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# Optimal Discounted Portfolio, Expected Wealth and Strategic Consumption for a Defined Contribution Pension Scheme

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**Abstract** - This paper deals with optimal discounted portfolio, expected wealth and strategic consumption process for a defined contribution (DC) pension scheme. The aims of this paper are to find: the optimal discounted portfolio and optimal discounted consumption choice to be adopted by the pension plan member (PPM) up to retirement period; the expected discounted wealth and variance of the expected wealth for the plan member. The financial market is composed by a riskless and a risky assets, and the effective salary of the plan member is assume to be stochastic. The expected discounted wealth and its variance are obtained. The discounted portfolio and consumption processes of the plan member are obtained. It is find that part of the discounted portfolio value is proportional to the ratio of the present value of the discounted expected future contributions to the optimal discounted wealth value overtime. It is also find that there is the need for gradual transfer of part of the portfolio value in risky asset to the riskless one against unforeseen shocks.

**Keywords** : *Strategic discounted consumption, optimal discounted portfolio, pension plan member, unforeseen shocks, stochastic salary, defined contribution, pension scheme.*

**GJSFR-F Classification** : *MSC 2010: 91G10*



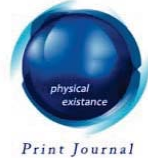
OPTIMAL DISCOUNTED PORTFOLIO, EXPECTED WEALTH AND STRATEGIC CONSUMPTION FOR A DEFINED CONTRIBUTION PENSION SCHEME

*Strictly as per the compliance and regulations of :*



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Ref

2. Blanchard, O. J. and Fischer, S. (1989), "Lectures on Macroeconomics". The MIT press.

# Optimal Discounted Portfolio, Expected Wealth and Strategic Consumption for a Defined Contribution Pension Scheme

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## I. INTRODUCTION

We consider optimal discounted portfolio and discounted consumption problem in a defined contributory pension scheme. This paper follows the work of [14] and [16]. In their paper, they considered the optimal portfolio and strategic consumption in the lifecycle of a PPM in pension scheme. But, the area of discounted portfolio, discounted consumption and the variance of expected discounted wealth of a PPM are yet to be considered to the best of my knowledge in the literature of financial mathematics and actuarial science. In this paper, we assume that the PPM consume continuously throughout his or her life time and consumption terminates when the individual dies.

In the literature, [9] considered a numerical solution as well as analytical results to the intertemporal consumption problem for portfolio management. [17] examined a tractable model of precautionary savings in continuous time and assumed that the uncertainty was about the timing of the income loss in addition to the assumption of non-stochastic asset return. [3] considered labour supply flexibility and portfolio choice of individual life cycle. They considered the objective of maximizing the expected discounted lifetime utility by assumed that the utility function has two argument (consumption and labour/leiture). [2] adopted the quadratic utility function that characterized a linear marginal utility function. They asserted that the utility function was not attractive in describing the behaviour of individual towards risk as it implies increasing absolute risk

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aversion. [3] concluded that labour income induced the individual to invest an additional amount of wealth to the risky asset. They established that labour income and investment choices are related, but they failed to analyze the optimal consumption process of the investor. [4] studied the generally the optimal management of a defined contribution pension plan where the guarantee depends on the level of interest rates at a fixed retirement date. [5] considered optimal dynamic asset allocation strategy for a DC pension plan by taking into account the stochastic features of the plan member's lifetime salary progression and the stochastic properties of the assets held in accumulating pension fund. They emphasized that salary risk (the fluctuation in the plan member's earning in response to economic shocks) is not fully hedgeable using existing financial assets. They further emphasized that wage-indexed bonds could be used to hedge productivity and inflation shocks. They further asserted that such bonds are not widely traded. They referred the optimal dynamic asset allocation strategy stochastic lifestyling. They compared it against various static and deterministic lifestyle strategies in order to calculate the costs of adopting suboptimal strategies. Their solution technique made use of the present value of future contribution premiums into the plan. This technique can be found in [1], [4], [8]. Deterministic lifestyling which is the gradual switch from equities to bonds according to present rules is a popular asset allocation strategy during the accumulation phase of DC pension plans and is designed to protect the pension fund from a catastrophic fall in the stock market just prior to retirement (see [1], [5], [6]). [5] and [9] analysed extensively the occupational DC pension funds, where the contribution rate is a fixed percentage of salary.

The classical dynamic lifetime portfolio selection in a continuous time model was developed by [10], [11]. [9] used a dynamic programming approach to derive a formula for optimal investment allocation in a DC scheme and compared three risk measures to analyze the terminal net replacement ratio achieved by members. They suggested that when the choice of investment strategy is determined, risk profiles of individual and different risk measures are both important factors which should be taken into consideration.

In this paper, we aim at finding the optimal discounted portfolio and optimal discounted consumption choice to be adopted by the pension plan member (PPM) up to retirement period and the entire lifetime; and the expected discounted wealth and variance of the expected discounted wealth for the plan member.

The structure of the remainder parts of the paper is as follows. Section 2 presents the formulation of the problem which include the financial models, wealth process and stochastic salary of a PPM. In section 3, we present the discounted wealth and discounted consumption process of a PPM. Section 4 presents the present value of discounted future contribution of a PPM. In section 5, we present the valuation of the discounted wealth process of a PPM. Section 6 presents the optimal discounted portfolio and consumption process of a PPM. In section 7, we present the optimal expected discounted wealth valuation and discounted consumption process of a PPM. Section 8 presents the accumulated expected discounted consumption process of a PPM. Finally, section 9 concludes the paper.

## II. PROBLEM FORMULATION

We consider a continuous-time financial market where there are two investment instruments: a riskless and a risky assets. The price dynamics of the two assets are given, respectively, by

Ref

3. Bodie, Z., Merton, R. C., and Samuelson, W. F., (1992) "Labour supply flexibility and portfolio choice in a life cycle model". *Journal of Economic Dynamics and Control*, 16, 427-449.

$$dB(t) = rB(t)dt, \quad B(0) = 1, \tag{1}$$

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \quad S(0) = s_0, \tag{2}$$

$t \in \mathfrak{R}$ . We allow  $r > 0, \mu > r$ , to be constants.  $\mu$  is the predictable 1-dimensional process of excess appreciation rate in relation to the stock,  $\sigma = (\sigma_1, \sigma_2)$  is a predictable process of volatility vector of stock,  $\sigma_1$  is volatility of stock arising from inflation and  $\sigma_2$  is the volatility stock arising from the stock market.  $(W(t) = (W_1(t), W_2(t))'; t \in \mathfrak{R}_+)$  is a standard 2-dimensional Brownian motion on a filtered probability space  $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \in [0, T]}, P)$  with  $W(0)$  a null vector almost surely.  $W_1(t)$  is the source of risk of inflation and  $W_2(t)$  is the source of risk of the stock market. We assume that the filtration  $\{\mathfrak{F}_t\}_{t \in [0, T]}$  is generated by the Brownian motion and is right continuous, and that each  $\mathfrak{F}_t$  contains all the  $P$ -null sets of  $\mathfrak{F}$ . We denote by  $L^2_{\mathfrak{F}}$  the set of square integrable  $\{\mathfrak{F}_t\}_{t \in [0, T]}$ -adapted processes,

$$L^2_{\mathfrak{F}} = \left\{ \begin{array}{l} \Delta \mid \text{The process } \Delta = \{\Delta(t)\}_{t \in [0, T]} \\ \text{adapted process such that} \\ \int_0^T E[\Delta^2(t)]dt < \infty \end{array} \right\}$$

and by  $L^2_{\mathfrak{F}_T}$  the set of square integrable  $\mathfrak{F}_T$ -measurable random variables,

$$L^2_{\mathfrak{F}_T} = \left\{ \Delta \mid \begin{array}{l} \Delta \text{ is an } F_T \text{-measurable random} \\ \text{variable such that } E[\Delta^2] < \infty \end{array} \right\}$$

.From (1), we have

$$B(t) = \exp(rt), \quad B(0) = 1; \tag{3}$$

We assume that the financial market is arbitrage-free, complete and continuously open between time 0 and  $T$ , i.e., there is only one process  $\theta$  satisfying

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

where,  $\theta_1$  is the market price of inflation risk and and

$$\theta_2 = \frac{\mu - r - \sigma_1 \theta_1}{\sigma_2},$$

is the market price of stock risk.

The exponential process

$$Z(t) = \exp \left[ -\theta'W(t) - \frac{1}{2} \|\theta\|^2 t \right], \tag{4}$$

is assumed to be a martingale

From (1) and (4), we have the discounted factor to be

$$\Lambda(t) = B(t)^{-1} Z(t) \tag{5}$$

Taking the differential (5), we have

$$d\Lambda(t) = -\Lambda(t)(rdt + \theta' dW(t)) \tag{6}$$

The dynamics of the PPM effective salary is given by

$$\begin{aligned} dY(t) &= Y(t)(\omega dt + \sigma' dW(t)), \\ Y(0) &= y > 0, \end{aligned} \tag{7}$$

where  $Y(t)$  is the salary of the PPM at time  $t$ ,  $\omega$  is the expected growth rate of salary of PPM and  $\sigma' = (\sigma_{Y1}, \sigma_{Y2})$  is the volatility vector of salary which is driven by the source of uncertainty of inflation,  $W_1(t)$  and the stock market,  $W_2(t)$ . The two sources of risk are partial correlated. We assume that  $\omega > 0$  is a constant and  $\sigma'$  is a constant vector.

### III. DISCOUNTED WEALTH AND DISCOUNTED CONSUMPTION PROCESS OF A PPM

In this section, we consider the discounted wealth process of the PPM in pension scheme.

Let  $\Delta(t)$  portfolio process invested in risky asset at time  $t$  and  $C(t)$  the consumption process at time  $t$ , then the pair  $(\Delta, C)$  is said to be self-financing if the corresponding wealth process  $X^{\Delta, C}(t), t \in [0, T]$ , satisfies

$$dX^{\Delta, C}(t) = \Delta(t)X^{\Delta, C}(t) \frac{dS(t)}{S(t)} + (1 - \Delta(t))X^{\Delta, C}(t) \frac{dB(t)}{B(t)} + (cY(t) - C(t))dt, \tag{8}$$

where  $\Delta_0 = 1 - \Delta(t)$  represents the proportion of the portfolio invested in cash account at time  $t$ .

Substituting the assets returns in (1) and (2) into (8), we obtain the following

$$dX^{\Delta, C}(t) = (X^{\Delta, C}(t)(r + \Delta(t)(\mu - r)) + cY(t) - C(t))dt + \sigma\Delta(t)X^{\Delta, C}(t)dW(t) \tag{9}$$

This is our stochastic differential equation which represents the wealth process of the PPM at time  $t$ .

Using (6) and (9), we have the discounted wealth process to be

$$d(\Lambda(t)X^{\Delta, C}(t)) = (c\Lambda(t)Y(t) - \Lambda(t)C(t))dt + \Lambda(t)X^{\Delta, C}(t)(\Delta(t)\sigma' - \theta)'dW(t) \tag{10}$$

### IV. PRESENT VALUE OF DISCOUNTED FUTURE CONTRIBUTION

In this section, we present the present value of pension plan member future contribution in pension scheme.

Definition 1: The present value of the expected discounted future contributions process is defined as

$$\tilde{\Phi}(t) = E_t \left[ \int_t^T \frac{\Lambda(s)}{\Lambda(t)} cY(s) ds \right], \tag{11}$$

where  $c > 0$  is the proportion of the salary contributed into the pension funds and  $E_t$  is the conditional expectation with respect to the Brownian filtration  $\{\mathfrak{F}(t)\}_{t \geq 0}$

$$\Lambda(t) \equiv Z(t) \exp[-rt], 0 \leq t \leq T. \tag{12}$$

is the stochastic discount factor which adjusts for nominal interest rate and market price of risk.

Definition 2: The expected discounted future consumption process is defined by

$$\Psi(t) = E \left[ \int_t^\infty \frac{\Lambda(u)}{\Lambda(t)} C(u) du \right], t \in [T, \infty). \tag{13}$$

We take limits of integration in Eq. (13) from  $t = T$  to  $t = \infty$  because, we assume that consumption starts when the PPM retired and consume his/her investment (benefits) till he/she is dead.

Theorem 1: Let  $\tilde{\Phi}(t)$  be the present value of expected future contributions into the pension funds, then

$$\tilde{\Phi}(t) = \frac{cY(t)}{\varphi} (\exp[\varphi(T-t)] - 1) \tag{14}$$

where  $\varphi = \omega - r - \sigma^I \theta$

Proof: (see Nwozo and Nkeki (2011)).

Find the differential of bothsides of Eq. (14), we obtain

$$d\tilde{\Phi}(t) = \tilde{\Phi}(t) \left[ (r + \sigma^I \theta) dt + \sigma^I dW(t) \right] - cY(t) dt \tag{15}$$

Using (6) and (15), we have

$$d(\Lambda(t)\tilde{\Phi}(t)) = -c\Lambda(t)Y(t)dt + \Lambda(t)\tilde{\Phi}(t)(\sigma^I - \theta)' dW(t) \tag{16}$$

## V. VALUATION OF THE DISCOUNTED WEALTH PROCESS OF A PPM

Definition 3: The value of wealth of the PPM is define as

$$V(t) = X(t) + \tilde{\Phi}(t). \tag{17}$$

So that the discounted value of wealth becomes

$$\Lambda(t)V(t) = \Lambda(t)X(t) + \Lambda(t)\tilde{\Phi}(t).$$

Then, dynamics of the value of the discounted wealth of the PPM is obtain as

$$d(\Lambda(t)V(t)) = d(\Lambda(t)X(t)) + d(\Lambda(t)\tilde{\Phi}(t)). \tag{18}$$

Substituting in (10) and (16) into (18), we obtain the change in wealth of the PPM as follows:

$$d(\Lambda(t)V(t)) = -\Lambda(t)C(t)dt + (\Lambda(t)X^{\Delta,C}(t)(\Delta(t)\sigma^I - \theta) + \Lambda(t)\tilde{\Phi}(t)(\sigma^I - \theta))' dW(t) \tag{19}$$



## VI. OPTIMAL PORTFOLIO AND CONSUMPTION PROCESS OF A PPM

In this section, we derived the optimal discounted portfolio and discounted consumption process of a PPM under dynamic programming principle. We assume that the PPM chooses power utility function in maximizing the expected utility of the terminal discounted wealth and consumption process. The choice of the power utility (which is a linear combination of two power utility functions. The first term is with respect to the wealth process while the second is with respect the consumption process) is motivated by the fact that pension scheme are in general large investment companies whose strategic plans are with respect to the size of funds they are managing.

We now give the optimal discounted portfolio and discounted consumption process for pension funds planners at time  $t$ .

The PPM's problem is to choose an admissible strategy so as to maximize the expected utility of accumulative consumptions and terminal discounted wealth,

$$U(t, V) = \sup_{\substack{\Delta \in \Pi(\cdot, \Phi) \\ C \in \mathfrak{R}}} E_t^{x, \Phi} J(t, x, \Phi, \tilde{C})$$

where

$$J(t, \cdot, \Phi, \tilde{C}) = \int_t^T e^{-\rho u} U(u, \tilde{C}(u)) du + U(V(T)),$$

We assume that  $U$  is concave and

$U((V, t) \in C^{1,2}(\mathfrak{R} \times [0, T])$ ,  $\rho$  is the PPM's preference discount factor,  $\Pi(x, \Phi)$  set of admissible portfolio strategy that are  $F_t$ - progressively measurable, that satisfy the integrability condition

$$E \left[ \int_0^T \Delta(s)^2 ds \right] < \infty, E \left[ \int_0^T \tilde{C}(s)^2 ds \right] < \infty,$$

$E_0^{x, \Phi}$  denotes the conditional expectation at time  $t$  given the initial endowment,

$$\Lambda(t) X^{\Delta, C}(t) = x, \Lambda(t) \tilde{\Phi}(t) = \Phi,$$

$$\Lambda(t) C(t) = \tilde{C}(t),$$

and the utility function is taken as

$$U(V) = \frac{V^\gamma}{\gamma}, 0 < \gamma < 1.$$

It turns out that  $U(t, V)$  satisfies the following Hamilton–Jacobi–Bellman equation

$$U_t + \frac{1}{2} (\Delta(t)\sigma' - \theta)' (\Delta(t)\sigma' - \theta) U_{xx} + \frac{1}{2} \Phi^2 (\sigma^I' - \theta)' (\sigma^I' - \theta) U_{\Phi\Phi} - \Phi (\sigma^I' - \theta)' (\Delta(t)\sigma' - \theta) U_{x\Phi} - \tilde{C}(t) U_x + U(\tilde{C}(t)) \exp(-\rho t) = 0. \tag{20}$$

This yields the HJB equation for the value function

$$(U_t V)_t + H(t, \Delta, C) = 0,$$

Subject to:

$$U(V) = \frac{(x + \Phi)^\gamma}{\gamma} - \frac{\tilde{C}(t)^\gamma}{\gamma}, 0 < \gamma < 1. \tag{21}$$

where

$$\begin{aligned} H(t, \Delta, \tilde{C}) = & \frac{1}{2} x^2 (\Delta(t) \sigma' - \theta)' (\Delta(t) \sigma' - \theta) U_{xx} \\ & + \frac{1}{2} \Phi^2 (\sigma^{I'} - \theta)' (\sigma^{I'} - \theta) U_{\Phi\Phi} \\ & + x \Phi (\sigma^{I'} - \theta)' (\Delta(t) \sigma' - \theta) U_{x\Phi} \\ & - \tilde{C}(t) U_x + U(\tilde{C}(t)) \exp(-\rho t) \end{aligned}$$

Hence, we have the following optimal discounted portfolio and optimal discounted consumption, respectively as follows:

$$\Delta^*(t) = \frac{\sigma^{-1} \theta}{x} - \frac{\Phi \sigma^{-1} (\sigma^{I'} - \theta) U_{x\Phi}}{x U_{xx}}, \tag{22}$$

$$\tilde{C}^*(t) = I(U_x \exp[\rho t]) \tag{29}$$

where,  $I = \left( \frac{dU(\tilde{C})}{d\tilde{C}} \right)^{-1}$ .

Substituting Eq. (22), and (29) into Eq.(20), we have

$$\begin{aligned} & U_t + \frac{1}{2} \theta' \theta (1 - x^2) U_{xx} - \\ & \frac{1}{2} \Phi^2 (\sigma^{I'} - \theta)' (\sigma^{I'} - \theta) \frac{U_{x\Phi}^2}{U_{xx}} \\ & + \frac{1}{2} \Phi^2 (\sigma^{I'} - \theta)' (\sigma^{I'} - \theta) U_{\Phi\Phi} \\ & - I(U_x \exp(\rho t)) U_x + \\ & \exp(-\rho t) U(I(U_x \exp(\rho t))) = 0, \end{aligned} \tag{36}$$



We assume that in this paper that

$$U(t, V) = \frac{(x + \Phi)^\gamma Q(t)^\gamma}{\gamma} \tag{23}$$

and

$$U(t, \tilde{C}) = \frac{(\tilde{C}(t)Q(t))^\gamma}{\gamma}.$$

Finding the partial derivative of (23) with respect to  $t, x, xx, x\Phi, \Phi\Phi$ , we have the following:

$$U_t = (x + \Phi)^\gamma Q(t)^{\gamma-1} Q'(t)$$

$$U_x = (x + \Phi)^{\gamma-1} Q(t)^\gamma$$

$$U_\Phi = (x + \Phi)^{\gamma-1} Q(t)^\gamma$$

$$U_{xx} = (\gamma - 1)(x + \Phi)^{\gamma-2} Q(t)^\gamma$$

$$U_{x\Phi} = (\gamma - 1)(x + \Phi)^{\gamma-2} Q(t)^\gamma$$

$$U_{\Phi\Phi} = (\gamma - 1)(x + \Phi)^{\gamma-2} Q(t)^\gamma$$

We observe that the assumption of concavity of  $U$  turns out to be true, as

$$U_{xx} = U_{\Phi\Phi} = U_{x\Phi} = (\gamma - 1)Q(t)^{\gamma-2} < 0 \text{ since, } \gamma < 1.$$

Substituting the partial derivatives into (36) we have

$$Q'(t) + \frac{1}{2V^2} \theta' \theta (1 - x^2) (\gamma - 1) Q(t) + \frac{1 - \gamma}{\gamma W} \left[ \left( Q(t)^{\frac{\gamma^2}{\gamma-1}} \exp \left[ \frac{\rho t}{\gamma - 1} \right] \right) \right] = 0 \tag{24}$$

(24) can be solve numerically.

Suppose that

$$U(t, v_0) = \frac{(x_0 + \Phi_0)^\gamma Q(t)^\gamma}{\gamma} \tag{25}$$

Then (25) becomes

$$Q'(t) + \frac{1}{2v_0^2} \theta' \theta (1 - x_0^2) (\gamma - 1) Q(t) + \frac{1 - \gamma}{\gamma v_0} \left[ \left( Q(t)^{\frac{\gamma^2}{\gamma-1}} \exp \left[ \frac{\rho t}{\gamma - 1} \right] \right) \right] = 0 \tag{26}$$

Solving (26), we have

$$Q(t) = \tilde{Q}(t)^{\frac{1-\gamma}{1-\gamma+\gamma^2}},$$

where,

$$\begin{aligned} \tilde{Q}(t) = & \frac{2v_0 e^{\frac{\rho t}{\gamma-1}} ((\gamma-1) - 2\gamma(\gamma-1))}{\psi\gamma(\gamma-1)} + \\ & \frac{2v_0 e^{\frac{\rho t}{\gamma-1}} (\gamma^2(\gamma-1) + \gamma^2 + 2\gamma^3 + \gamma^4)}{\psi\gamma(\gamma-1)} + \\ & \frac{2v_0 e^{\frac{\rho t}{\gamma-1} + \frac{r(T-t)}{\gamma-1}} ((\gamma-1)(1 + (\gamma-1)\gamma))}{\psi\gamma} \times \\ & e^{\frac{1}{1-\gamma} \left( r - \rho + \frac{(\alpha_0^2 - 1)(1 + (\gamma-1)\gamma)(1-\gamma)\theta'\theta}{2v_0^2} \right) (T-t)} \end{aligned}$$

and

$$\psi = (\alpha_0^2 - 1)(\gamma - 1)(1 + (\gamma - 1)\gamma)\theta'\theta + 2v_0^2\rho.$$

Therefore, the optimal discounted portfolio for the PPM at time  $t$  is obtained as

$$\begin{aligned} \Delta^*(t) &= \frac{\sigma^{-1}\theta(\Lambda(t)X^*(t) + \Lambda(t)\Phi(t))}{\Lambda(t)X^*(t)} - \frac{\sigma^{-1}\sigma^{I'}\Phi(t)}{X^*(t)}, \\ \Delta_0 &= 1 + \frac{\sigma^{-1}\sigma^{I'}\Phi(t)}{X^*(t)} - \frac{\sigma^{-1}\theta(\Lambda(t)X^*(t) + \Lambda(t)\Phi(t))}{\Lambda(t)X^*(t)}, \end{aligned} \tag{27}$$

the optimal discounted consumption process  $C^*(t)$  of the PPM at time  $t$ , given that (25) holds, is obtained

$$\begin{aligned} \tilde{C}^*(t) &= (\alpha_0 + \Phi_0)(g(t) + h(t) \times \\ & e^{\frac{1}{1-\gamma} \left( r - \rho + \frac{(\alpha_0^2 - 1)(1 + (\gamma-1)\gamma)(1-\gamma)\theta'\theta}{2v_0^2} \right) (T-t)}), \end{aligned} \tag{28}$$

where

$$g(t) = \frac{2v_0 e^{\frac{\rho t}{\gamma-1}} ((\gamma-1) - 2\gamma(\gamma-1))}{\psi\gamma(\gamma-1)} + \frac{2v_0 e^{\frac{\rho t}{\gamma-1}} (\gamma^2(\gamma-1) + \gamma^2 + 2\gamma^3 + \gamma^4)}{\psi\gamma(\gamma-1)}$$

and

$$h(t) = \frac{2v_0 e^{\frac{\rho t}{\gamma-1} + \frac{r(T-t)}{\gamma-1}} ((\gamma-1)(1+(\gamma-1)\gamma))}{\psi\gamma}$$

At time  $t = 0$ , (28) becomes

$$\tilde{C}^*(0) = (x_0 + \Phi_0)(g(0) + h(0) \times e^{\frac{1}{1-\gamma} (r-\rho + \frac{(\gamma^2-1)(1+(\gamma-1)\gamma)(1-\gamma)\theta'\theta}{2v_0^2})T}) \tag{29}$$

where

$$g(0) = \frac{2v_0((\gamma-1) - 2\gamma(\gamma-1))}{\psi\gamma(\gamma-1)} + \frac{2v_0(\gamma^2(\gamma-1) + \gamma^2 + 2\gamma^3 + \gamma^4)}{\psi\gamma(\gamma-1)},$$

and

$$h(0) = \frac{2v_0 e^{\frac{rT}{\gamma-1}} ((\gamma-1)(1+(\gamma-1)\gamma))}{\psi\gamma}$$

At time  $t = T$ , (28) becomes

$$\tilde{C}^*(T) = (x_0 + \Phi_0)(g(T) + h(T)), \tag{29}$$

where

$$g(T) = \frac{2v_0 e^{\frac{\rho T}{\gamma-1}} ((\gamma-1) - 2\gamma(\gamma-1))}{\psi\gamma(\gamma-1)} + \frac{2v_0 e^{\frac{\rho T}{\gamma-1}} (\gamma^2(\gamma-1) + \gamma^2 + 2\gamma^3 + \gamma^4)}{\psi\gamma(\gamma-1)}$$

and

$$h(T) = \frac{2v_0 e^{\frac{\rho T}{\gamma-1}} ((\gamma-1)(1+(\gamma-1)\gamma))}{\psi\gamma}$$

Now, for  $0 < \gamma < 1$ , we have intuitively that the growth rate (GRC) of the optimal expected discounted consumption is obtain as

$$GRC = \frac{1}{1-\gamma} (r - \rho + \frac{(x_0^2 - 1)(1+(\gamma-1)\gamma)(1-\gamma)\theta'\theta}{2v_0^2}). \tag{30}$$

This is referred to as the Euler equation for the intertemporal maximization of discounted consumption under uncertainty. The positive term  $\theta'\theta$  captures the uncer-

tainty of the financial market. When the financial market is risky, it will induce investors to shift consumption over time. Suppose there is no initial wealth, then (30) becomes

$$GRC = \frac{1}{1-\gamma} \left( r - \rho + \frac{(1 + (\gamma - 1)\gamma)(\gamma - 1)\theta' \theta}{2\Phi_0^2} \right).$$

From (27), the first term is the variational form of the classical portfolio strategy while the last term is an intertemporal hedging strategy that offset any shock to the stochastic salary of the PPM. At  $t = 0$ , we have

$$\Delta^*(0) = \frac{\sigma^{-1}\theta(x_0 + \Phi_0)}{x_0} - \frac{\sigma^{-1}\sigma^{I'}\Phi_0}{x_0},$$

$$\Delta_0 = 1 + \frac{\sigma^{-1}\sigma^{I'}\Phi_0}{x_0} - \frac{\sigma^{-1}\theta(x_0 + \Phi_0)}{x_0}.$$

### VII. OPTIMAL EXPECTED DISCOUNTED WEALTH VALUATION AND DISCOUNTED CONSUMPTION PROCESS OF A PPM

In this section, we present the optimal expected value of wealth of the PPM in pension plan at time  $t$ . From (19), we have that

$$d(\Lambda(t)V(t)) = -\Lambda(t)C(t)dt + \theta(1 - \Lambda(t))V(t) + \Lambda(t)\Phi(t)dW(t) \tag{31}$$

$$d(\Lambda(t)V(t))^2 = (-2\Lambda(t)V(t)\Lambda(t)C(t)dt + \theta'\theta(\Lambda(t)V(t))^2 - \theta'\theta\Lambda(t)V(t) - \theta'\theta\Lambda(t)V(t)\Lambda(t)\Phi(t) + \tag{32}$$

$$\theta'\theta(1 + (\Lambda(t)\Phi(t))^2 + (\Lambda(t)\Phi(t))dt + 2\theta'\Lambda(t)V(t)(1 - \Lambda(t)V(t) + \Lambda(t)\Phi(t))dW(t)$$

$$dE(\Lambda(t)V(t)) = -E(\Lambda(t)C(t))dt \tag{33}$$

$$dE(\Lambda(t)V(t))^2 = E(-2\Lambda(t)V(t)\Lambda(t)C(t) + \theta'\theta(\Lambda(t)V(t))^2 - \theta'\theta\Lambda(t)V(t) - \theta'\theta\Lambda(t)V(t)\Lambda(t)\Phi(t) + \tag{34}$$

$$\theta'\theta(1 + (\Lambda(t)\Phi(t))^2 + (\Lambda(t)\Phi(t))dt$$

Solving, (33) and (34), we have

$$E(\Lambda(t)V(t)) = v_0 - E\left(\int_0^t (\Lambda(s)C(s))ds\right) \tag{35}$$

$$E(\Lambda(t)V(t))^2 = v_0^2 e^{\theta\theta t} + \int_0^t K(s) e^{\theta\theta(t-s)} ds \tag{36}$$

where,

$$\begin{aligned} K(t) &= 2E\left(\Lambda(t)C(t)\left(v_0 - \int_0^t \Lambda(s)C(s)ds\right)\right) + \\ &\theta'\theta E\left(v_0 - \int_0^t \Lambda(s)C(s)ds\right) + \\ &\frac{c\theta\theta e^{\omega t} (e^{\phi(T-t)} - 1)}{\phi} E\left(v_0 - \int_0^t \Lambda(s)C(s)ds\right) + \\ &\theta'\theta \left(1 + \frac{c^2 e^{2\omega t} (e^{\phi(T-t)} - 1)^2}{\phi^2} + \frac{c e^{\omega t} (e^{\phi(T-t)} - 1)}{\phi}\right). \\ \text{Var}(\Lambda(t)V(t)) &= E(\Lambda(t)V(t))^2 - (E(\Lambda(t)V(t)))^2 \\ &= v_0^2 e^{\theta\theta t} + \int_0^t K(s) e^{\theta\theta(t-s)} ds - \\ &\left(v_0 - E\left(\int_0^t (\Lambda(s)C(s)ds)\right)\right)^2 \end{aligned} \tag{37}$$

At  $t = T$ , we have

$$\begin{aligned} E(\Lambda(T)V(T)) &= v_0 - E\left(\int_0^T (\Lambda(s)C(s)ds)\right) \\ \text{Var}(\Lambda(T)V(T)) &= E(\Lambda(T)V(T))^2 - (E(\Lambda(T)V(T)))^2 = \\ &v_0^2 e^{\theta\theta T} + \int_0^T K(s) e^{\theta\theta(T-s)} ds \\ &- \left(v_0 - E\left(\int_0^T (\Lambda(s)C(s)ds)\right)\right)^2. \end{aligned} \tag{38}$$

### VIII. ACCUMULATED EXPECTED DISCOUNTED CONSUMPTION PROCESS OF A PPM

The accumulated expected discounted consumption process of the PPM up to retirement period is given by

$$\begin{aligned} \tilde{\Psi}(T) &= E\left[\int_0^T \tilde{C}(u)du\right] = \int_0^T (x_0 + \Phi_0)(g(t) + h(t)e^{GRC(T-t)})dt \\ &= (x_0 + \Phi_0) \int_0^T g(t)dt + (x_0 + \Phi_0) \int_0^T h(t)e^{GRC(T-t)}dt \end{aligned} \tag{39}$$

Solving (39), we have

$$\begin{aligned} \tilde{\Psi}(T) &= \frac{2v_0^2((\gamma - 1) - 2\gamma(\gamma - 1))}{\psi\gamma\rho} (1 - e^{\frac{\rho T}{\gamma-1}}) \\ &+ \frac{2v_0^2(\gamma^2(\gamma - 1) + \gamma^2 + 2\gamma^3 + \gamma^4)}{\psi\gamma\rho} (1 - e^{\frac{\rho T}{\gamma-1}}) \\ &+ \frac{2v_0^2((1 - \gamma)(1 + (\gamma - 1)\gamma))e^{\frac{(\rho+\alpha)T}{\gamma-1}}}{\alpha\psi\gamma} (1 - e^{\frac{\alpha T}{1-\gamma}}) \end{aligned}$$

where

$$\alpha = \frac{(x_0^2 - 1)(1 + (\gamma - 1)\gamma)(1 - \gamma)\theta'\theta}{2v_0^2}.$$

If we allow  $T$  to tend to infinity, we have

$$\begin{aligned} \lim_{T \rightarrow \infty} E_0 \left[ \int_0^T \tilde{C}(u) du \right] &= E_0 \left[ \int_0^\infty \tilde{C}(u) du \right] = \\ \lim_{T \rightarrow \infty} \tilde{\Psi}(T) &= \tilde{\Psi}_\infty = \frac{2v_0^2((\gamma - 1) - 2\gamma(\gamma - 1))}{\psi\gamma\rho} \\ &+ \frac{2v_0^2(\gamma^2(\gamma - 1) + \gamma^2 + 2\gamma^3 + \gamma^4)}{\psi\gamma\rho}. \end{aligned}$$

This shows that throughout the life-cycle of the PPM, his or her accumulated expected discounted consumption will remain nonempty. If  $\tilde{\Psi}_\infty$  is nonnegative, it implies that the PPM is unable to finished his or her consumption before he or she dies. If  $\tilde{\Psi}_\infty$  is negative, it implies that the PPM is finished his or her consumption why still alive. If  $\tilde{\Psi}_\infty$  is zero, it implies that the period the PPM finishes his or her consumption was the period he or she dies. Interestingly, the nature of  $\tilde{\Psi}_\infty$  absolutely depend on the value of  $\gamma, \rho$ . This implies that the level of risk the PPM takes will depend on how large or how small the value of the accumulated discounted consumption will be and consumption rate. We observe that at  $\rho = 0$  or

$$\rho = \frac{(x_0^2 - 1)(\gamma - 1)(1 + (\gamma - 1)\gamma)\theta'\theta}{2v_0^2},$$

optimal expected discounted consumption will be unbounded. The optimal expected discounted consumption will be negative if

$$\rho < \frac{(x_0^2 - 1)(\gamma - 1)(1 + (\gamma - 1)\gamma)\theta'\theta}{2v_0^2},$$

and non negative if

$$\rho > \frac{(x_0^2 - 1)(\gamma - 1)(1 + (\gamma - 1)\gamma)\theta' \theta}{2v_0^2}.$$

## IX. CONCLUSION

This paper dealt with optimal discounted portfolio, expected wealth and strategic life-time consumption process for a defined contributory pension scheme. The expected discounted wealth and its variance are obtained. The discounted variational classical portfolio and consumption process of the plan member are established. The accumulated discounted consumption process of the PPM throughout his or her life-cycle was established. It was found that part of the discounted portfolio value is proportional to the ratio of the present value of the discounted expected future contributions to the optimal discounted portfolio value overtime. It was found that there is the need for gradual transfer of part of the portfolio value in risky asset to the riskless one against unforeseen shocks.

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