Generalizations of 2D-Canonical Sine-Sine Transform

By S.B.Chavhan
YeshwantMahavidyalaya, India

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Generalizations of 2D-Canonical Sine-Sine Transform

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I. Introduction

Integral transforms have been successfully used for almost two centuries is solving many problems in mathematical physical, applied mathematics and engineering science. Historically origin of the integral transform is P.S. Laplace and J.Fourier. Laplace transform is useful, for evaluating certain definite integral [2].

The definition of canonical sine-sine transform as follows [1].

\[ \{2DCST \, f(t,x)\}(s,w) = \left\{ f(t,x), K_s(t,x)K_w(s,w) \right\} \]

In the present paper, 2D sine-sine transform is extended in the distribution sense. The plan of the paper is as follows. The definitions are given in section 2. In section 3, testing functions space is defined by Galfand-shilov technique [3],[4].Section 4 some result on countable union space are proved. In section 5, inversion and uniqueness theorem are stated. In section 6, modulations theorems are given. The notations and terminology as per zemanian [5],[6].

II. Definition Two Dimensional Canonical Sine-Sine Transform

Let \( E(R \times R) \) denote the dual of \( E(R \times R) \). Therefore the generalized canonical sine transform of \( f(t,x) \in E(R \times R) \) is defined as

\[ \{2DCST \, f(t,x)\}(s,w) \]

\[ = (-1)^{\frac{1}{2}} \frac{1}{2\pi b} \frac{1}{2\pi b} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin \left( \frac{w}{b} t \right) \sin \left( \frac{s}{b} x \right) e^{\frac{j}{2} \left( \frac{s}{b} x \right)^2} e^{\frac{j}{2} \left( \frac{w}{b} t \right)^2} f(t,x) dt dx \]

Author : Department of Mathematics & Statistics, YeshwantMahavidyalaya, Nanded-431602 (India).
E-mail : chavhan_sathish49@yahoo.in

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Where
\[ \sup_{t, x} \{ K_k(t, s)K_k(x, w) \} = -\infty < t < \infty \begin{vmatrix} D_i^r D_s^\beta \phi(t, x) \end{vmatrix} \] < \infty \]

\[ -\infty < x < \infty \]

### III. Different S-Type Testing Function Spaces

In this section we have defined s-type testing function spaces by imposing conditions not only on the decreases of the fundamental functions at infinity, but also on the growth of their derivatives as the order of derivative increases. Clearly, \( SS^a \) space will be an extension of testing function space \( D \), so that these spaces have been successfully used in pseudo differential operator theory.

**a) The space \( SS^a \):**

It is given by
\[ SS^a = \left\{ \phi : \phi \in E, \sigma_{i, k, p} \phi(t, x) = \sup \left| D_i^r D_s^\beta \phi(t, x) \right| \leq C_i A_i \right\} \] (3.1)

The constant \( C_{i, p} \) and \( A \) depend on \( \phi \).

**b) The space \( SS^b \):**

\[ SS^b = \left\{ \phi : \phi \in E, \sigma_{i, k, p} \phi(t, x) = \sup \left| D_i^r D_s^\beta \phi(t, x) \right| \leq C_i B_i \right\} \] (3.2)

The constants \( C_i, B \) and \( B \) depend on \( \phi \).

**c) The space \( SS^c \):**

This space is formed by combining the condition (3.1) and (3.2)
\[ SS^c = \left\{ \phi : \phi \in E, \sigma_{i, k, p} \phi(t, x) = \sup \left| D_i^r D_s^\beta \phi(t, x) \right| \leq C_i A_i \right\} \] (3.3)

\( l, k, p = 0, 1, 2 \ldots \ldots \) Where \( A, B, C \) depend on \( \phi \).

**d) The space \( SS^d \):**

It is defined as,
\[ SS^d = \left\{ \phi : \phi \in E, \sigma_{i, k, p} \phi(t, x) = \sup \left| D_i^r D_s^\beta \phi(t, x) \right| \leq C_i A_i (m + \mu)^l \right\} \] (3.4)

For any \( \mu > 0 \) where \( m \) is the constant, depending on the function \( \phi \).

**e) The space \( SS^e \):**

This space is given by
\[ SS^e = \left\{ \phi : \phi \in E, \sigma_{i, k, p} \phi(t, x) = \sup \left| D_i^r D_s^\beta \phi(t, x) \right| \leq C_i (n + \delta)^l \right\} \] (3.5)

For any \( \delta > 0 \) where \( n \) the constant is depends on the function \( \phi \).
The space \(SS_{r,m}^{a,b,\gamma,n}\)

This space is defined by combining the conditions in (3.4) and (3.5).

\[
SS_{r,m}^{a,b,\gamma,n} = \left\{ \phi \in E_r / \xi_{i,k,p}(t,x) \leq \sup_{i} \left| t^i D_x^i \phi(t,x) \right| \leq C_{m} (m + \mu)^{l} \right\}
\]

IV. Results on Countable Unions-Type Space

**Proposition 4.1:** If \(m_1 < m_2\) then \(SS_{r,m_1}^{a,b} \subset SS_{r,m_2}^{a,b}\). The topology of \(SS_{r,m_1}^{a,b}\) is equivalent to the topology induced on \(SS_{r,m_2}^{a,b}\) by \(SS_{r,m_1}^{a,b}\)

\[
i.e \ T_{r,m_1}^{a,b} \sim T_{r,m_2}^{a,b} / SS_{r,m_1}^{a,b}
\]

**Proof:** For \(\phi \in SS_{r,m_1}^{a,b}\) and \(\delta_{i,k,p}(\phi) \leq C_{k,p}(m_1 + \mu)^{l}\gamma\)

\[
\leq C_{k,p}(m_2 + \mu)^{l}\gamma\ 
\]

Thus, \(SS_{r,m_1}^{a,b}\) \(\subset SS_{r,m_2}^{a,b}\)

The space \(SS_{r}^{a,b}\) can be expressed as union of countable normed spaces.

**Proposition 4.2:** \(SS_{r}^{a,b} = \bigcup_{i=1}^{\infty} SS_{r,m_i}^{a,b}\) and if the space \(SS_{r}^{a,b}\) is equipped with strict inductive topological \(S_{a,b,m}\) defined by injective map from \(SS_{r,m_i}^{a,b}\) to \(SS_{r}^{a,b}\) then the sequence \(\{\phi_n\}\) in \(SS_{r}^{a,b}\) converges to zero.

**Proof:** we show that \(SS_{r}^{a,b} = U \ SS_{r,m_i}^{a,b}\)

Clearly \(\bigcup_{i=1}^{\infty} SS_{r,m_i}^{a,b} \subset SS_{r}^{a,b}\) for proving the other inclusion, let \(\phi \in SS_{r}^{a,b}\) then

\[
\delta_{i,k,p}(\phi(t,x)) = \sup_{i} \left| t^i D_x^i \phi(t,x) \right| \leq C_{k,p} A^{i}\gamma,
\]

where \(A\) is some positive constant, choose an integer \(m = m_1\) and \(\mu = 0\) such that \(C_{k,p} A^{i} \leq C_{k,p}(m + \mu)^{i}\).

Then (4.1) we get \(\phi \in SS_{r,m_1}^{a,b}\) implying that \(SS_{r}^{a,b} = U \ SS_{r,m_i}^{a,b}\)

**Proposition 4.3:** If \(\gamma_1 < \gamma_2\) and \(\beta_1 < \beta_2\) then \(SS_{r_1}^{a,b,\gamma_1} \subset SS_{r_2}^{a,b,\gamma_2}\) and the topology of \(SS_{r_1}^{a,b,\gamma_1}\) is equivalent to the topology induced on \(SS_{r_2}^{a,b,\gamma_2}\) by \(SS_{r_2}^{a,b,\gamma_1}\).

**Proof:** Let \(\phi \in SS_{r_1}^{a,b,\gamma_1}\)

\[
\xi_{i,k,p}(\phi) = \sup_{i} \left| t^i D_x^i \phi(t,x) \right|
\]
\[ \leq CA^{\gamma_1}B^k k^{k\beta_1} \]
\[ \leq CA^{\gamma_2}B^p k^{p\beta_2} \quad \text{where} \, k, p = 0, 1, 2, 3 \]

Hence \( \phi \in SS_{\gamma_2}^{a,b,\beta_2} \). Consequently, \( SS_{\gamma_1}^{a,b,\beta_1} \subset SS_{\gamma_2}^{a,b,\beta_2} \). The topology of \( SS_{\gamma_1}^{a,b,\beta_1} \) is equivalent to the topology \( T_{\gamma_2}^{a,b,\beta_2} / SS_{\gamma_2}^{a,b,\beta_2} \).

It is clear from the definition of topologies of these spaces.

**Proposition 4.4:** \( SS_{\gamma_1}^{a,b,\beta_1} \) and if the space \( SS_{\gamma_1}^{a,b,\beta_1} \) is equipped with the strict inductive limit topology defined by the injective maps from \( SS_{\gamma_1}^{a,b,\beta_1} \) to \( SS_{\gamma_1}^{a,b,\beta_1} \) then the sequence \( \{ \phi_n \} \) in \( SS_{\gamma_1}^{a,b,\beta_1} \) converges to zero iff \( \{ \phi_n \} \) is contained in some \( SS_{\gamma_1}^{a,b,\beta_1} \) and converges to zero.

**Proof:** \( SS_{\gamma_1}^{a,b,\beta_1} = U_{\gamma_1,\beta_1} SS_{\gamma_1}^{a,b,\beta_1} \)

Clearly
\[ U_{\gamma_1,\beta_1} SS_{\gamma_1}^{a,b,\beta_1} \subset SS^a \]

For proving other inclusion, let \( \phi(t,x) \in SS_{\gamma_1}^{a,b,\beta_1} \) then
\[ \eta_{i,k,p}(\phi) = \sup_{i_k} \left| f^T D^i f \right| \]

is bounded by some number. We can choose integers \( \gamma_m \) and \( \beta_m \) such that
\[ \eta_{i,k,p}(\phi) \leq CA^{\gamma_1}B^k k^{k\beta_1} \]

\( \therefore \phi \in SS_{\gamma_1}^{a,b,\beta_1} \) for some integer \( \gamma_1 \) and \( \beta_1 \)

Hence \( SS_{\gamma_1}^{a,b,\beta_1} \subset U_{\gamma_1,\beta_1} SS_{\gamma_1}^{a,b,\beta_1} \). Thus \( SS_{\gamma_1}^{a,b,\beta_1} = U_{\gamma_1,\beta_1} SS_{\gamma_1}^{a,b,\beta_1} \).

**V. INVERSION AND UNIQUENESS THEOREMS**

**Theorem 5.1: (Inversion)** If \( \{ 2DCSST f(t,x) \} \) is canonical sine-sine transform of \( f(t,x) \) then inverse of transform is given by
\[
\begin{align*}
    f(t,x) & = \sqrt{\frac{2\pi}{b}} \sqrt{\frac{2\pi}{b}} \int_{-\infty}^{\infty} e^{\frac{i}{\pi}(\frac{w}{b})^2} e^{\frac{i}{\pi}(\frac{t}{b})^2} \int_{-\infty}^{\infty} \sin \left( \frac{s}{b} \right) \sin \left( \frac{w}{b} \right) e^{\frac{i}{\pi}(\frac{s}{b})^2} e^{\frac{i}{\pi}(\frac{w}{b})^2} \{ 2DCSST f(t,x) \} (s,w) ds dw \\
\end{align*}
\]

**Theorem 5.2: (Uniqueness)** If \( \{ 2DCSST f(t,x) \} \) and \( \{ 2DCSST g(t,x) \} \) are 2D canonical sine-sine transform and \( \sup pf \subset s_a \) and \( \sup pg \subset s_b \) then, \( f = g \) in the sense of equality in \( D(I) \).
Proof: By inversion theorem

\[
-f - g = \frac{1}{\sqrt{2\pi b}} \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\pi b}} e^{-\frac{z^2}{2\pi b}} \sin \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi z}{b} \right) \left[ \{2DCSST f(t,x)\}(s, w) d\mu \right] d\nu
\]

Thus \( f = g \) in \( D(U) \)

VI. Modulation Theorems for Canonical Sine-Sine Transform

Theorem 6.1: If \( \{2DCSST f(t,x)\}(s, w) \) is canonical sine-sine transform of \( f(t,x) \) then

\[
\{2DCSST \cos \mu f(t,x)\}(s, w) = \frac{1}{2} \left[ e^{-i(\mu b)s} \{2DCSST f(t,x)\}(s + \mu b, w) + e^{i(\mu b)s} \{2DCSST f(t,x)\}(s - \mu b, w) \right]
\]

Proof: Definition of two dimensional canonical sine-sine transform \( f(t,x) \) is

\[
\{2DCSST f(t,x)\}(s, w) = \frac{1}{\sqrt{2\pi b}} \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\pi b}} e^{-\frac{z^2}{2\pi b}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\pi b}} e^{-\frac{x^2}{2\pi b}} \sin \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi z}{b} \right) \sin \left( \frac{\pi t}{b} \right) \sin \left( \frac{\pi x}{b} \right) f(t,x) dt dx
\]

\[
\{2DCSST \cos \mu f(t,x)\}(s, w)
\]

\[
= \frac{1}{\sqrt{2\pi b}} \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\pi b}} e^{-\frac{z^2}{2\pi b}} \int_{-\infty}^{\infty} \cos \mu \left( e^{-\frac{t^2}{2\pi b}} e^{-\frac{x^2}{2\pi b}} \right) f(t,x) dt dx
\]

\[
= \frac{1}{2} \left[ \frac{1}{\sqrt{2\pi b}} \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\pi b}} e^{-\frac{z^2}{2\pi b}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\pi b}} e^{-\frac{x^2}{2\pi b}} \sin \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi z}{b} \right) \sin \left( \frac{\pi t}{b} \right) \sin \left( \frac{\pi x}{b} \right) f(t,x) dt dx
\]

\[
+ \frac{1}{\sqrt{2\pi b}} \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\pi b}} e^{-\frac{z^2}{2\pi b}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\pi b}} e^{-\frac{x^2}{2\pi b}} \sin \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi z}{b} \right) \sin \left( \frac{\pi t}{b} \right) \sin \left( \frac{\pi x}{b} \right) f(t,x) dt dx
\]

\[
= \frac{1}{2} \left[ e^{-i(\mu b)s} \{2DCSST f(t,x)\}(s + \mu b, w) + e^{i(\mu b)s} \{2DCSST f(t,x)\}(s - \mu b, w) \right]
\]

Thus \( f = g \) in \( D(U) \).
Theorem 6.2 If \( \{2DCSST f(t,x)\}(s,w) \) is canonical sine-sine transform of \( f(t,x) \) then

\[
\{2DCSST \sin \mu t f(t,x)\}(s,w) = \frac{i e^{-\frac{i}{2} \mu (s+w)}}{2} \left[ e^{-i(s+w)} \{2DCSST f(t,x)\}(s+\mu b,w) - e^{i(s+w)} \{2DCSST f(t,x)\}(s-\mu b,w) \right]
\]

Theorem 6.3 If \( \{2DCSST f(t,x)\}(s,w) \) is canonical sine-sine transform of \( f(t,x) \) then

\[
\{2DCSST e^{i\mu t} f(t,x)\}(s,w) = \frac{e^{-\frac{i}{2} \mu (s+w)}}{2} \left[ e^{-i(s+w)} \{2DCSST f(t,x)\}(s+\mu b,w) - e^{i(s+w)} \{2DCSST f(t,x)\}(s-\mu b,w) \right] + e^{i(s+w)} \{2DCSST f(t,x)\}(s-\mu b,w)
\]

Proof: Since \( \{2DCSST e^{i\mu t} f(t,x)\}(s,w) = \{2DCSST (\cos \mu t + i \sin \mu t) f(t,x)\}(s,w) \)

\[
\{2DCSST e^{i\mu t} f(t,x)\}(s,w) = \{2DCSST \cos \mu t f(t,x)\}(s,w) + i \{2DCSST \sin \mu t f(t,x)\}(s,w)
\]

\[
\cdot \frac{e^{-\frac{i}{2} \mu (s+w)}}{2} \left[ e^{-i(s+w)} \{2DCSST f(t,x)\}(s+\mu b,w) - e^{i(s+w)} \{2DCSST f(t,x)\}(s-\mu b,w) \right]
\]

\[
+ \frac{e^{-\frac{i}{2} \mu (s+w)}}{2} \left[ e^{-i(s+w)} \{2DCSST f(t,x)\}(s+\mu b,w) + e^{i(s+w)} \{2DCSST f(t,x)\}(s-\mu b,w) \right]
\]

\[
- e^{-i(s+w)} \{2DCSST f(t,x)\}(s+\mu b,w) + e^{i(s+w)} \{2DCSST f(t,x)\}(s-\mu b,w)
\]

\[
\cdot \frac{e^{-\frac{i}{2} \mu (s+w)}}{2} \left[ e^{-i(s+w)} \{2DCSST f(t,x)\}(s+\mu b,w) - e^{i(s+w)} \{2DCSST f(t,x)\}(s-\mu b,w) \right]
\]

\[
+ e^{i(s+w)} \{2DCSST f(t,x)\}(s-\mu b,w)
\]

VII. Conclusion

In this paper 2D canonical sine–sine transform is generalized in the distributional sense. Uniqueness theorem is proved and various testing functions specs defined by using Gelfand-shilov technique, topology properties are discussed. And lastly modulation theorems are proved.

References Références Referencias