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The Integration of Certain Products of Special Functions

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Abstract - The aim of the present paper is to obtain a finite integral involving a product of Fujiwara's polynomial [7], M-series [15], a general class of polynomial [10], with the H-function of several complex variables [11]. The results are quite general in nature hence encompass many new, known and unknown results hitherto in the literature.

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THE INTEGRATION OF CERTAIN PRODUCTS OF SPECIAL FUNCTIONS

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10. Srivastava, H.M., A contour integral involving Fox's function, Indian J. Math., 14 (1972), 1-6.

The Integration of Certain Products of Special Functions

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Abstract - The aim of the present paper is to obtain a finite integral involving a product of Fujiwara's polynomial [7], M-series [15], a general class of polynomial [10], with the H-function of several complex variables [11]. The results are quite general in nature hence encompass many new, known and unknown results hitherto in the literature.

I. Introduction

Srivastava [10] introduced a general class of polynomials (see also Srivastava and Singh [14])

$$S_q^p[x] = \sum_{s=0}^{[q/p]} \frac{(-q)_{ps}}{s!} A_{q,s}\, x^s,$$

$$= \Phi_3(s) \quad q = 0,1,2,\ldots \tag{1.1}$$

where p is an arbitrary positive integer and the coefficients $A_{q,s}(q,s \geq 0)$ are arbitrary coefficients, real or complex.

The series representation of the multivariable H-function (Srivastava and Panda [11]) studied by Olkha and Chaurasia ([8], [9]) is given as follows:

$$H[z_1,\ldots,z_r] = H^{0,\lambda':(u',v');\ldots;(u^{(r)},v^{(r)})}_{A',C':[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left[\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] \;:[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}]; \\ [(c):\psi_1,\ldots,\psi^{(r)}]:[d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}]; \end{matrix} \; z_1,\ldots,z_r \right]$$

$$= \sum_{m_i=1}^{u^{(i)}} \sum_{n_i=0}^{\infty} \Phi_1 \Phi_2 \frac{\prod_{i=1}^{r} (z_i)^{U_i} (-1)^{\sum_{i=1}^{r}(n_i)}}{\prod_{i=1}^{r} (\delta_{m_i}^{(i)})\; n_i!} \tag{1.2}$$

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where

$$\Phi_1 = \frac{\prod_{j=1}^{\lambda'} \Gamma\left[1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} U_i\right]}{\prod_{j=\lambda'+1}^{A'} \Gamma\left[a_j - \sum_{i=1}^{r} \theta_j^{(i)} U_i\right] \prod_{j=1}^{C'} \left[1 - c_j + \sum_{i=1}^{r} \psi_j^{(i)} U_i\right]}, \tag{1.3}$$

$$\Phi_2 = \frac{\prod_{\substack{j=1 \\ j \neq m_i}}^{u^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} U_i) \prod_{j=1}^{v^{(i)}} \Gamma(1 - b_j^{(i)} + \phi_j^{(i)} U_i)}{\prod_{j=u^{(i)}+1}^{D^{(i)}} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} U_i) \prod_{j=v^{(i)}+1}^{B^{(i)}} \Gamma(b_j^{(i)} - \phi_j^{(i)} U_i)} \tag{1.4}$$

and

$$U_i = \frac{d_{m_i}^{(i)} + n_i}{\delta_{m_i}^{(i)}}, \; i = 1, \ldots, r \tag{1.5}$$

which is valid under the conditions

$$\delta_{m_i}^{(i)} [d_j^{(i)} + p_i] \neq \delta_j^{(i)} [d_{m_i}^{(i)} + n_i] \tag{1.6}$$

$$\text{for } j \neq m_i, m_i = 1, \ldots, u^{(i)}; p_i, n_i = 0, 1, 2, \ldots; z \neq 0 \tag{1.7}$$

$$\nabla_i = \sum_{j=1}^{\lambda'} \theta_j^{(i)} - \sum_{j=1}^{C'} \psi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} < 0, \; \forall i = 1, \ldots, r \tag{1.8}$$

Srivastava and Panda [12] introduced the multivariable H-function as follows:

$$H[y_1, \ldots, y_R] = H^{0,\lambda:(\alpha',\beta');\ldots;(\alpha^{(R)},\beta^{(R)})}_{A,C:[M',N'];\ldots;[M^{(R)},N^{(R)}]} \left[\begin{matrix} [(g):\gamma';\ldots;\gamma^{(R)}]:[q';\eta'];\ldots;[q^{(R)},\eta^{(R)}]; \\ (f):\xi';\ldots;\xi^{(R)}]:[p',\epsilon');\ldots;[p^{(R)},\epsilon^{(R)}]; \end{matrix} \; y_1, \ldots, y_R \right] \tag{1.9}$$

For the sake of brevity

$$T_i = \sum_{j=1}^{\lambda} \gamma_j^{(i)} - \sum_{j=1}^{C} \xi_j^{(i)} + \sum_{j=1}^{M^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{N^{(i)}} \epsilon_j^{(i)} \leq 0, \tag{1.10}$$

$$\Omega_i = \sum_{j=\lambda+1}^{A} \gamma_j^{(i)} - \sum_{j=1}^{C} \xi_j^{(i)} + \sum_{j=1}^{\beta^{(i)}} \eta_j^{(i)} - \sum_{j=\beta^{(i)}+1}^{M^{(i)}} \eta_j^{(i)} + \sum_{j=1}^{\alpha^{(i)}} \epsilon_j^{(i)} - \sum_{j=\alpha^{(i)}+1}^{N^{(i)}} \epsilon_j^{(i)} > 0 \tag{1.11}$$

Ref.

12. Srivastava, H.M. and Panda, R., Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. 283/284 (1976), 265-274.

$$|\arg(y_i)| < \frac{1}{2}\Omega_i \pi, \forall = 1,\ldots,R \qquad (1.12)$$

$$_{p'}\overset{\alpha}{M}_{q'}[y] = \sum_{s'=0}^{\infty} \frac{(a_1)_{s'}\ldots(a_{p'})_{s'}}{(b_1)_{s'}\ldots(b_{q'})_{s'}} \frac{y^{s'}}{\Gamma(\alpha s'+1)} \qquad (1.13)$$

Ref.

Here $\alpha \in C, \mathrm{Re}(\alpha) > 0, (a_j)_{k'}, (b_j)_{k'}$ are the Pochammer symbols. The series in (1.13) is defined when none of the parameters b_js, j = 1,2,...,q, is a negative integer or zero. If any numerator parameter a_j is negative integer or zero, then the series terminates to a polynomial in y. The series is convergent if p' ≤ q' and — y — < 1. For other details see [ ].

II. Main Theorem

The transformation is valid under the following conditions:

(i) $h_i, h'_i, T_i, \Omega_i, D^* = \tau(b-a), k>0, i=1,\ldots,R, i'=1,\ldots,r, k'>0$

(ii) $\mathrm{Re}(\rho) > -1, \mathrm{Re}\left(\sigma + \sum_{i=1}^{R} h_i \frac{p_j^{(i)}}{\epsilon_j^{(i)}} + \sum_{i'=1}^{r} h'_{i'} \frac{d^{(i)}}{\delta_j^{(i)}}\right) > -1$

(iii) $F_n(\rho,\omega;t)$ is Fujiwar's polynomial[7].

(iv) p is an arbitrary positive integer and the coefficients $A_{q,s}(q,s \geq 0)$ are arbitrary coefficients, real or complex.

(v) $|\arg(y_i)| < \frac{1}{2}\Omega_i, T_i, \Omega_i$ are given in (1.10) and (1.11).

(vi) $p' \leq q', |y| < 1$.

Thus, the following transformation holds

$$\int_a^b (t-a)^{\rho}(b-t)^{\sigma} F_n(\rho,\omega;t) S_q^p[x(b-t)^k]\, _{p'}\overset{\alpha}{M}_{q'}[y(b-t)^{k'}]$$

$$.\ H[z_1(b-t)^{h'},\ldots,z_r(b-t)^{h'_r}] H[y_1(b-t)^h, y_R(b-t)^{h_R}]dt$$

$$= \sum_{m_i=1}^{u^{(i)}} \sum_{n_i=0}^{\infty} \Phi_1 \Phi_2 \Phi_3(s) \Phi_4(s') \Gamma(1+\rho+n)(b-a)^{\rho+\sigma+1+\sum_{i=1}^{r} h'_i U_i + k_s + k's'}$$

$$.\ \frac{(-1)^{\sum_{i=1}^{r}(n_i)+n+ks+k's'} \prod_{i=1}^{r} (z_i)^{U_i} (D^*)^{\eta}}{\prod_{i=1}^{r} ((\delta_{m_i}^{(i)})n_i!)\, n!}$$

7. Fujiwara, J., A unified presentation of classical orthogonal polynomials, Math. Japan, 11 (1966), 133-148.

$$\cdot\ H^{0,\lambda+2\ :(\alpha',\beta');\ldots;(\alpha^{(R)},\beta^{(R)})}_{A+2,C+2:[M',N'];\ldots;[M^{(R)},N^{(R)}]}\Bigg[$$

$$\left[\omega-\sigma-\sum_{i=1}^{r} h_i^{'}U_i-ks-k's':h_1,\ldots,h_R\right],\left[-\sigma-\sum_{i=1}^{r} h_i^{'}U_i-ks-k's':h_1,\ldots,h_r\right],$$

$$\left[\omega+n-\sigma-\sum_{i=1}^{r} h_i^{'}U_i-ks-k's':h_1,\ldots,h_R\right],\left[-1-\rho-n-\sigma-\sum_{i=1}^{r} h_i^{'}U_i-ks-k's':h_1,\ldots,h_R\right],$$

$$\left.\begin{matrix}[(g):\gamma',\ldots,\gamma^{(R)}]:[q';\eta'];\ldots;[q^{(R)},\eta^{(R)}];\\ [(f):\xi',\ldots,\xi^{(R)}]:[p';\epsilon'];\ldots;[p^{(R)},\epsilon^{(R)}];\end{matrix}\ y_1(b-a)^{h_1},\ldots,y_R(b-a)^{h_R}\right]. \qquad (2.1)$$

III. Proof

To derive (2.1), we express the general class of polynomials, M-series, the multivariable H-function in series form with the help of (1.2), (1.1) and (1.13) and then changing the order of integration and summation which is valid with the conditions stated and evaluating the remaining integral with the help of a known result of Chaurasia and Sharma ([2], p.269, eqn. (2.1)), we arrive at the desired result.

IV. Special Cases

(i) Assigning suitable values to the parameters with appealing to a known result ([11], p.139, eqn.(4.11)), after a little simplification, we have the following result

Theorem (A)

The transformation is valid under the following conditions

(a) $\mathrm{Re}(\rho) > -1,\ \mathrm{Re}(\sigma) > -1$

(b) $h_j > 0, h_{i'}^{'} > 0, k > 0, k' > 0,\ j = 1,\ldots,R,\ i' = 1,\ldots,r,\ D^* = \tau(b-a)$

where

$$\Delta_j = 1+\sum_{i=1}^{\mu} \xi_i^{(j)} + \sum_{i=1}^{B^{(j)}} \epsilon_i^{(j)} - \sum_{i=1}^{\lambda} \gamma_i^{(j)} - \sum_{i=1}^{\alpha^{(j)}} \eta_i^{(j)} \quad (j = 1,\ldots,R)$$

(c) The equality holds when — y_j — < L_j, j = 1,...,R with the L_j defined by equation (5.3), p.157 in [12].

(d) p is an positive integer and the coefficients $A_{q,s}\ (q,s \geq 0)$ are arbitrary coefficients, real or complex.

(e) $F_n(\rho,\omega;t)$ is Fujiwara polynomial [7].

(f) $p' \leq q'$ and $|y| < 1$.

$$\int_a^b (t-a)^{\rho}(b-t)^{\sigma} F_n(\rho,\omega;t)\, S_q^p[x(b-t)^k]\ {}_{p'}\overset{\alpha}{M}_{q'}[y(b-t)^{k'}]$$

Ref.

2. Chaurasia, V.B.L. and Sharma, S.C., An integral involving extended Jacobi polynomials and H-function of several complex variables, Vij. Pari. Anu. Pat. 27(3) (1984), 267-272.

Notes

$$\cdot\ F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left[z_1(b-t)^{h'},\ldots,z_r(b-t)^{h'_r}\right]$$

$$\cdot\ F^{\lambda:\alpha';\ldots;\alpha^{(R)}}_{\mu:\beta';,,,;\beta^{(R)}}\left[y_1(b-t)^{h_1},\ldots,y_R(b-t)^{h_R}\right]dt$$

$$=\sum_{m_1,\ldots,m_r=0}^{\infty}\Phi_3(s)\,\Phi_4(s')\frac{\prod_{i=1}^{A}(a_i)_{m_1\theta_1+\ldots+m_r\theta_i^{(r)}}\prod_{i=1}^{B'}(b')_{m_1\phi_i'}\ldots}{\prod_{i=1}^{C}(c)_{m_1\psi_1+\ldots+m_r\psi_i^{(r)}}\prod_{i=1}^{D'}(d')_{m_1\delta_i'}\ldots}$$

$$\cdot\ \frac{\prod_{i=1}^{B^{(r)}}(b_i^{(r)})_{m_r\phi_i^{(r)}}}{\prod_{i=1}^{D^{(r)}}(d_i^{(r)})_{m_r\delta_i^{(r)}}}\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!}\frac{(-1)^n}{n!}(b-a)^{=r+s+1+\sum_{i=1}^{r}h'm_i+sk+s'k'}$$

$$\cdot\ \frac{\Gamma(1+\rho+n)\Gamma\left(1+\sigma+\sum_{i=1}^{r}h'm_i+sk+s'k'\right)}{\Gamma\left(1+\sigma-\omega-n+\sum_{i=1}^{r}h'm_i+sk+s'k'\right)}$$

$$\cdot\ \frac{\Gamma\left(1+\sigma-\omega+\sum_{i=1}^{r}h'm_i+sk+s'k'\right)}{\Gamma\left(1+\omega+n+\sigma+\sum_{i=1}^{r}h'm_i+sk+s'k'\right)}$$

$$\cdot\ F^{\lambda+2:\alpha';\ldots;\alpha^{(R)}}_{\mu+2:\beta';\ldots;\beta^{(R)}}\left[\begin{matrix}\left[1+\sigma+\sum_{i=1}^{r}h'm_i+sk+s'k':h_1,\ldots,h_R\right],\\ \left[1+\sigma-\omega-\eta+\sum_{i=1}^{r}h'm_i+sk+s'k':h_1,\ldots,h_R\right],\end{matrix}\right.$$

$$\left.\begin{matrix}\left[1+\sigma-\omega+\sum_{i=1}^{r}h'm_i+sk+s'k':h_1,\ldots,h_R\right],[(g):\gamma',\ldots,\gamma^{(R)}]:[(q'):\eta'];\ldots'[(q^{(R)}:\eta^{(R)}];\\ \left[2+\omega+n+\sigma+\rho+\sum_{i=1}^{r}h'm_i+sk+s'k':h_1,\ldots,h_R\right],[(f):\xi',\ldots,\xi^{(R)}]:[(p'):\in'];\ldots;[(p^{(R)}:\in^{(R)}];\end{matrix}\ y_1(b-a)^{h_1},\ldots,y_R(b-a)^{h_R}\right]\tag{4.1}$$

(ii) Taking r = 1 = R in (2.1), we have the following result

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Theorem (B)

The transformation is valid under the following conditions

(a) $\mathrm{Re}(1+\rho)>0, h, h', k, k', T>0, |\arg(y)|<\frac{1}{2}T\pi, D^{*}=\tau(b-a)$

(b) $\mathrm{Re}\left(\sigma+h'\frac{p_j}{\epsilon_j}+h\frac{d_{j'}}{\delta_{j'}}+1\right)>0, j=1,\ldots,u, j'=1,\ldots,\alpha.$

(c) p is an positive integer and the coefficient $A_{q,s}(q,s\geq 0)$ are arbitrary coefficients, real or complex.

(d) $F_n(\rho,\omega;t)$ is Fujiwara polynomial [7].

(e) $p' \leq q'$ and $|y| < 1$.

Thus, the following transformation holds

$$\int_a^b (t-a)^{\rho}(b-t)^{\sigma} F_n(\rho,\omega;t)\, S_q^p[x(b-t)^k]\,{}_{p'}\overset{\alpha}{M}_{q'}[y(b-t)^{k'}]$$

$$.\; H_{B,D}^{u,v}\left[\begin{matrix}[b':\phi]\\ [d':\delta]\end{matrix}\middle| z(b-t)^{h'}\right] H_{M,N}^{\alpha',\beta'}\left[\begin{matrix}[q'':\eta\\ [p'':\in\end{matrix}\middle| y'(b-t)^{h}\right] dt$$

$$=\sum_{m_1=0}^{u}\sum_{n_1=0}^{\infty}\Phi_1^{*}\Phi_3(s)\Phi_4(s')(-1)^{n_1}z^{U}(D^{*})^{n}\frac{(b-a)^{\rho+\sigma+1+h'U+sk+s'k'}\Gamma(1+\rho+n)}{n!\;\; n_1!\;\;\delta n_1}$$

$$.\; H_{M+2,N+2}^{\alpha,\beta+2}\left[\begin{matrix}[\omega-\sigma-h'U-sk-s'k':h],[-\sigma-h'U-sk-s'k':h],[b':\phi];\\ [(d':\delta],[\omega+n-\sigma-\eta'U-sk-s'k':h],[-1-\rho-\eta-\sigma-h'U-sk-s'k':h];\end{matrix}\; y'(b-a)^{h}\right]. \quad (4.2)$$

(iii) When $k'\to 0, q\to 0$, the result in (2.1), (4.1) and (4.2) reduce to the result obtained by Chaurasia and Chand [3].

(iv) Putting $q\to 0, h_i'\to 1, y\to 0, i=1,\ldots,r$ in (2), we have a result due to Chaurasia and Sharma [3].

(v) The results derived by the equations (3.2) and (3.3) in [2] can b e obtained from our results.

(vi) Setting $a=-1, b=1=\lambda, q\to 0, y\to 0, h_i'=1, i=1,\ldots,r$ in (2.1), we get a known result of Srivastava and Panda [11].

(vii) Taking $q\to 0, y\to 0, h_i'=1, i=1,\ldots,r$ the result in (4.1) reduces to a known result derived by Chaurasia and Sharma in [3].

(viii) The results (2.1),(4.1) and (4.2) established by Chaurasia and Singh in [4] can be reduced as a particular cases of our results.

A great number of interesting transformation formulae as special cases of our results can be derived, but we omit them here for lack of space.

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Ref.

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Notes

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