Lorentzian Para Sasakian Manifolds Admitting Special Semi Symmetric Recurrent Metric Connection

By Sunil Kumar Srivastava & R. P. Kushwaha
D.D.U Gorakhpur University, India

Abstract - Several author as Agashe and Chafle [1], Sengupta, De. U.C, Binh [4] and many other introduced semi symmetric non metric connection in different way. In this paper we have studied LP sasakian manifold with special semi-symmetric recurrent metric connection [2] and discuss it exiectance in LP sasakian manifold. In section 3 we establish the relation between the Riemannian connection and special semi-symmetric recurrent metric connection on LP sasakian manifold [4]. The section 4 deals with $\xi$-conformaly flat and $\emptyset$ concircularly flat of n dimensional LP sasakian manifold and we proved that $\xi$-conformaly flatness with special semi-symmetric recurrent metric connection and Riemannian manifold coincide.

Keywords : lorentzian para saskian manifold, semi symmetric recurrent metric connection, $\xi$-conformaly flat, $\emptyset$ concircularly flat.

GJSFR-F Classification : MSC 2000: 53C25

© 2013. Sunil Kumar Srivastava & R. P. Kushwaha. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.
Lorentzian Para Sasakian Manifolds Admitting Special Semi Symmetric Recurrent Metric Connection

Sunil Kumar Srivastava σ & R. P. Kushwaha σ

Abstract - Several author as Agashe and Chafle [1], Sengupta, De. U.C, Binh [4] and many other introduced semi symmetric non metric connection in different way. In this paper we have studied LP sasakian manifold with special semi-symmetric recurrent metric connection [2] and discuss it existance in LP sasakian manifold. In section 3 we establish the relation between the Riemannian connection and special semi-symmetric recurrent metric connection on LP sasakian manifold [4]. The section 4 deals with ξ-conformaly flat and ϕ concircularly flat of n dimensional LP sasakian manifold and we proved that ξ-conformaly flatness with special semi-symmetric recurrent metric connection and Riemannian manifold coincide.

Keywords : lorentzian para sasakian manifold, semi symmetric recurrent metric connection, ξ-conformaly flat, ϕ concircularly flat.

1. INTRODUCTION

An n-dimensional differentiable manifold $M^n$ is a lorentzian para-sasakian manifold if it admits tensor field of type (1,1), a contravariant vector field $\xi$, a covariant vector field $\eta$ and a lorentzian metric $g$ satisfying:

$$\phi^2 X = X + \eta(x)\xi$$  \hspace{1cm} (1.1)

$$\eta(\xi) = -1$$  \hspace{1cm} (1.2)

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$  \hspace{1cm} (1.3)

$$g(X, \xi) = \eta(X)$$  \hspace{1cm} (1.4)

$$(D_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$$  \hspace{1cm} (1.5)

$$D_X \xi = \phi X$$  \hspace{1cm} (1.6)

For arbitrary vector field and $Y$, where D denotes the operator of covariant differentiation with respect to lorentzian metric $g$ [5].

Author σ : Deptt. of Science & Humanities Columbia Institute of Engg. & Technology Raipur India.
Author σ : Deptt. of Mathematics & Statistics D.D.U Gorakhpur University, Gorakhpur India. E-mail : Sunilk537@gmail.com

© 2013 Global Journals Inc. (US)
In a LP sasakian manifold with structure \((\emptyset, \xi, \eta, g)\) the following relation hold.

(a) \(\emptyset(\xi) = 0\) \hspace{1cm} (b) \(\eta(\emptyset X) = 0\) \hspace{1cm} \text{rank}\emptyset = n - 1 \quad (1.7)

Let us put \(F(X,Y) = g(\emptyset X,Y)\) \quad (1.8)

The tensor field \(F\) is symmetric (0,2) tensor field ie \(F(X,Y) = F(Y,X)\) \quad (1.9)

And \(\quad (D_X\eta)(Y) = F(X,Y) = g(\emptyset X,Y)\) \quad (1.10)

Also in a LP sasakian manifold the following relation holds;

\[ g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \quad (1.11) \]

And \(S(Y,\xi) = (n - 1)\eta(X) \quad (1.12)\)

For any vector field \(X,Y,Z\) where \(R(X,Y)Z\) is the Riemannian curvature tensor and \(S\) is the Ricci tensor.

Let \((M^n, g)\) be an LP sasakian manifold with Levi-Civita connection \(D\). we define a linear connection \(\overline{D}\) on \(M^n\) by

\[ \overline{D}_X Y = D_X Y - \eta(X)Y \quad (1.13) \]

Where \(\eta\) is 1-form associated with vector field \(\xi\) on \(M^n\), given by

\[ g(X,\xi) = \eta(X) \quad (1.14) \]

Using (1.13) the torsion tensor \(T\) of \(M^n\) with respect to connection \(\overline{D}\) is given by

\[ T(X,Y) = \overline{D}_X Y - \overline{D}_Y X - [X,Y] = \eta(Y)X - \eta(X)Y \quad (1.15) \]

A linear connection satisfying (1.15) is called semi symmetric connection.

Further from (1.13), we have

\[ (\overline{D}_X g)(Y,Z) = 2\eta(X)g(Y,Z) \quad (1.16) \]

A linear connection satisfying (1.16) is called semi symmetric recurrent metric connection The word special is used to distinguish it from other connection.

II. **Existence of Special Semi-Symmetric Recurrent Metric Connection**

Let \(\overline{D}\) be a linear connection in \(M^n\), given by

\[ \overline{D}_X Y = D_X Y + H(X,Y) \quad (2.1) \]

Where \(H\) is a tensor of type (1,2).

Now, we determine the tensor field \(H\) such that \(\overline{D}\) satifies (1.15) and (1.16).
From (2.1), we have
\[ \bar{T}(X,Y) = H(X,Y) - H(Y,X) \]  
(2.2)

Let \( G(X,Y,Z) = (\bar{D}_X g)(Y,Z) \) then
\[ g(H(X,Y),Z) + g(H(X,Z),Y) = -G(X,Y,Z) \]  
(2.3)

From (2.1), (2.2) and (2.4), we have
\[ g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X) \]
\[ = g(H(X,Y),Z) - g(H(Y,X),Z) + g(H(Z,X),Y) - g(H(X,Z),Y) + g(H(Z,Y),X) - g(H(Y,Z),X) \]
\[ = 2g(H(X,Y),Z) + 2\eta(X)g(Y,Z) - 2\eta(Z)g(X,Y) + 2\eta(Y)g(X,Z) \]
Or
\[ H(X,Y) = \frac{1}{2}\left\{ \bar{T}(X,Y) + \bar{T}'(X,Y) + \bar{T}''(X,Y) \right\} - \eta(X)Y - \eta(Y)X + g(X,Y)\xi \]  
(2.5)

Where
\[ g(T''(X,Y),Z) = g(T(Z,X),Y) \]  
(2.6)

Using (1.5), (2.6) we get
\[ \bar{T}'(X,Y) = \eta(X)Y - g(X,Y)\xi \]  
(2.7)

Then in view of (1.15), (2.5) and (2.7), we get
\[ H(X,Y) = -\eta(X)Y \]

This implies
\[ \bar{D}_X Y = D_X Y - \eta(X)Y \]

Conversely, a connection \( \bar{D} \) given by (1.13), satisfies (1.15) and (1.16) show that \( \bar{D} \) is a special semi symmetric recurrent metric connection.

So we state the following theorem.

**Theorem 2.1:** let \((M^n, g)\) be an LP sasakian manifold with lorentzian para contact metric structure \((\emptyset, \xi, \eta, g)\) admits a special semi-symmetric connection which is given by
\[ \bar{D}_X Y = D_X Y - \eta(X)Y \]

**III. Curvature Tensor of** \( M^n \) **with Respect to Special Semi –Symmetric Recurrent Metric Metric Connection** \( \bar{D} \)

The Curvature tensor of \( M^n \) with respect to special semi-symmetric recurrent metric metric connection \( \bar{D} \) is given by
\[ \bar{R}(X,Y,Z) = \bar{D}_X \bar{D}_Y Z - \bar{D}_X \bar{D}_Y Z - \bar{D}_{[X,Y]} Z \]
Using (1.13) and (1.10) in above we have
\[ \bar{R}(X,Y,Z) = R(X,Y,Z) \]
(3.1)

Hence we conclude.
**Proposition 3.1:** The Curvature tensor of $M^n$ with respect to special semi-symmetric recurrent metric connection $\bar{D}$ coincide with the curvature tensor of connection $D$ of Riemannian manifold. Taking the inner product of (3.1) with $W$, we have

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W)$$

(3.2)

Where

$$\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y,Z),$$

From (3.2), we have

$$\bar{R}(X,Y,Z,W) = -R(Y,X,Z,W)$$

(3.3)

$$\bar{R}(X,Y,Z,W) = -R(X,Y,W,Z)$$

(3.4)

Combining above two relation, we have

$$\bar{R}(X,Y,Z,W) = R(Y,X,W,Z)$$

(3.5)

We also have,

$$\bar{R}(X,Y,Z) + \bar{R}(Y,Z,X) + \bar{R}(Z,X,Y) = 0$$

(3.6)

This is the Bianchi first identity for $\bar{D}$.

Hence we conclude that the curvature tensor of $M^n$ with respect to special semi-symmetric recurrent metric connection $\bar{D}$ satisfies the first Bianchi identity. Contracting (3.2) over $X$ and $W$, we obtain

$$\bar{S}(Y,Z) = S(Y,YZ)$$

(3.7)

Where $\bar{S}$ and $S$ denote the Ricci tensor of the connection $\bar{D}$ and $D$ respectively.

From (3.7) we obtain a relation between the scalar curvature of $M^n$ with respect to the Riemannian connection and special semi-symmetric recurrent metric connection which is given by

$$\bar{r} = r$$

(3.8)

So we have following

**Proposition 3.2:** For n dimensional LP sasakian manifold with special semi symmetric recurrent metric connection $\bar{D}$

(1) The curvature tensor $\bar{R}$ is given by (3.1)

(2) Ricci tensor $\bar{S}$ is given by (3.7)

(3) $\bar{r} = r$
IV. CONCIRCULAR CURVATURE TENSOR OF LP SASAKIAN MANIFOLD WITH RESPECT TO SPECIAL SEMI-SYMMETRIC RECURRENT METRIC CONNECTION

Analogous to the definition of concircular curvature tensor in a Riemannian manifold we define concircular curvature tensor with respect to the special semi-symmetric recurrent metric connection $D$ as

$$
\tilde{C}(X,Y,Z) = \tilde{R}(X,Y,Z) - \frac{r}{n(n-1)} \{g(Y,Z)X - g(X,Z)Y\} \tag{4.1}
$$

Using (3.1) and (3.7) in (4.1), we have

$$
\tilde{C}(X,Y,Z) = C(X,Y,Z) \tag{4.2}
$$

So we have

**Proposition 4.1**: $\tilde{C}(X,Y,Z) = C(X,Y,Z)$ that is manifold coincide with Riemannian Manifold.

The notion of an $\xi$-conformally flat contact manifold was given by Zhen, Cabrezizo and Fernander [3]. In an analogous we define an $\xi$-conformally flat n-dimensional LP sasakian manifold.

**Definition 5.2**: An n-dimensional LP sasakian manifold is called $\xi$-conformally flat if the condition $\tilde{C}(X,Y)\xi = 0$ holds on $M^n$.

From (4.2) it is clear that $\tilde{C}(X,Y)\xi = C(X,Y)\xi$

So we have the following theorem.

**Theorem 4.1**: In an n-dimensional LP Sasakian manifold, an $\xi$–conformally flatness with respect to special semi-symmetric recurrent metric connection and Riemannian connection coincide.

**Definition 4.3**: An n dimensional LP Sasakian manifold satisfying the condition

$$
\varnothing^2 \tilde{C}(\varnothing X,\varnothing Y)\varnothing Z = 0 \tag{4.3}
$$

Is called $\varnothing$– concircularly flat.

Letus suppose that $M^n$ be n dimensional $\varnothing$ – concircularly flat LP sasakian manifold with respect to special semi-symmetric recurrent metric connection. It can easily be seen that $\varnothing^2 \tilde{C}(\varnothing X,\varnothing Y)\varnothing Z = 0$ if and only if

$$
g(\tilde{C}(\varnothing X,\varnothing Y)\varnothing Z,\varnothing W) = 0 \tag{4.4}
$$

For all $X,Y, Z$, Won $T(M)$.

Using (4.1), $\varnothing$– concircularly flat means

$$
g(\tilde{R}(\varnothing X,\varnothing Y)\varnothing Z,\varnothing W) = \frac{r}{n(n-1)} \{g(\varnothing Y,\varnothing Z)g(\varnothing X,\varnothing W) - g(\varnothing X,\varnothing Z)g(\varnothing Y,\varnothing W)\} \tag{4.5}
$$

Let \( \{e_1, e_2, ..., \xi\} \) be a local orthogonal basis of the vector in \( M^n \) using the fact that \( \emptyset e_1, \emptyset e_2, ..., \xi \) is also a local orthogonal basis, putting \( X = W = e_i \) in (4.5) and summaning with respect to \( i \), we have

\[
S(\emptyset Y, \emptyset Z) = \frac{r}{n(n-1)} \{g(\emptyset Y, \emptyset Z)\} \tag{4.6}
\]

Putting \( Y = \emptyset Y, Z = \emptyset Z \) in (4.6) and using the fact \( S \) is symmetric, we have

\[
g(\tilde{R}(\emptyset X, \emptyset Y)\emptyset Z, \emptyset W) = 0
\]

Hence we have

**Theorem 4.2:** an n-dimensional LP sasakian manifold is \( \emptyset \) – concircularly flat with respect to special semi-symmetric recurrent metric connection and manifold coincide with Riemannian Manifold.

**REFERENCES Références Referencias**