi\rangle \in M$.

A $\mathbb{M}(2)$ -module with a **$\mathbb{M}(2)$ -norm** is called a **normed $\mathbb{M}(2)$ -module**. It is easy to check that $||\cdot||$ in (2.6) is a **$\mathbb{M}(2)$ -norm** on M and that the $\mathbb{M}(2)$ -module M is **complete** with respect to the following metric on M :

$$d(|\phi\rangle, |\psi\rangle) = |||\phi\rangle - |\psi\rangle||$$

if and only if V_1 and V_2 are complete (see [8]).

Definition 2.3. A bicomplex Hilbert space is a $\mathbb{M}(2)$ -inner product space M which is complete with respect to the induced $\mathbb{M}(2)$ -norm (2.6).

III. MAIN RESULTS

Throughout the text, by a **bicomplex Hilbert space** we shall mean an infinite dimensional bicomplex Hilbert space. A normed $\mathbb{M}(2)$ -module with a Schauder $\mathbb{M}(2)$ -basis is called a **countable $\mathbb{M}(2)$ -module**.

Definition 3.1. A bicomplex Hilbert space M is said to be *separable by a basis* if it has a Schauder $\mathbb{M}(2)$ -basis.

We note that by Theorem 3.10 in [8], any Schauder $\mathbb{M}(2)$ -basis of M can be orthonormalized.

Remark 3.2. A topological space S is called *separable* if it admits a countable dense subset W .

Ref.

[8] R. Gervais Lavoie, L. Marchildon and D. Rochon, *Infinite dimensional Hilbert spaces*, Ann. Funct. Anal. **1**, 75–91 (2010).

Proposition 3.3. Let $\langle \cdot | \cdot \rangle$ be a bicomplex inner product in the bicomplex Hilbert space M and let $\| \cdot \|$ be the induced norm. If the sequences $\{|\psi_n\rangle\}$ and $\{|\phi_n\rangle\}$ in M converge to $\{|\psi\rangle\}$ and $\{|\phi\rangle\}$ respectively, then the sequence of inner products $\{\langle \psi_n | \phi_n \rangle\}$ converges to $\langle \psi | \phi \rangle$.

Proof. First observe that: $\langle \psi_n | \phi_n \rangle - \langle \psi | \phi \rangle$

$$\begin{aligned} &= \langle \psi_n | \phi_n \rangle - \langle \psi | \phi_n \rangle + \langle \psi | \phi_n \rangle - \langle \psi | \phi \rangle \\ &= \langle \psi_n - \psi | \phi_n \rangle + \langle \psi | \phi_n - \phi \rangle \\ &= \langle \psi_n - \psi | \phi_n - \phi \rangle + \langle \psi_n - \psi | \phi \rangle + \langle \psi | \phi_n - \phi \rangle. \end{aligned}$$

From this we get by the **bicomplex Schwarz inequality** ([8], Theorem 3.8): $|\langle \psi_n | \phi_n \rangle - \langle \psi | \phi \rangle|$

$$\begin{aligned} &= |\langle \psi_n - \psi | \phi_n - \phi \rangle + \langle \psi_n - \psi | \phi \rangle + \langle \psi | \phi_n - \phi \rangle| \\ &\leq |\langle \psi_n - \psi | \phi_n - \phi \rangle| + |\langle \psi_n - \psi | \phi \rangle| + |\langle \psi | \phi_n - \phi \rangle| \\ &\leq [\sqrt{2} \| |\psi_n\rangle - |\psi\rangle \| \cdot \| |\phi_n\rangle - |\phi\rangle \| + \sqrt{2} \| |\psi_n\rangle - |\psi\rangle \| \cdot \| |\phi\rangle \| \\ &\quad + \sqrt{2} \| |\psi\rangle \| \cdot \| |\phi_n\rangle - |\phi\rangle \|]. \end{aligned}$$

The proposition now follows easily.

Theorem 3.4 (Best Approximation Theorem). Let $\{|\psi_n\rangle\}$ be an arbitrary orthonormal sequence in the bicomplex Hilbert space $M = H_1 \oplus H_2$, and let $\alpha_1, \dots, \alpha_n$ be a set of bicomplex numbers. Then for all $|\psi\rangle \in M$,

$$\left\| |\psi\rangle - \sum_{l=0}^n \alpha_l |\psi_l\rangle \right\| \geq \left\| |\psi\rangle - \sum_{l=0}^n \langle \psi_l | \psi \rangle |\psi_l\rangle \right\|.$$

Proof. By definition of the bicomplex inner product, the set $\{|\psi_{n\mathbf{k}}\rangle\}$ is also an arbitrary orthonormal sequence in the Hilbert space H_k for $k = 1, 2$. Therefore, using the classical Best Approximation Theorem (see [13], P.61) on the Hilbert spaces H_1 and H_2 , we obtain for $k = 1, 2$:

$$\left\| |\psi_{\mathbf{k}}\rangle - \sum_{l=0}^n P_k(\alpha_l) |\psi_{l\mathbf{k}}\rangle \right\|_k \geq \left\| |\psi_{\mathbf{k}}\rangle - \sum_{l=0}^n \langle \psi_{l\mathbf{k}} | \psi_{\mathbf{k}} \rangle_{\widehat{k}} |\psi_{l\mathbf{k}}\rangle \right\|_k.$$

Hence, by definition of the $\mathbb{M}(2)$ -norm, we have that

$$\begin{aligned} \left\| |\psi\rangle - \sum_{l=0}^n \alpha_l |\psi_l\rangle \right\| &= \frac{1}{\sqrt{2}} \sqrt{\sum_{k=1}^2 \left\| |\psi_{\mathbf{k}}\rangle - \sum_{l=0}^n P_k(\alpha_l) |\psi_{l\mathbf{k}}\rangle \right\|_k^2} \\ &\geq \frac{1}{\sqrt{2}} \sqrt{\sum_{k=1}^2 \left\| |\psi_{\mathbf{k}}\rangle - \sum_{l=0}^n \langle \psi_{l\mathbf{k}} | \psi_{\mathbf{k}} \rangle_{\widehat{k}} |\psi_{l\mathbf{k}}\rangle \right\|_k^2} \\ &= \left\| |\psi\rangle - \sum_{l=0}^n \langle \psi_l | \psi \rangle |\psi_l\rangle \right\|. \end{aligned}$$

An important consequence of the Best Approximation Theorem is that an orthonormal basis for a dense subspace of a bicomplex Hilbert space is actually an orthonormal basis in the full bicomplex Hilbert space. This is very useful result for the construction of specific orthonormal basis in separable Hilbert spaces. The precise result is as follows.

Theorem 3.5. Let N be a dense subspace of the bicomplex Hilbert space M , and assume that $\{|\mu_l\rangle\}$ is an orthonormal Schauder $\mathbb{M}(2)$ -basis for N . Then $\{|\mu_l\rangle\}$ is also an orthonormal Schauder $\mathbb{M}(2)$ -basis for M .

Ref.

[13] V. L. Hansen, Functional Analysis: Entering Hilbert Space, World Scientific, Singapore, 2006.

Proof. Since $\{|m_l\rangle\}$ is a Schauder $\mathbb{M}(2)$ -basis for N , any $|\psi\rangle \in N$ admits a unique expansion as an infinite series $|\psi\rangle = \sum_{l=1}^{\infty} \alpha_l |m_l\rangle$. In fact,

$$|\psi\rangle = \sum_{l=1}^{\infty} \langle m_l | \psi \rangle |m_l\rangle.$$

This follows by Proposition 3.3 and the short computation

$$\langle m_l | \psi \rangle = \left\langle m_l \left| \lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_k m_k \right. \right\rangle = \lim_{n \rightarrow \infty} \left\langle m_l \left| \sum_{k=1}^n \alpha_k m_k \right. \right\rangle = \alpha_l,$$

valid for all $l \in \mathbb{N}$. Now, to complete the proof, let us prove that any ket $|\phi\rangle \in M$ admits the same expansion form:

$$|\phi\rangle = \sum_{l=1}^{\infty} \langle m_l | \phi \rangle |m_l\rangle. \quad (3.1)$$

To prove this assertion, let an arbitrary $\epsilon > 0$ be given. Since N is dense in M , we can choose $|\psi\rangle \in N$, such that $\| |\phi\rangle - |\psi\rangle \| < \frac{\epsilon}{2}$. Now write $|\psi\rangle = \sum_{l=1}^{\infty} \langle m_l | \psi \rangle |m_l\rangle$, and choose $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow \left\| |\psi\rangle - \sum_{l=1}^n \langle m_l | \psi \rangle |m_l\rangle \right\| < \frac{\epsilon}{2}.$$

By the Best Approximation Theorem, we then get for all $n \geq n_0$,

$$\begin{aligned} \left\| |\phi\rangle - \sum_{l=1}^n \langle m_l | \phi \rangle |m_l\rangle \right\| &\leq \left\| |\phi\rangle - \sum_{l=1}^n \langle m_l | \psi \rangle |m_l\rangle \right\| \\ &\leq \left\| |\phi\rangle - |\psi\rangle \right\| + \left\| |\psi\rangle - \sum_{l=1}^n \langle m_l | \psi \rangle |m_l\rangle \right\| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned}$$

Hence,

$$|\phi\rangle = \lim_{n \rightarrow \infty} \sum_{l=1}^n \langle m_l | \phi \rangle |m_l\rangle = \sum_{l=1}^{\infty} \langle m_l | \phi \rangle |m_l\rangle.$$

This prove that $\{|m_l\rangle\}$ is an orthonormal Schauder $\mathbb{M}(2)$ -basis for M .

The next result shows that all separable bicomplex Hilbert spaces are separable by a basis.

Lemma 3.6. *Every separable bicomplex Hilbert space M has an orthonormal Schauder $\mathbb{M}(2)$ -basis.*

Proof. By the definition of separability, M contains a countable, dense subset W of kets in M . Consider the linear subspace U in M consisting of all finite bicomplex linear combinations of kets in W - the *bicomplex linear span* of W . Clearly, U is a dense sub- $\mathbb{M}(2)$ -module in M . By the construction of U we can eliminate kets from the countable set W one after the other to get a (bicomplex) linearly independent set $\{|\phi_n\rangle\}$ (finite, or countable) of kets in U that spans U . However, a sub- $\mathbb{M}(2)$ -module U in M of finite dimension is a complete space, thus a closed set in M , and then $U = \bar{U} = M$ a contradiction with our hypothesis. Therefore, the set $\{|\phi_n\rangle\}$ is a countable (bicomplex) linearly independent set of kets in U . Now, since no $|\phi_n\rangle$ (and thus no $\langle \phi_n | \phi_n \rangle$) can belongs to the null cone, the classical Gram-Schmidt process can be applied (see [7], P.574). Hence, we can turn the sequence $\{|\phi_n\rangle\}$ into an orthonormal sequence $\{|\psi_n\rangle\}$ with the property that for all $n \in \mathbb{N}$,

$$\text{span}\{|\phi_n\rangle\}_{l=1}^n = \text{span}\{|\psi_l\rangle\}_{l=1}^n$$

Ref.

[7] R. Gervais Lavoie, L. Marchildon and D. Rochon, *Finite-dimensional bicomplex Hilbert spaces*, Adv. Appl. Clifford Algebras **21**, 561–581 (2011).

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Since $\{|\psi_l\rangle\}$ is orthonormal, we can use $\{|\psi_l\rangle\}$ as a Schauder $\mathbb{M}(2)$ -basis to generate a linear subspace N in M (for the unicity, see the proof of Theorem 3.5). Then N is a dense sub- $\mathbb{M}(2)$ -module in M , since U is a dense sub- $\mathbb{M}(2)$ -module in N . The latter follows since any ket $|\psi\rangle \in N$ can be expanded into a series $|\psi\rangle = \sum_{l=1}^{\infty} \alpha_l |\psi_l\rangle$, showing that $|\psi\rangle = \lim_{n \rightarrow \infty} \sum_{l=1}^n \alpha_l |\psi_l\rangle$, and hence that $|\psi\rangle$ is the limit of a sequence of kets in U .

By construction, $\{|\psi_l\rangle\}$ is an orthonormal Schauder $\mathbb{M}(2)$ -basis for N and hence by Theorem 3.5 also for M .

Theorem 3.7. *If M is a separable bicomplex Hilbert space, then H_k ($k = 1, 2$) is an infinite dimensional separable complex Hilbert space.*

Proof. From Lemma 3.6, $M = H_1 \oplus H_2$ has an orthonormal Schauder $\mathbb{M}(2)$ -basis $\{|\psi_l\rangle\}$. It is easy to see that $\{|\psi_{lk}\rangle\}$ is also an orthonormal Schauder basis for H_k ($k = 1, 2$). Hence, H_k ($k = 1, 2$) is separable by a basis. Now, from Theorem 3.3.6. in [13], H_k ($k = 1, 2$) is an infinite dimensional separable complex Hilbert space.

Definition 3.8. Denote by l_2^2 , the space of all (real, complex or bicomplex) sequences $\{w_l\}$ such that

$$\sum_{l=1}^{\infty} |w_l|^2 < \infty.$$

The bicomplex l_2^2 space is clearly an $\mathbb{M}(2)$ -module. The norm of the associated vector space $(l_2^2)'$ over $\mathbb{C}(\mathbf{i}_1)$ is defined by

$$\|\{w_l\}\|_2 = \left(\sum_{l=1}^{\infty} |w_l|^2 \right)^{\frac{1}{2}}.$$

Theorem 3.9. l_2^2 is a bicomplex Hilbert space.

Proof. Let us prove that $(l_2^2)' = (\mathbf{e}_1 l^2) \oplus (\mathbf{e}_2 l^2)$. This comes automatically from the fact that any bicomplex sequence $\{w_l\}$ can be decomposed as the following sum of two sequences in $\mathbb{C}(\mathbf{i}_1)$:

$$\{w_l\} = \mathbf{e}_1 \{z_{1l} - z_{2l} \mathbf{i}_1\} + \mathbf{e}_2 \{z_{1l} + z_{2l} \mathbf{i}_1\}.$$

To complete the proof, we need to verify that the norm $\|\cdot\|_2$ coincides with the induced $\mathbb{M}(2)$ -norm of the bicomplex Hilbert space $(\mathbf{e}_1 l^2) \oplus (\mathbf{e}_2 l^2)$. Let $\|\cdot\|$ be the induced $\mathbb{M}(2)$ -norm of the bicomplex Hilbert space $(\mathbf{e}_1 l^2) \oplus (\mathbf{e}_2 l^2)$. Thus

$$\|\{w_l\}\| = \frac{1}{\sqrt{2}} \sqrt{|\{z_{1l} - z_{2l} \mathbf{i}_1\}|_1^2 + |\{z_{1l} + z_{2l} \mathbf{i}_1\}|_2^2}$$

where $|\cdot|_1 = |\cdot|_2$ is the classical norm on l^2 . Hence,

$$\begin{aligned} \|\{w_l\}\| &= \frac{1}{\sqrt{2}} \sqrt{|\{z_{1l} - z_{2l} \mathbf{i}_1\}|_1^2 + |\{z_{1l} + z_{2l} \mathbf{i}_1\}|_1^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{\sum_{l=1}^{\infty} |z_{1l} - z_{2l} \mathbf{i}_1|^2 + \sum_{l=1}^{\infty} |z_{1l} + z_{2l} \mathbf{i}_1|^2} \\ &= \sqrt{\sum_{l=1}^{\infty} \frac{|z_{1l} - z_{2l} \mathbf{i}_1|^2 + |z_{1l} + z_{2l} \mathbf{i}_1|^2}{2}} \\ &= \|\{w_l\}\|_2. \end{aligned}$$

We are now ready for the proof of the main result on the structure of infinite dimensional, separable bicomplex Hilbert space. We show that the space of square summable bicomplex sequences l_2^2 . Define the projection $T_k : M \rightarrow V_k$ as

$$T_k|\phi\rangle := \mathbf{e}_k T(|\phi\rangle), \quad \forall |\phi\rangle \in M, \quad k = 1, 2.$$

With this definition we have the following Lemma.

Lemma 3.10. *Let M_1, M_2 be two $\mathbb{M}(2)$ -modules and $T : M_1 \rightarrow M_2$ be a bicomplex linear function. Then $\forall |\phi\rangle \in M_1$ we have*

$$T_k(|\phi\rangle) = T(|\phi_k\rangle), \quad (k = 1, 2).$$

Proof.

$$\begin{aligned} T_k(|\phi\rangle) &= \mathbf{e}_k(T(|\phi\rangle)) \\ &= \mathbf{e}_k(T(|\phi_1\rangle + |\phi_2\rangle)) \\ &= T(|\phi_k\rangle). \end{aligned}$$

Theorem 3.11 (Riesz-Fischer). *Every separable bicomplex Hilbert space M is isometrically isomorphic to the bicomplex Hilbert space l_2^2 .*

Proof. From Lemma 3.6, since $M = H_1 \oplus H_2$ is a separable bicomplex Hilbert space, it has an orthonormal Schauder $\mathbb{M}(2)$ -basis:

$$\{|m_1\rangle, \dots, |m_l\rangle, \dots\}.$$

Then each $|\psi\rangle \in M$ admits a unique decomposition as

$$|\psi\rangle = \sum_{l=1}^{\infty} w_l |m_l\rangle, \quad w_l \in \mathbb{M}(2).$$

Since the infinite series above converges, by Theorem 3.11 in [8], the series $\sum_{l=1}^{\infty} |w_l|^2$ converges in $\mathbb{R}$ and thus $\{w_l\} \in l_2^2$. Now, define a map $T : M \rightarrow l_2^2$ as

$$T(|\phi\rangle) = \{w_l\}_{l=1}^{\infty} \quad \forall |\phi\rangle \in M.$$

T is a well defined map: Let $|\phi\rangle, |\psi\rangle \in M$ be such that $|\phi\rangle = |\psi\rangle$. Hence, $\sum_{l=1}^{\infty} w_l |m_l\rangle = \sum_{l=1}^{\infty} w_l' |m_l\rangle$ and then by the uniqueness of the representation we find that $w_l = w_l'$ for each $l \in \mathbb{N}$, which further implies that $T(|\phi\rangle) = T(|\psi\rangle)$. Next, we show that T is **bicomplex** linear. Let $|\phi\rangle, |\psi\rangle \in M$ and $\alpha, \beta \in \mathbb{T}$. Then,

$$\begin{aligned} T(\alpha|\phi\rangle + \beta|\psi\rangle) &= T\left(\alpha \sum_{l=1}^{\infty} w_l |m_l\rangle + \beta \sum_{l=1}^{\infty} w_l' |m_l\rangle\right) \\ &= T\left(\sum_{l=1}^{\infty} (\alpha w_l) |m_l\rangle + \sum_{l=1}^{\infty} (\beta w_l') |m_l\rangle\right) \\ &= T\left(\sum_{l=1}^{\infty} (\alpha w_l + \beta w_l') |m_l\rangle\right) \\ &= \{\alpha w_l + \beta w_l'\} \\ &= \alpha\{w_l\} + \beta\{w_l'\} \\ &= \alpha T(|\phi\rangle) + \beta T(|\psi\rangle). \end{aligned}$$

Now, since $\{|w_l\rangle\}$ is an orthonormal basis in M , by Equation (3.1) in Theorem 3.5, every ket $|\phi\rangle \in M$ admits the unique expansion

$$|\phi\rangle = \sum_{l=1}^{\infty} \langle m_l | \phi \rangle |m_l\rangle.$$

Hence, T is **injective**, since $T(|\phi\rangle) = \{\langle m_l | \phi \rangle\} = 0$ implies $\langle m_l | \phi \rangle = 0$ for all $l \in \mathbb{N}$, and thus $|\phi\rangle = 0$. Moreover, T is **surjective**, since for any element $\{\alpha_l\} \in l_2^2$, the series $|\xi\rangle = \sum_{l=1}^{\infty} \alpha_l |m_l\rangle$ is convergent (Theorem 3.11 in [8]). Finally we shall show that T is an isometry. By Lemma 3.10, we have that

$$\begin{aligned} \|T(|\phi\rangle)\| &= |\sqrt{(T(|\phi\rangle), T(|\phi\rangle))}| \\ &= |\sqrt{\mathbf{e}_1(T(|\phi_1\rangle), T(|\phi_1\rangle))_1 + \mathbf{e}_2(T(|\phi_2\rangle), T(|\phi_2\rangle))_2}| \end{aligned} \quad (3.2)$$

By Theorem 3.7, the classical Riesz-Fischer Theorem can be applied to H_k where $T : H_k \rightarrow \mathbf{e}_k l^2$ for $k = 1, 2$. Then we find that

$$\begin{aligned} (T(|\phi_k\rangle), T(|\phi_k\rangle))_{\widehat{k}} &= |T(|\phi_k\rangle)|_k^2 \\ &= |\phi_k|_k^2 \\ &= \langle \phi_k | \phi_k \rangle_{\widehat{k}} \end{aligned}$$

for $k = 1, 2$, where $|\cdot|_1 = |\cdot|_2$ is the classical norm on l^2 . Thus, from Equation (3.2), we get that

$$\|T(|\phi\rangle)\| = \|\phi\|.$$

This proves that T is an isometry. Hence M is isometrically isomorphic to the bicomplex Hilbert space l_2^2 .

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