Some Results on Compactness in Bicomplex Space

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Some Results on Compactness in Bicomplex Space

Sukhdev Singh\textsuperscript{\textdegree} & Rajiv K. Srivastava\textsuperscript{\textdegree}

Abstract: In this paper, we have studied the continuity and compactness of the bicomplex space and its subsets. We have studied the compactness of some subsets of the bicomplex space in the idempotent order topology. We have also given a result regarding homeomorphism in the idempotent order topology and the complex order topology on the bicomplex space.

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1. Introduction

Throughout the paper, $C_0$, $C_1$ and $C_2$ denote sets of real numbers, complex numbers and bicomplex numbers, respectively. A bicomplex number is defined as (cf. [1], [3])

$$\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2,$$

where $x_p \in C_0$, $1 \leq p \leq 4$, $z_1, z_2 \in C_1$, $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$.

With usual binary compositions, $C_2$ becomes a commutative algebra with identity. Besides the additive and multiplicative identities 0 and 1, there exist exactly two non-trivial idempotent elements denoted by $e_1$ and $e_2$ defined as $e_1 = (1 + i_1 i_2) / 2$ and $e_2 = (1 - i_1 i_2) / 2$. Note that $e_1 + e_2 = 1$ and $e_1, e_2 = 0$.

A bicomplex number $\zeta = z_1 + i_2 z_2$ can be uniquely expressed as a complex combination of $e_1$ and $e_2$ as (cf. [3])

$$\zeta = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2$$

$$= 1_\zeta e_1 + 2_\zeta e_2$$

where $1_\zeta = z_1 - i_1 z_2$ and $2_\zeta = z_1 + i_1 z_2$. 


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The complex coefficients $^1\xi$ and $^2\xi$ are called the idempotent components and the combination $^1\xi e_1 + ^2\xi e_2$ is known as idempotent representation of bicomplex number $\xi$. The auxiliary complex spaces $A_1$ and $A_2$ are defined as follows:

$$A_1 = \{z_1 - i_1 z_2 ; z_1, z_2 \in C_1\} = \{^1\xi : \xi \in C_2\}$$

and

$$A_2 = \{z_1 + i_1 z_2 ; z_1, z_2 \in C_1\} = \{^2\xi : \xi \in C_2\}.$$

The idempotent representation $(z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2 = ^1\xi e_1 + ^2\xi e_2$ associates with each point $\xi = z_1 + i_2 z_2$ in $C_2$, the points $^1\xi = z_1 - i_1 z_2$ and $^2\xi = z_1 + i_1 z_2$ in $A_1$ and $A_2$, respectively and to each pair of points $(z, w) \in A_1 \times_c A_2$, there corresponds a unique bicomplex point $\xi = ze_1 + we_2$.

Srivastava [3] initiated the topological study of $C_2$. He defined three topologies on $C_2$, viz., norm topology $\tau_1$, complex topology $\tau_2$ and idempotent topology $\tau_3$ and has proved some results on these topological structures.

In thepresentpaper, we shall confine ourselves mainly to $C_2$ equipped with $\tau_5$ and $\tau_6$. For the sake of ready reference, we give below relevant literature of $\tau_5$ and $\tau_6$ (for details cf. [4]).

Denote by $<_C$, the dictionary ordering of the bicomplex numbers expressed in the complex form. The order topology induced by this ordering is called as Complex Order Topology. Complex order topology $\tau_5$ is generated by the basis $B_5$ comprising members of the following families of subsets of $C_2$:

(i) $K_1 = \{(z_1 + i_2 z_2, w_1 + i_2 w_2) : z_1 < w_1\}$

(ii) $K_2 = \{(z_1 + i_2 z_2, z_1 + i_2 w_2) : z_2 < w_2\}$.

**Remark 1.1:** Note that, since $z_1 < w_1$ and $z_2 < w_2$ in the dictionary order topology in $C_1$, $K_1$ and $K_2$ can also be described as $K_1 = M_1 \cup M_2$ and $K_2 = M_3 \cup M_4$, where

(i) $M_1 = \{(z_1 + i_2 z_2, w_1 + i_2 w_2) : \text{Re} z_1 < \text{Re} w_1\}$

(ii) $M_2 = \{(z_1 + i_2 z_2, w_1 + i_2 w_2) : \text{Re} z_1 = \text{Re} w_1, \text{Im} z_1 < \text{Im} w_1\}$

(iii) $M_3 = \{(z_1 + i_2 z_2, z_1 + i_2 w_2) : \text{Re} z_2 < \text{Re} w_2\}$

(iv) $M_4 = \{(z_1 + i_2 z_2, w_1 + i_2 w_2) : \text{Re} z_2 = \text{Re} w_2, \text{Im} z_2 < \text{Im} w_2\}$.

Note further that $M_1$, $M_2$, $M_3$ and $M_4$ are in fact, families of space segments, frame segments, plane segments and line segments, respectively.
Similarly, denote by \( \prec_{\text{id}} \) the dictionary ordering of the bicomplex numbers expressed in the idempotent form. The order topology induced by this ordering is called as \textit{Idempotent Order Topology}. Hence, idempotent order topology \( \tau_6 \) is generated by the basis \( B_6 \) comprising of members of the following families of subsets of \( C_2 \):

\[
\begin{align*}
(\text{i}) & \quad L_1 = \left\{ \left( 1 \xi e_1 + 2 \xi e_2, 1 \eta e_1 + 2 \eta e_2 \right)_{\text{id}} : 1 \xi < 1 \eta \right\} \\
(\text{ii}) & \quad L_2 = \left\{ \left( 1 \xi e_1 + 2 \xi e_2, 1 \eta e_1 + 2 \eta e_2 \right)_{\text{id}} : 2 \xi < 2 \eta \right\}
\end{align*}
\]

the set \( (\xi, \eta)_{\text{id}} \) denoting the open interval with respect to the ordering \( \prec_{\text{id}} \) and \( \prec \) denoting the dictionary order relation in \( A_1 \) and \( A_2 \).

The set of the type \( \{ \xi : a < \text{Re} 1 \xi < b \} \) is called an ID–space segment. A set of the type \( \{ \xi : \text{Re} 1 \xi = a \} \) is called an ID–frame and is denoted as \( (\text{Re} 1 \xi = a) \). A set of the type \( \{ \xi : \text{Re} 1 \xi = a, b < \text{Im} 1 \xi < c \} \) is called as an open ID–frame segment. The terms ID–plane, ID–plane segment, ID–line and ID–line segment are define analogously (for details cf. [5]).

Note that \( L_1 \) and \( L_2 \) can also be described as \( L_1 = N_1 \cup N_2 \) and \( L_2 = N_3 \cup N_4 \), where

\[
\begin{align*}
(\text{i}) & \quad N_1 = \left\{ \left( 1 \xi e_1 + 2 \xi e_2, 1 \eta e_1 + 2 \eta e_2 \right)_{\text{id}} : \text{Re} 1 \xi < \text{Re} 1 \eta \right\} \\
(\text{ii}) & \quad N_2 = \left\{ \left( 1 \xi e_1 + 2 \xi e_2, 1 \eta e_1 + 2 \eta e_2 \right)_{\text{id}} : \text{Re} 1 \xi = \text{Re} 1 \eta, \text{Im} 1 \xi < \text{Im} 1 \eta \right\} \\
(\text{iii}) & \quad N_3 = \left\{ \left( 1 \xi e_1 + 2 \xi e_2, 1 \eta e_1 + 2 \eta e_2 \right)_{\text{id}} : \text{Re} 2 \xi < \text{Re} 2 \eta \right\} \\
(\text{iv}) & \quad N_4 = \left\{ \left( 1 \xi e_1 + 2 \xi e_2, 1 \eta e_1 + 2 \eta e_2 \right)_{\text{id}} : \text{Re} 2 \xi = \text{Re} 2 \eta, \text{Im} 2 \xi < \text{Im} 2 \eta \right\}
\end{align*}
\]

Note further that \( N_1, N_2, N_3 \) and \( N_4 \) are families of open ID–space segments, open ID–frame segments, open ID–plane segments and open ID–line segments, respectively.

In other words,

\[
B_5 = \bigcup_{p=1}^{2} K_p = \bigcup_{p=1}^{4} M_p
\]

and

\[
B_6 = \bigcup_{p=1}^{2} L_p = \bigcup_{p=1}^{4} N_p
\]

\textbf{Theorem 1.1 [7]}: Every order topology is Hausdorff.

\textbf{Remarks 1.2}: The auxiliary complex spaces \( A_1 \) and \( A_2 \) are Hausdorff space with respect to the order topology generated by dictionary order relation \( \prec \) on them.
II. Compactness in Bicomplex Space

In this section, we have given some results regarding compactness on some subsets of the bicomplex space.

**Lemma 2.1:** Suppose that \( S = \{ ze_1 + w e_2 : z \in A_1, w = z^{-1}, 0 < z < 1 \} \).

Any subset \( (z_0 e_1 + w_1 e_2, z_0 e_1 + w_2 e_2)_{ID} \), where \( 0 < z_0 < 1 \), of \( C_2 \) contains at most one point of the set \( S \).

**Proof:** We have \( S = \{ ze_1 + w e_2 : z \in A_1, w = z^{-1}, 0 < z < 1 \} \).

Let \( z_0 \in C_1 \), where \( 0 < z < 1 \).

As \( z_0 \neq 0 \).

\( \Rightarrow \exists \) an element \( w \in C_1 \) such that \( z_0^{-1} \neq w \).

Therefore, \( \exists z_0 e_1 + w e_2 \in C_2 \) such that

\[ z_0 e_1 + w e_2 \in (z_0 e_1 + w_1 e_2, z_0 e_1 + w_2 e_2)_{ID}, \]

where \( w_1, w_2 \in A_2 \) and \( w_1 < w < w_2 \).

Further, if \( (z_0 e_1 + w_1 e_2, z e_1 + w_2 e_2)_{ID} \) is a subset of \( C_2 \) such that \( w < w_1 \)

where \( w = z_0^{-1} \), then

\[ (z_0 e_1 + w_1 e_2, z e_1 + w_2 e_2)_{ID} \cap S = \emptyset. \]

**Theorem 2.1:** The set \( S = \{ ze_1 + w e_2 : z \in A_1, w = z^{-1}, 0 < z < 1 \} \) is a compact subset of \( C_2 \).

**Proof:** To prove \( S \) is a compact, we show that the set \( S \) is bounded and closed with respect to the idempotent order topology in \( C_2 \). Since, the coefficients of \( e_1 \) of the elements of \( S \) are bounded by the lower and upper bounds 0 and 1, respectively.

Therefore, the elements of the set \( S \) are bounded by the bicomplex numbers \( w_1 e_2 \) and \( e_1 + w_1 e_2 \), \( w_1 \in A_2 \) as lower and upper bounds respectively, i.e.,

\[ ze_1 + w e_2 \in (w_1 e_2, e_1 + w_1 e_2)_{ID}, \] where \( 0 < z < 1 \).

Hence, the set \( S \) is bounded.

To prove that \( S \) is closed, we shall show that \( S^c \) is open in \( C_2 \).

Let \( \xi = z_0 e_1 + w_0 e_2 \in S^c \).

Then there are three possibilities:

1) \( z_0^{-1} \neq w_0 \) and \( z_0 \in (0, 1) \)

2) \( z_0^{-1} = w_0 \) and \( z_0 \notin (0, 1) \)

3) \( z_0^{-1} \neq w_0 \) and \( z_0 \notin (0, 1) \).
**Case 1:** When $z_0^{-1} \neq w_0$ and $z_0 \in (0, 1)$.

Since both $z_0^{-1}$ and $w_0$ are members of the auxiliary complex space $A_2$ and $A_2$ is a Hausdorff space under the dictionary order topology.

Therefore, $\exists \ u_1, u_2 \in A_2$ such that

$$z_0^{-1} \in (u_1, u_2).$$

Similarly, $\exists \ v_1, v_2 \in A_2$ such that

$$w_0 \in (v_1, v_2) \text{ and } (u_1, u_2) \cap (v_1, v_2) = \emptyset.$$

So, we have

$$\xi = z_0 e_1 + w_0 e_2 \in (z_0 e_1 + v_1 e_2, z_0 e_1 + v_2 e_2)_{\text{ID}}.$$

Now if the interval $(z_0 e_1 + v_1 e_2, z_0 e_1 + v_2 e_2)_{\text{ID}}$ has non-empty intersection with the set $S$ then it will contain only one point namely, $z_0 e_1 + z_0^{-1} e_1$.

But

$$z_0 e_1 + z_0^{-1} e_1 \notin (z_0 e_1 + v_1 e_2, z_0 e_1 + v_2 e_2)_{\text{ID}}.$$

Hence,

$$z_0 e_1 + z_0^{-1} e_1 \notin (z_0 e_1 + v_1 e_2, z_0 e_1 + v_2 e_2)_{\text{ID}} \setminus S = \emptyset.$$

$$\Rightarrow \xi = z_0 e_1 + w_0 e_2 \in (z_0 e_1 + v_1 e_2, z_0 e_1 + v_2 e_2)_{\text{ID}} \subset S^c.$$

**Case 2:** When $z_0^{-1} = w_0$ and $z_0 \notin (0, 1)$.

Two sub cases will arise as follows:

Either $z_0 \leq 0$ or $z_0 \geq 1$.

**Sub-case 2 (i):** If $z_0 \leq 0$.

If $z_0 = 0$. Then $\xi = z_0 e_1 + w_0 e_2$ does not exist. Therefore we can say that $z_0 \neq 0$.

If $z_0 < 0$, then as $z_0 (\neq 0) \in A_1$ and as $A_1$ is a Hausdorff space under dictionary order topology.

Therefore, there exist $a_1, a_2 \in A_1$ such that

$$0 \in (a_1, a_2)$$

and similarly there exist

$$w_1, w_2 \in A_1 \text{ such that } z_0 \in (w_1, w_2).$$

and

$$(a_1, a_2) \cap (w_1, w_2) = \emptyset.$$

Also,

$$(0, 1) \cap (w_1, w_2) = \emptyset.$$

Therefore, $w_1 < 0$ as well as $w_2 < 0$. 

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Hence, for any \( b_1, b_2 \in A_2 \) we have
\[
z_0 e_1 + w_0 e_2 \in (w_1 e_1 + b_1 e_2, w_1 e_1 + b_2 e_2)_{ID} \subset S^c
\]

**Sub-case 2(ii):** If \( z_0 \geq 1 \).

Either \( z_0 = 1 \) or \( z_0 > 1 \).

If \( z_0 = 1 \), then \( \xi = 1 \).

So there exists two bicomplex numbers \( \eta = e_1 + \left(\frac{1}{2}\right)e_2 \) and \( \zeta = e_1 + \left(\frac{3}{2}\right)e_2 \)

such that \( \xi \in (\eta, \zeta)_{ID} \subset S^c \).

If \( z_0 > 1 \), then \( z_0^{-1}(= w_0) < 1 \).

Since \( z_0 (\neq 1) \in A_1 \) and as \( w_0 \) and 1 are two points of \( A_2 \),

Also \( A_2 \) is Hausdorff under dictionary order topology.

\[ \Rightarrow \exists u_1, u_2 \in A_2 \text{ such that } w_0 \in (u_1, u_2) \]

and similarly, \( \exists v_1, v_2 \in A_2 \) such that
\[ 1 \in (v_1, v_2) \]

and
\[ (u_1, u_2) \cap (v_1, v_2) = \phi. \]

Therefore,
\[
z_0 e_1 + w_0 e_2 \in (z_0 e_1 + u_1 e_2, z_0 e_1 + u_2 e_2)_{ID}
\]

and
\[
(z_0 e_1 + u_1 e_2, z_0 e_1 + u_2 e_2)_{ID} \cap S \neq \phi
\]

\[ \Rightarrow (z_0 e_1 + u_1 e_2, z_0 e_1 + u_2 e_2)_{ID} \subset S^c. \]

**Case 3:** When \( z_0^{-1} \neq w_0 \) and \( z_0 \notin (0, 1) \).

As \( z_0^{-1} \neq w_0 \) and \( z_0^{-1} \), \( w_0 \in A_2 \) and \( A_2 \) is Hausdorff space in the dictionary order topology.

Therefore, there exist \( c_1, c_2 \in A_2 \) such that
\[
z_0^{-1} \in (c_1, c_2)
\]

Similarly there exist \( d_1, d_2 \in A_2 \) such that
\[
w_0 \in (d_1, d_2) \text{ and } (c_1, c_2) \cap (d_1, d_2) = \phi.
\]

Therefore we have obtained an interval
\[
(z_0 e_1 + d_1 e_2, z_0 e_1 + d_2 e_2)_{ID}
\]
such that

\[ z_0 e_1 + w_0 e_2 \in (z_0 e_1 + d_1 e_2, z_0 e_1 + d_2 e_2)_{id} \]

and

\[ (z_0 e_1 + d_1 e_2, z_0 e_1 + d_2 e_2)_{id} \subseteq S^c. \]

We conclude that set \( S \) is closed as well as bounded.

Hence, \( S \) is compact subset of \( C_2 \).

**Theorem 2.2:** The set

\[ S = \{(z, \sin z^{-1}) : 0 < z < 1\} \]

is a compact subset of \( C_2 \).

**Proof:** We have \( S = \{(z, \sin z^{-1}) : 0 < z < 1\} \)

\[ \Rightarrow S = \{(z, \sin w) : 0 < z < 1, w = z^{-1}\} \]

\[ S = \{ze_1 + (\sin w)e_2 : 0 < z < 1, w = z^{-1}\}. \]

\[ \Rightarrow \bar{z} = z_0 e_1 + w_0 e_2 \in (z_0 e_1 + v_1 e_2, z_0 e_1 + v_2 e_2)_{id} \subseteq S^c \]

To prove \( S \) is a compact subset of \( C_2 \), we show that \( S \) is closed and bounded.

Since, \( 0 < z < 1 \),

\[ 0 < z \text{ as well as } z < 1. \]

Therefore, \( z \) is bounded with lower and upper bounds \( 0 \) and \( 1 \), respectively.

Hence, \( S \) is bounded subset of \( C_2 \) under the idempotent order relation with lower and upper bounds \( w_1 e_2 \) and \( e_1 + w_2 e_2 \), respectively.

Now to show \( S \) is closed in \( C_2 \).

Let \( \bar{z} = u e_1 + v e_2 \in S^c \).

Then there are three possibilities as follows:

(i) \( u \not\in (0, 1) \) and \( v = \sin u^{-1} \)

(ii) \( u \in (0, 1) \) and \( v \neq \sin u^{-1} \)

(iii) \( u \not\in (0, 1) \) and \( v \neq \sin u^{-1} \).

**Case (i):** If \( u \not\in (0, 1) \) and \( v = \sin u^{-1} \)

Now as \( u \not\in (0, 1) \).
Therefore, either \( u \leq 0 \) or \( 1 \leq u \).

If \( u = 0 \), then \( \sin u^{-1} \) is not defined. So that, \( u \neq 0 \).

Now suppose that \( u < 0 \).

Since \( u \) and \( 0 \) are two distinct points of \( A_1 \) and \( A_1 \) is Hausdorff space with respect to the dictionary order topology.
Therefore, there exists \( u_1, u_2 \in A \) such that
\[
u \in (u_1, u_2).
\]
Similarly there exists \( z_1, z_2 \in A \) such that
\[
0 \in (z_1, z_2)
\]
and also \((u_1, u_2) \cap (z_1, z_2) = \phi\). So that
\[
ue_1 + ve_2 \in (u_1 e_1 + v_1 e_2, u_2 e_1 + v_2 e_2)_{ID}
\]
and
\[
(u_1 e_1 + v_1 e_2, u_2 e_1 + v_2 e_2)_{ID} \cap S = \phi.
\]
Therefore, \( S^c \) is an open set. Hence \( S \) is a closed set.
Similarly we can prove that \( S \) is closed if \( 1 \leq u \).

**Case b):** If \( u \in (0 + i_1 0, 1 + i_1 0) \) and \( v \neq \sin u^{-1} \).

Then \( v \) and \( \sin u^{-1} \) are two distinct points of \( A_2 \) and \( A_2 \) is Hausdorff space with respect to the dictionary order topology.

Therefore there exists \( v_1, v_2 \in A_2 \) such that
\[
v \in (v_1, v_2).
\]
Similarly, there exists \( w_1, w_2 \in A_2 \) such that \( \sin u^{-1} \in (w_1, w_2) \)
and
\[
(v_1, v_2) \cap (w_1, w_2) = \phi.
\]
Therefore,
\[
u e_1 + v e_2 \in (u e_1 + v_1 e_2, u e_1 + v_2 e_2)_{ID}
\]
and
\[
(u e_1 + v_1 e_2, u e_1 + v_2 e_2)_{ID} \cap S = \phi
\]
\[
\Rightarrow \ u e_1 + v e_2
\]
\[
\in (u e_1 + v_1 e_2, u e_1 + v_2 e_2)_{ID} \subset S^c
\]
So that \( S^c \) is an open set.
Hence, \( S \) is a closed set.

**Case c):** If \( u \notin (0 + i_1 0, 1 + i_1 0) \) and \( v \neq \sin u^{-1} \).

By the similar procedure as in case (a), we have \( S \) is a closed subset of \( C_2 \).

Hence we conclude that \( S \) is a closed and bounded subset of \( C_2 \).

Therefore, \( S \) is a compact subset of \( C_2 \).
References