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Lie Algebraic Approach and Complex Invariant Coupled Oscillator Systems

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Abstract - In classical mechanics, the system of coupled harmonic oscillators is shown to possess the symmetry applicable toa six-dimensional space in complex coordinates, two-dimensional phase space consisting of two position and twomomentum variables. In search into the features of a dynamical system, with the possibility of its complex invariant,we explore this dynamical systems. Dynamical algebraic approach is used to study two-dimensional complex systems(coupled oscillator system) on the extended complex phase plane (ECPS). Scope and importance of invariants in theanalysis of complex trajectories for dynamical systems is discussed.

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GJSFR-F Classification : *AMS Classification: 37K10, 03D15, 34C45*

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Jasvinderpal Singh Virdi

Abstract - In classical mechanics, the system of coupled harmonic oscillators is shown to possess the symmetry applicable toa six-dimensional space in complex coordinates, two-dimensional phase space consisting of two position and twomomentum variables. In search into the features of a dynamical system, with the possibility of its complex invariant,we explore this dynamical systems. Dynamical algebraic approach is used to study two-dimensional complex systems(coupled oscillator system) on the extended complex phase plane (ECPS). Scope and importance of invariants in theanalysis of complex trajectories for dynamical systems is discussed.

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I. Introduction

The coupled oscillator provides many soluble models in different branches of physics because of its mathematical simplicity. It stays with us in many different forms because it provides the mathematical basis for many soluble models in physics, including the Lee model in quantum field theory [1], the Bogoliubov transformation in superconductivity, relativistic models of elementary particles [2]. In physics coupled harmonic oscillator system in two-dimensions [3], in the Bells inequality experiments employing coupled harmonic oscillators [4]. This also has been used for description of motion of a charged particle in a magnetic field [5, 6] or . They are also studied in context of electrical circuits with time-varying capacitors and inductors, particularly with reference to their memory property, has become of considerable interest in recent years [7]. Blasone *et al* [8] studied momentum-dependent terms in the Hamiltonian structure in the context of the so-called holographic principle and in the treatment of quantum gravity as a dissipative and deterministic system. Hamiltonian for such system is given by

$$H = \frac{1}{2}[\alpha_1 p_x^2 + \alpha_2 p_y^2 + \beta_1 x^2 + \beta_2 y^2] + \alpha_3(p_x y + x p_y). \qquad (1)$$

Invariants for above Hamiltonian provided they exist and can be computed, and even the complex invariants [9] exist, are a very useful tool to understand the theoretical structure of this dynamical systems. Since invariants of real Hamiltonian systems have been played a vital role in understanding the underlying dynamics of the systems and so we expect that the complex invariants can also be helpful in exploring some deep insights into features of complex dynamical systems. In the past, complex invariants have been discussed in context of understanding fermion masses and quark mixing, and CP-conserving two-Higgs-doublet model scalar potentials in the Particle physics [10, 11].

In this paper, we construct a complex invariants corresponding coupled oscillators based on the ECPS approach in complex domain [12]. Recently, with a view to explore some role of invariants for complex systems, Kaushal et al. [13] found invariants for some one dimensional systems within the framework of an ECPS. Some quantum mechanical studies within the ECPS are also reported [14]. But most of such studies are restricted in one dimension only. Such studies in higher dimensions is desirable from the intrinsic mathematical interest, to check the validity of various methods/theories and to find solutions of some realistic physical problems. With this motivation, recently we generalized the ECPS in two dimensions and studied the coupled oscillator.

Ref

1. T. D. Lee, Some special examples in renormalizable field theory, Phys. Rev. **95**, 1329 (1954).

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We make use of the Lie algebraic approach to derive at least one invariant for the TD versions of above coupled oscillator system. In fact, the Lie algebraic approach (ref therein [15]) commands several advantages over the rationalization method, particularly for the TD systems, not only in terms of the closure property of the Poisson bracket algebra of phase space functions but also for its straightforward extension to the corresponding quantum system.

a) Lie algebraic approach

If we define

$$x = x_1 + ip_3; \quad y = x_2 + ip_4; \quad p_x = p_1 + ix_3; \quad p_y = p_2 + ix_4. \tag{2}$$

then, a two dimensional real phase space (x, y, p_x, p_y, t), may be transformed into the corresponding extended complex phase plane (ECPS) $(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t)$ The above transformations add four additional degrees of freedom, (x_3, x_4, p_3, p_4, t), which can make mathematical analysis of a problem a bit more involved. But nevertheless, these type of transformations are used in many studies [12, 13, 14, 13]. From eq.(2) one can easily obtain one can easily obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}; \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}; \quad \frac{\partial}{\partial p_x} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}; \quad \frac{\partial}{\partial p_y} = \frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}. \tag{3}$$

Now consider a complex phase space function $I(x, y, p_x, p_y, t)$ as

$$I = I_1(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t) + iI_2(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t). \tag{4}$$

Further, the invariance of I implying

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H] = 0, \tag{5}$$

where [., .] is the Poisson bracket, which in view of the definition, eq.(2), turns out to be

$$\begin{aligned} [I, H]_{(x,p)} &= [I, H]_{(x_1,p_1)} - i[I, H]_{(x_1,x_3)} - i[I, H]_{(p_3,p_1)} - [I, H]_{(p_3,x_3)} \\ &+ [I, H]_{(x_2,p_2)} - i[I, H]_{(x_2,x_4)} - i[I, H]_{(p_4,p_2)} - [I, H]_{(p_4,x_4)}. \end{aligned} \tag{6}$$

b) Example

Consider a coupled harmonic oscillator systems in two-dimensions, whose Hamiltonian is given by (1). Using (2), the above Hamiltonian (see [12] for detail method) can be expressed as

$$\begin{aligned} H &= \frac{\alpha_1}{2}p_1^2 - \frac{\alpha_1}{2}x_3^2 + i\alpha_1 p_1 x_3 + \frac{\alpha_1}{2}p_2^2 - \frac{\alpha_1}{2}x_4^2 + i\alpha_1 p_2 x_4 + \frac{\alpha_2}{2}x_1^2 - \frac{\alpha_2}{2}p_3^2 + i\alpha_2 p_3 x_1 p_4 \\ &+ \frac{\alpha_2}{2}x_2^2 - \frac{\alpha_2}{2}p_4^2 + i\alpha_2 x_2 + \alpha_3(p_1x_2 + ip_1p_4 + ix_3x_2 - x_3p_4 + x_1p_2 + ix_1x_4 + ip_2p_3 - p_3x_4) \\ &= \sum_{m=1}^{20} h_m(t)\Gamma_m(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4), \end{aligned} \tag{7}$$

and the various Γ's and h(t)'s for the above complex H are given as

$$\begin{aligned} &\Gamma_1 = \frac{p_1^2}{2}; \ \Gamma_2 = \frac{x_3^2}{2}; \ \Gamma_3 = p_1x_3; \ \Gamma_4 = \frac{p_2^2}{2}; \ \Gamma_5 = \frac{x_4^2}{2}; \ \Gamma_6 = p_2x_4; \ \Gamma_7 = \frac{x_1^2}{2}; \ \Gamma_8 = \frac{p_3^2}{2}; \\ &\Gamma_9 = x_1p_3; \ \Gamma_{10} = \frac{x_2^2}{2}; \ \Gamma_{11} = x_2p_4; \ \Gamma_{12} = \frac{p_4^2}{2}; \ \Gamma_{13} = p_1x_2; \ \Gamma_{14} = p_1p_4; \ \Gamma_{15} = x_3x_2; \\ &\Gamma_{16} = x_3p_4, \ \ \Gamma_{17} = x_1p_2; \ \ \Gamma_{18} = x_1x_4; \ \ \Gamma_{19} = p_3p_2; \ \ \Gamma_{20} = p_3x_4. \end{aligned} \tag{8}$$

With

$$\begin{aligned} &h_1 = h_2 = \alpha_1; \ h_3 = i\alpha_1; \ h_4 = h_5 = \alpha_2; \ h_6 = i\alpha_2; \ h_7 = h_8 = \beta_1; \ h_9 = i\beta_1; \ h_{10} = h_{11} = \beta_2 \\ &h_{12} = i\beta_2; \ h_{14} = h_{15} = i\alpha_3; \ h_{13} = \alpha_3; \ h_{16} = -\alpha_3; \ h_{17} = \alpha_3; \ h_{18} = h_{19} = i\alpha_3; \ h_{20} = -\alpha_3. \end{aligned} \tag{9}$$

Ref

14. R. S. Kaushal, On the quantum mechanics of complex Hamiltonian systems in one dimension, J. Phys. A: Math. Gen. **34** (2001) L709-721; R. S. Kaushal and Parthasarathi, Quantum mechanics of complex Hamiltonian systems in one dimension, J. Phys. A **35**, (2002) 8743-8754.

The dynamical algebra in this case is not closed. To find closure property for the above system, we have to add sixteen more phase space functions (Γ_l)'s. The additional (Γ_l)'s are as follow

$$\Gamma_{21}=p_1p_3;\ \Gamma_{22}=p_1x_1;\ \Gamma_{23}=x_2p_1;\ \Gamma_{24}=p_1p_4,\ \Gamma_{25}=p_3x_3;\ \Gamma_{26}=x_1x_3;\ \Gamma_{27}=x_2x_3;\ \Gamma_{28}=x_3p_4;$$
$$\Gamma_{29}=x_2p_2;\ \Gamma_{30}=p_4p_2;\ \Gamma_{31}=p_2p_3;\ \Gamma_{32}=p_2x_1,\ \Gamma_{33}=x_2x_4;\ \Gamma_{34}=x_4p_4;\ \Gamma_{35}=x_1x_4;\ \Gamma_{36}=p_3x_4, \quad (10)$$

with corresponding $h_l(t)=0$. Now in the light of Poisson bracket (6) for complex systems, we get large number (288 no. of) nonvanishing Poisson brackets (for more detail see [12]). Therefore, their use (5) yields the following set of PDEs in λ's as described in (**??**) section two:

$$\dot\lambda_1=4\alpha_1(i\lambda_{21}-\lambda_{22})-4\alpha_3(\lambda_{23}-i\lambda_{24}), \quad (11)$$

$$\dot\lambda_2=-4\alpha_1(\lambda_{25}+i\lambda_{26})-4\alpha_3(i\lambda_{27}+\lambda_{28}), \quad (12)$$

$$\dot\lambda_3=-2\alpha_1(\lambda_{21}+i\lambda_{22}+i\lambda_{25}-\lambda_{26})-2\alpha_3(i\lambda_{23}+\lambda_{24}+\lambda_{27}-i\lambda_{28}), \quad (13)$$

$$\dot\lambda_4=-4\alpha_2(\lambda_{29}+i\lambda_{30})+4\alpha_3(i\lambda_{31}-\lambda_{32}), \quad (14)$$

$$\dot\lambda_5=-4\alpha_2(\lambda_{34}+i\lambda_{33})-4\alpha_3(\lambda_{36}+i\lambda_{35}), \quad (15)$$

$$\dot\lambda_6=-2\alpha_2(i\lambda_{29}+\lambda_{30}-i\lambda_{34}+\lambda_{33})-2\alpha_3(\lambda_{31}+i\lambda_{32}+\lambda_{35}-i\lambda_{36}), \quad (16)$$

$$\dot\lambda_7=-4\beta_1(-\lambda_{22}+i\lambda_{26})+4\alpha_3(\lambda_{32}-i\lambda_{35}), \quad (17)$$

$$\dot\lambda_8=4\beta_1(i\lambda_{21}+\lambda_{25})+4\alpha_3(\lambda_{36}+i\lambda_{31}), \quad (18)$$

$$\dot\lambda_9=2\beta_1(\lambda_{21}+i\lambda_{22}-i\lambda_{25}+\lambda_{26})+2\alpha_3(\lambda_{31}+i\lambda_{32}+\lambda_{35}-i\lambda_{36}), \quad (19)$$

$$\dot\lambda_{10}=4\alpha_3(\lambda_{23}-i\lambda_{27})-4\beta_2(\lambda_{29}+\lambda_{33}), \quad (20)$$

$$\dot\lambda_{11}=2\alpha_3(i\lambda_{23}+\lambda_{24}+\lambda_{27}-i\lambda_{28})-2\alpha_3(\lambda_{29}+i\lambda_{30}+\lambda_{34}-i\lambda_{33}), \quad (21)$$

$$\dot\lambda_{12}=4\alpha_3(i\lambda_{24}+\lambda_{28})-4\beta_2(\lambda_{30}+\lambda_{34}), \quad (22)$$

$$\dot\lambda_{13}=2\alpha_3(\lambda_{22}-i\lambda_{33}-i\lambda_{26}+\lambda_{29})+2\beta_1(i\lambda_{23}-i\lambda_{27})+2\beta_2(\lambda_{32}-i\lambda_{35}), \quad (23)$$

$$\dot\lambda_{14}=2\alpha_3(\lambda_{26}+\lambda_{30}-i\lambda_{34}+i\lambda_{22})+2\beta_1(\lambda_{24}-i\lambda_{28})+\beta_2(-\lambda_{32}+i\lambda_{35}), \quad (24)$$

$$\dot\lambda_{15}=2\alpha_3(\lambda_{21}-i\lambda_{25}+\lambda_{33}+i\lambda_{29})+2\beta_1(i\lambda_{23}+\lambda_{27})+\beta_2(\lambda_{31}-\lambda_{36}), \quad (25)$$

$$\dot\lambda_{16}=2\beta_2(i\lambda_{21}+\lambda_{25}+i\lambda_{30}+i\lambda_{34})+2\beta_1(i\lambda_{24}+\lambda_{28})-\beta_2(\lambda_{31}+i\lambda_{36}), \quad (26)$$

$$\dot\lambda_{17}=2\alpha_3(i\lambda_{21}-\lambda_{22}-\lambda_{29}+i\lambda_{30})+2\alpha_1(i\lambda_{31}-\lambda_{32})-2\alpha_2(i\lambda_{23}+\lambda_{24}), \quad (27)$$

$$\dot\lambda_{18}=-2\alpha_3(\lambda_{21}+i\lambda_{22}+i\lambda_{29}+\lambda_{30})+2\alpha_1(-\lambda_{31}+i\lambda_{32})+\alpha_2(-i\lambda_{23}+\lambda_{24}), \quad (28)$$

Ref

12. J.S. Virdi, and S.C. Mishra, Exact complex integrals in two dimensions for shifted harmonic oscillators, Pramana J. Phys. **79** (2012) 19-40; J.S. Virdi, F. Chand, C.N. Kumar, and S.C. Mishra, Complex dynamical invariants for two-dimensional complex potentials, Pramana J. Phys. **79**, (2012) 173-183.

$$\dot{\lambda}_{19} = 2\alpha_3(i\lambda_{25} - \lambda_{26} - i\lambda_{29} - \lambda_{30}) - 2\alpha_1(\lambda_{31} + i\lambda_{32}) + 2\alpha_2(-\lambda_{27} + i\lambda_{28}), \quad (29)$$

$$\dot{\lambda}_{20} = -2\alpha_3(\lambda_{25} + i\lambda_{26} + i\lambda_{29} + \lambda_{30}) - 2\alpha_1(\lambda_{31} + i\lambda_{32}) + 2\alpha_2(-i\lambda_{27} - \lambda_{28},) \quad (30)$$

$$\dot{\lambda}_{21} = 2\beta_1(i\lambda_1 + \lambda_3) + 2\alpha_1(-\lambda_9 + i\lambda_8) + 2\alpha_3(-i\lambda_{16} - \lambda_{15}) + 2\alpha_3(i\lambda_{17} + \lambda_{18}), \quad (31)$$

$$\dot{\lambda}_{22} = 2\beta_1(\lambda_1 - i\lambda_3) + 2\alpha_1(i\lambda_9 + \lambda_7) + 2\alpha_3(i\lambda_{14} - \lambda_{13}) + 2\alpha_3(\lambda_{17} - i\lambda_{18}), \quad (32)$$

$$\dot{\lambda}_{23} = 2\beta_2(i\lambda_{17} - i\lambda_{18}) + 2\alpha_1(-\lambda_{13} + i\lambda_{15}) + 2\alpha_3(-i\lambda_3 + \lambda_1) + 2\alpha_3(i\lambda_{11} - \lambda_{10}), \quad (33)$$

$$\dot{\lambda}_{24} = 2\alpha_3(i\lambda_1 + \lambda_3) + 2\alpha_3(-\lambda_{11} + i\lambda_{12}) + 2\alpha_1(-i\lambda_{16} - \lambda_{14}) + 2\beta_2(\lambda_{18} - \lambda_{17}), \quad (34)$$

$$\dot{\lambda}_{25} = 2\beta_1(-\lambda_2 - i\lambda_3) - 2\alpha_1(i\lambda_9 + \lambda_8) + 2\alpha_3(-\lambda_{16} - i\lambda_{15}) + 2\alpha_3(i\lambda_{19} + \lambda_{20}), \quad (35)$$

$$\dot{\lambda}_{26} = 2\beta_1(-i\lambda_2 + \lambda_3) + 2\alpha_1(i\lambda_7 - \lambda_9) + 2\alpha_3(-i\lambda_{13} - \lambda_{14}) + 2\alpha_3(\lambda_{19} - i\lambda_{20}), \quad (36)$$

$$\dot{\lambda}_{27} = 2\alpha_3(-i\lambda_2 + \lambda_3) + 2\alpha_3(-i\lambda_{10} - \lambda_{11}) - 2\alpha_1(i\lambda_{13} + \lambda_{15}) + 2\beta_2(\lambda_{19} + i\lambda_{20}), \quad (37)$$

$$\dot{\lambda}_{28} = 2\alpha_3(\lambda_2 + i\lambda_3) + 2\alpha_3(-i\lambda_{11} - i\lambda_{12}) - 2\alpha_1(\lambda_{14} - i\lambda_{16}) + 2\beta_2(-\lambda_{19} + i\lambda_{20}), \quad (38)$$

$$\dot{\lambda}_{29} = 2\beta_2(\lambda_4 + \lambda_6) + 2\alpha_2(-\lambda_{10} + i\lambda_{11}) + 2\alpha_3(i\lambda_{15} - \lambda_{13}) + 2\alpha_3(\lambda_{17} - i\lambda_{19}), \quad (39)$$

$$\dot{\lambda}_{30} = -2\alpha_3(\lambda_4 + i\lambda_6) + 2\alpha_2(-\lambda_{11} - i\lambda_{12}) + 2\alpha_3(-i\lambda_{16} - i\lambda_{14}) + 2\alpha_3(i\lambda_{17} + \lambda_{19}), \quad (40)$$

$$\dot{\lambda}_{31} = 2\alpha_3(i\lambda_4 + \lambda_6) + 2\alpha_2(i\lambda_8 + \lambda_9) + 2\alpha_2(-i\lambda_{16} - \lambda_{15}) + 2\beta_1(i\lambda_{17} + \lambda_{19}), \quad (41)$$

$$\dot{\lambda}_{32} = 2\alpha_3(\lambda_4 - i\lambda_6) + 2\alpha_3(\lambda_7 + i\lambda_9) + 2\alpha_2(i\lambda_{14} - \lambda_{13}) + 2\beta_1(\lambda_{17} - i\lambda_{19}), \quad (42)$$

$$\dot{\lambda}_{33} = 2\beta_2(\lambda_5 + \lambda_6) + 2\alpha_2(-\lambda_{10} - i\lambda_{11}) + 2\alpha_3(-i\lambda_{13} - \lambda_{15}) + 2\alpha_3(\lambda_{18} - i\lambda_{20}), \quad (43)$$

$$\dot{\lambda}_{34} = 2\beta_2(-\lambda_5 - \lambda_6) + 2\alpha_2(-i\lambda_{11} + \lambda_{12}) + 2\alpha_3(-\lambda_{14} - i\lambda_{16}) + 2\alpha_3(i\lambda_{18} + \lambda_{20}), \quad (44)$$

$$\dot{\lambda}_{35} = 2\beta_1(-\lambda_{18} - \lambda_{20}) + 2\alpha_2(-i\lambda_{13} - \lambda_{14}) + 2\alpha_3(-i\lambda_5 + \lambda_6) + 2\alpha_3(i\lambda_7 - \lambda_9), \quad (45)$$

$$\dot{\lambda}_{36} = 2\alpha_3(\lambda_5 + i\lambda_6) - 2\alpha_3(i\lambda_9 + \lambda_8) + 2\alpha_2(-\lambda_{16} + i\lambda_{15}) + 2\beta_1(i\lambda_{18} + \lambda_{20}). \quad (46)$$

As such the solution to these 36 coupled equation turns out to be difficult. Therefore, we make the following choices about λ's which facilitate to find solutions of above equations.
From eqs.(11), (12) and (13), we get $2\dot{\lambda}_3 = i\dot{\lambda}_1 - i\dot{\lambda}_2$. If we consider $\lambda_3 = c_3$ (a constant), and by taking $\lambda_1 = \lambda_2 = \eta_1(t)$; which immediately gives

$$\lambda_1 = \eta_1(t) + c_1; \qquad \lambda_2 = \eta_1(t) + c_2. \quad (47)$$

From eqs.(14), (15) and (16), we get $2\dot{\lambda}_6 = i\dot{\lambda}_4 - i\dot{\lambda}_5$. If we consider $\lambda_6 = c_6$ (a constant), and by taking $\lambda_4 = \lambda_5 = \eta_2(t)$; which immediately gives

$$\lambda_4 = \eta_2(t) + c_4; \qquad \lambda_5 = \eta_2(t) + c_5. \quad (48)$$

Notes

Again From eqs.(17), (18) and (19), we get $2\dot{\lambda}_9 = i\dot{\lambda}_7 - i\dot{\lambda}_8$. If we set $\lambda_9 = c_9$ (a constant), and consider $\lambda_7 = \lambda_8 = \eta_3(t)$; which immediately gives

$$\lambda_7 = \eta_3(t) + c_7; \qquad \lambda_8 = \eta_3(t) + c_8. \tag{49}$$

From eqs.(20), (21) and (22), we get $2\dot{\lambda}_{11} = i\dot{\lambda}_{10} - i\dot{\lambda}_{12}$. If we set $\lambda_{11} = c_{11}$ (a constant), and consider $\lambda_{12} = \lambda_{10} = \eta_4(t)$; which immediately gives

$$\lambda_{12} = \eta_4(t) + c_{12}; \qquad \lambda_{10} = \eta_4(t) + c_{10}. \tag{50}$$

Now, in order to find solutions for λ_{13}, λ_{14}, λ_{15} and λ_{16} we have to make simplification for complications of above set of 24 eqs (23-46). (i.e. $\alpha_1 = \alpha_2 = \alpha_3$, and $\beta_1 = \beta_2 = \alpha_3$). From eqs.(17), (20) and (23), we get $2\dot{\lambda}_{13} = \dot{\lambda}_7 + i\dot{\lambda}_{10}$. If we consider $\lambda_{13} = c_{13}$ (a constant), and considering the eqn from above relation (with $\lambda_{10} = \eta_4(t) + c_{10},\ \lambda_7 = \eta_3(t) + c_7;$) gives

$$\lambda_{13} = \eta(t) + c_{13}. \tag{51}$$

where $\eta(t) = \frac{1}{2i}[\eta_4(t) + \eta_3(t);\,]$ is an another function of time, and ($c_{13} = c_7 + c_{10}$; a constant.)
From eqs.(18), (20) and (25), we get $2i\dot{\lambda}_{15} = \dot{\lambda}_{10} + i\dot{\lambda}_8$, If we consider $\lambda_{15} = c_{15}$ (a constant), and considering the eqn from above relation (with $\lambda_{10} = \eta_4(t) + c_{10},\ \lambda_8 = \eta_3(t) + c_8;$) gives

$$\lambda_{15} = \eta(t) + c_{15}, \tag{52}$$

where $\eta(t) = \frac{1}{2i}[\eta_4(t) + \eta_3(t)]$ is an another function of time and $c_{15} = c_{10} + c_8$; a constant.
In order to find solutions for λ_{16}, from eqs.(18), (22) and (26), we get $2i\dot{\lambda}_{16} = \dot{\lambda}_8 + \dot{\lambda}_{12}$. If we consider $\lambda_{16} = c_{16}$; (a constant), and considering the relation (with $\lambda_8 = \eta_3(t) + c_8,\ \lambda_{12} = \eta_4(t) + c_{12};$) gives

$$\lambda_{16} = \eta(t) + c_{16}, \tag{53}$$

where $\eta(t) = \frac{1}{2i}[\eta_4(t) + \eta_3(t)]$; and $c_{16} = c_8 + c_{12}$; a constant.
From eqs.(17), (22) and (24), we get $2i\dot{\lambda}_{14} = \dot{\lambda}_7 + \dot{\lambda}_{12}$, If we consider $\lambda_{14} = c_{14}$; (a constant), and considering relation (with $\lambda_7 = \eta_3(t) + c_7,\ \lambda_{12} = \eta_4(t) + c_{12};$) gives

$$\lambda_{14} = \eta(t) + c_{14}, \tag{54}$$

where $\eta(t) = \frac{1}{2i}[\eta_4(t) + \eta_3(t)]$; and $c_{16} = c_7 + c_{12}$ a constant.
Now, to find solutions for λ_{17}, λ_{18}, λ_{19} and λ_{20}, refer from eqs.(11), (14) and (27), we get $2\dot{\lambda}_{17} = \dot{\lambda}_1 + \dot{\lambda}_4$; and considering the relation $\lambda_1 = \eta_1(t) + c_1,\ \ \lambda_4 = \eta_2(t) + c_4$; gives

$$\lambda_{17} = \phi(t) + c_{17}, \tag{55}$$

where $\phi(t) = \frac{1}{2}\int[\dot{\eta}_1(t) + \dot{\eta}_2(t)]dt$; and $c_{17} = c_1 + c_4$, a constant. From eqs.(11), (15) and (28), we get $2\dot{\lambda}_{18} = i\dot{\lambda}_1 - i\dot{\lambda}_5$; and considering the relation $\lambda_1 = \eta_1(t) + c_1,\ \lambda_5 = \eta_2(t) + c_5$; will results

$$\lambda_{18} = \varphi(t) + c_{18}, \tag{56}$$

where $\varphi(t) = \frac{1}{2}\int[i\dot{\eta}_1(t) - i\dot{\eta}_2(t)]dt$; and $c_{18} = c_1 + c_5$, a constant. Similarly to find solutions of λ_{19}, from eqs.(12), (14) and (29), we get $2\dot{\lambda}_{19} = i\dot{\lambda}_2 - i\dot{\lambda}_4$; and considering the relation $\lambda_2 = \eta_1(t) + c_2,\ \ \lambda_4 = \eta_2(t) + c_4$; gives

$$\lambda_{19} = \chi(t) + c_{19}, \tag{57}$$

where $\chi(t) = \frac{1}{2}\int[i\dot{\eta}_1(t) - i\dot{\eta}_2(t)]dt$ and $c_{19} = c_2 + c_4$, a constant.
Again from eqs.(12), (15) and (30), we get $2\dot{\lambda}_{20} = \dot{\lambda}_2 + \dot{\lambda}_4$; and considering the relation $\lambda_2 = \eta_1(t) + c_1,\ \ \lambda_4 = \eta_2(t) + c_4$; gives

$$\lambda_{20} = \quad (t) + c_{20}, \tag{58}$$

where $\quad (t) = \frac{1}{2}\int[\dot{\eta}_1(t) + \dot{\eta}_2(t)]dt$; and $c_{20} = c_2 + c_4$, a constant.
Solutions for $(\lambda_{21} - \lambda_{28})$ can be obtained respectively as from eqs.(31-38), we obtain following equations

$$i\dot{\lambda}_{21} + \dot{\lambda}_{22} = 2(\alpha_1\lambda_7 - \alpha_1\lambda_8 - \alpha_3\lambda_{13} + i\alpha_3\lambda_{14} - i\alpha_3\lambda_{15} + \alpha_3\lambda_{16}) = 0. \quad (59)$$

$$\dot{\lambda}_{23} + i\dot{\lambda}_{24} = 2(-\alpha_3\lambda_{10} + \alpha_3\lambda_{12} - \alpha_1\lambda_{13} + i\alpha_1\lambda_{15} - i\alpha_1\lambda_{14} + \alpha_1\lambda_{16}) = 0. \quad (60)$$

$$\dot{\lambda}_{25} - i\dot{\lambda}_{26} = 2(\alpha_1\lambda_7 - \alpha_1\lambda_8 - \alpha_3\lambda_{13} + i\alpha_3\lambda_{14} - i\alpha_3\lambda_{15} + \alpha_3\lambda_{16}) = 0. \quad (61)$$

$$\dot{\lambda}_{27} + i\dot{\lambda}_{28} = 2i(-\alpha_3\lambda_{10} + \alpha_3\lambda_{12} - \alpha_1\lambda_{13} + i\alpha_1\lambda_{15} - i\alpha_1\lambda_{14} + \alpha_1\lambda_{16}) = 0. \quad (62)$$

Notes

Since

$$\lambda_7 = \lambda_8, \lambda_{10} = \lambda_{12}; \qquad \lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{16} = \eta(t), \quad (63)$$

or if we set

$$\dot{\lambda}_{21} = -i\dot{\lambda}_{22} = \dot{\xi}(t); \quad \dot{\lambda}_{23} = -i\dot{\lambda}_{24} = \dot{\theta}(t);$$
$$\dot{\lambda}_{25} = i\dot{\lambda}_{26} = \dot{\delta}(t); \quad \dot{\lambda}_{27} = -i\dot{\lambda}_{28} = \dot{\zeta}(t). \quad (64)$$

which immediately gives

$$\lambda_{21} = \xi(t) + c_{21}; \quad \lambda_{22} = -i\xi(t) + c_{22}; \quad \lambda_{23} = \theta(t) + c_{23}; \quad \lambda_{24} = -i\theta(t) + c_{24};$$
$$\lambda_{25} = \delta(t) + c_{25}; \quad \lambda_{26} = -i\delta(t) + c_{26}; \quad \lambda_{27} = \zeta(t) + c_{27}; \quad \lambda_{28} = i\zeta(t) + c_{28}. \quad (65)$$

Solutions for $(\lambda_{29} - \lambda_{36})$ can be obtained respectively as, from eqs.(39-46), we obtain following equations

$$\dot{\lambda}_{31} - i\dot{\lambda}_{32} = -2(-i\alpha_3\lambda_7 + i\alpha_3\lambda_8 + i\alpha_2\lambda_{13} + \alpha_2\lambda_{14} - \alpha_2\lambda_{15} - i\alpha_2\lambda_{16}) = 0. \quad (66)$$

$$\dot{\lambda}_{29} + i\dot{\lambda}_{30} = 2(-\alpha_2\lambda_{10} + \alpha_2\lambda_{12} - \alpha_3\lambda_{13} + i\alpha_3\lambda_{15} - i\alpha_3\lambda_{14} + \alpha_3\lambda_{16}) = 0. \quad (67)$$

$$\dot{\lambda}_{33} + i\dot{\lambda}_{34} = 2i(-\alpha_2\lambda_{10} + \alpha_2\lambda_{12} - \alpha_3\lambda_{13} + i\alpha_3\lambda_{15} - i\alpha_3\lambda_{14} + \alpha_3\lambda_{16}) = 0. \quad (68)$$

$$\dot{\lambda}_{35} + i\dot{\lambda}_{36} = -2(-i\alpha_3\lambda_7 + i\alpha_3\lambda_8 + i\alpha_2\lambda_{13} + \alpha_2\lambda_{14} - \alpha_2\lambda_{15} - i\alpha_2\lambda_{16}) = 0. \quad (69)$$

Since

$$\lambda_7 = \lambda_8, \lambda_{10} = \lambda_{12}; \qquad \lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{16} = \eta(t), \quad (70)$$

or if we set

$$\dot{\lambda}_{29} = -i\dot{\lambda}_{30} = \dot{\gamma}(t); \quad \dot{\lambda}_{31} = i\dot{\lambda}_{32} = \dot{\mu}(t); \quad \dot{\lambda}_{33} = -i\dot{\lambda}_{34} = \dot{\rho}(t); \quad \dot{\lambda}_{35} = -i\dot{\lambda}_{36} = \dot{\sigma}(t). \quad (71)$$

Which immediately gives

$$\lambda_{29} = \gamma(t) + c_{29}; \quad \lambda_{30} = i\gamma(t) + c_{30}; \quad \lambda_{31} = \mu(t) + c_{31}; \quad \lambda_{32} = -i\mu(t) + c_{32};$$

$$\lambda_{33} = \rho(t) + c_{33}; \quad \lambda_{34} = i\rho(t) + c_{34}; \quad \lambda_{35} = \sigma(t) + c_{35}; \quad \lambda_{36} = i\sigma(t) + c_{36}. \quad (72)$$

where the arbitrary function η's, $\phi, \varphi, \chi, \psi, \xi, \theta, \delta, \zeta, \gamma, \mu, \rho$ and σ's and integration constants, c_i's, $(i = 1, ...36)$. are obtained through the equations, when we have solved eqs.[(11)-(46)]. Therefore, substitution of the solutions for λ_i's yield the, complex invariant for a two dimensional complex oscillator systems as

$$I = \frac{1}{2}\eta_1(p_1^2 + x_3^2) + \frac{1}{2}\eta_2(p_2^2 + x_4^2) + \frac{1}{2}\eta_4(x_1^2 + p_3^2) + \frac{1}{2}\eta_3(x_2^2 + p_4^2) + \eta(x_1x_3 + p_3p_4 + x_2p_3 + x_1p_4)$$

$$+\phi p_1p_2 + \varphi p_1x_4 + \chi x_3p_2 + \quad x_3x_4 + \xi(p_1p_3 - ip_1x_1) + \theta(p_1x_2 - ip_1p_4) + \delta(p_3x_3 - ix_1x_3)$$

$$+\zeta(ip_4x_3 + x_2x_3) + \gamma(ip_2p_4 + p_2x_2) + \mu(p_3p_2 - ix_1p_2) + \rho(ip_4x_4 + x_2x_4) + \sigma(ip_3x_4 + x_1x_4). \quad (73)$$

is our desired invariant.

II. Conclusion

We have shown in this paper, the coupled harmonic oscillator system with two space and one time variable share the same mathematical framework as the coupled harmonic oscillators in one dimension ECPS. The role of a linear invariant designed, however, for a rotating TD harmonic oscillator in N-dimensions is investigated by Malkin and Man'ko [16] in the context of coherent states. While the use of the quantum analogue of such TD systems in one dimension has been known [17, 18] for more than three decades.

Notes

References Références Referencias

1. T. D. Lee, Some special examples in renormalizable field theory, Phys. Rev. **95**, 1329 (1954).
2. H. Van Dam, Y. J. Ng, and L. C. Biedenharn, A comment on fermionic tachyons and Poincre representations, Phys. Lett. B **158**, (1985) 227-258.
3. J. Fu, S. Jimnez, Y. Tang, L. Vzquez, Construction of exact invariants of time-dependent linear nonholonomic dynamical systems, J. Phys. A: Math. Gen. **372**, (2008) 10-23.
4. H Goldstien, C Poole and J Safco, "*Classical mechanics*" (Pearson Edu. Inc., Singapore, 2002).
5. L. A. Pipes and L. R. Harvill, "*Applied Mathematics for Engineers and Physicists*" (New York: McGraw-Hill: 1970).
6. B. Yurke and D. Stoler, Bell's-inequality experiment employing four harmonic oscillators Phys. Rev. A **51**, (1995) 3437-3454.
7. L. O. Chua, C. A. Desoer, E. S. Kuh, "*Linear and Nonlinear Circuits*" (Singapore: McGraw-Hill).
8. M. Blasone, P. Jizba and G. Vitiello, "*Quantum Field Theory and Its Macroscopic Manifestations*" (Imperial collge press: London 2011).
9. M. S. Abdalla, A.I. Elkasapy, Lie algebraic approach of a charged particle in presence of a constant magnetic field via the quadratic invariant, Annals of Physics **325**, (2010) 16671678; R.K. Colegrave, P. Croxson and M.A. Mannan, Complex invariants for the time-dependent harmonic oscillator, Phy. Lett. A. **131** (1988) 407-411: J.S. Virdi, F. Chand, C.N. Kumar, and S.C. Mishra, Complex dynamical invariants for two-dimensional nonhermitian Hamiltonian systems, Can. Jour. Phys. **90**, (2012) 151-157.
10. Kusenko A. and Shrock R., General determination of phases in quark mass matrices, Phy Lett B **323**, (1994) 18-24; Phys. Rev. D **50**, (1994) R30-54.
11. J. F. Gunion and H. E. Haber, Conditions for CP violation in the general two-Higgs-doublet model, Phys. Rev. D **72**, (2005) 095002-095117.
12. J.S. Virdi, and S.C. Mishra, Exact complex integrals in two dimensions for shifted harmonic oscillators, Pramana J. Phys. **79** (2012) 19-40; J.S. Virdi, F. Chand, C.N. Kumar, and S.C. Mishra, Complex dynamical invariants for two-dimensional complex potentials, Pramana J. Phys. **79**, (2012) 173-183.
13. R. S. Kaushal and H J Korsch, Some remarks on complex Hamiltonian systems, Phys. Lett. A **276**, (2000) 47-57; R. S. Kaushal and Singh S, Construction of complex invariants for classical dynamical systems, Ann. Phys. **288**, (2001) 253-274; R. S. Kaushal and Singh S,, Complex dynamical invariants for one-dimensional classical systems, Phys. Scr. **67**, (2003) 181-186; R. S. Kaushal, Classical and quantum mechanics of complex hamiltonian systems: An extended complex phase space approach, Pramana J. Phys. **73**, (2009) 287-299.
14. R. S. Kaushal, On the quantum mechanics of complex Hamiltonian systems in one dimension, J. Phys. A: Math. Gen. **34** (2001) L709-721; R. S. Kaushal and Parthasarathi, Quantum mechanics of complex Hamiltonian systems in one dimension, J. Phys. A **35**, (2002) 8743-8754.
15. R. S. Kaushal, "*Classical and Quantum Mechanics of Noncentral Potentials*": A Survey of Two-Dimensional Systems" (New Delhi: Narosa and Berlin: Springer 1998).
16. I. A. Malkin and V. I. Man'ko, Linear adiabatic invariants and coherent states, Phys. Lett. A **22**, (1970) 243-254; I. A. Malkin and V. I. Man' ko, and D. A. Trifonov, Coherent states and transition probabilities in a time-dependent electromagnetic field, Phys. Rev. D **2**, (1970) 1371-1184.
17. H. N. Hollinhorst, Quantum limits on resonant-mass gravitational-radiation detectors, Phys. Rev. D **19**, (1979) 1669-1685.
18. V. V. Dodonov, I. A. Malkin and V. I. Man'ko, Even and odd coherent states and excitations of a singular oscillator, Physica **72**, (1974) 597-612; Int. Journ. Theo. Phys. **14**, (1975) 37-56.