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Some New Properties of Generalized Polynomials and $\bar{H}$-Function Associated with Feynman Integrals

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Abstract - In the present paper we study the integrals involving generalized polynomials (multivariable) and the $\bar{H}$- function. The $\bar{H}$- function was proposed by Inayat-Hussain which contain a certain class of Feynman integrals, the exact partition function of the Gaussian model in statistical mechanics and several other functions as its particular cases. Our integrals are unified in nature and act as key formulae from which we can derive as particular cases, integrals involving a large number of simpler special functions and polynomials. For the sake of illustration, we give here some particular cases of our main integral which are also new and of interest by themselves. At the end, we give applications of our main findings by interconnecting them with the Riemann–Liouville type of fractional integral operator. The results obtained by us are basic in nature and are likely to find useful applications in several fields notably electricals networks, probability theory and statistical mechanics.

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SOME NEW PROPERTIES OF GENERALIZED POLYNOMIALS AND H-FUNCTION ASSOCIATED WITH FEYNMAN INTEGRALS

Strictly as per the compliance and regulations of :

Ref.

6. Grosche C. and Steiner F., Handbook of Feynman path integrals, Springer tracts in modern physics Vol. 145, Springer-Verlag Berlin Heidelberg, New York, (1998).

Some New Properties of Generalized Polynomials and $\bar{H}$-Function Associated with Feynman Integrals

V.G.Gupta [α] & Nawal Kishor Jangid [σ]

Abstract - In the present paper we study the integrals involving generalized polynomials (multivariable) and the $\bar{H}$ -function. The $\bar{H}$ - function was proposed by Inayat-Hussain which contain a certain class of Feynman integrals, the exact partition function of the Gaussian model in statistical mechanics and several other functions as its particular cases. Our integrals are unified in nature and act as key formulae from which we can derive as particular cases, integrals involving a large number of simpler special functions and polynomials. For the sake of illustration, we give here some particular cases of our main integral which are also new and of interest by themselves. At the end, we give applications of our main findings by interconnecting them with the Riemann–Liouville type of fractional integral operator. The results obtained by us are basic in nature and are likely to find useful applications in several fields notably electricals networks, probability theory and statistical mechanics.

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I. Introduction

Feynman path integrals are reformulation of quantum mechanics and are more fundamental than the conventional one in terms of operators because in the domain of quantum cosmology the conventional formulation may fail but Feynman path integrals still apply [6]. Inayat-Hussain [9] has pointed out the usefulness of Feynman integrals in the study and development of simple and multiple variable hypergeometric series which in turn are very useful in statistical mechanics. Hussain has introduced in another paper [10] the $\bar{H}$-function which is a new generalization of the familiar H-function of Fox [4]. The $\bar{H}$-function contains the exact partition function of the Gaussian model in statistical mechanics, functions useful in testing hypothesis and several others as its special cases.

The $\bar{H}$-function has been defined and represented as follows [2].

$$\bar{H}_{P,Q}^{M,N}[z] = \bar{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_{-i\infty}^{+i\infty}\phi(\xi)z^{\xi}d\xi \tag{1.1}$$

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Where
$$\phi(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

which contains fractional powers of some of the gamma functions. Here, and throughout the paper $a_j(j=1,...,P)$, and $b_j(j=1,...,Q)$ are complex parameters, $\alpha_j\geq 0(j=1,...,P)$, $\beta_j\geq 0(j=1,...,Q)$ (not all zero simultaneously) and the exponents $A_j(j=1,...,N)$ and $B_j(j=M+1,...,Q)$ can take non-integer values.

The contour in (1.1) is along imaginary axis $\mathrm{Re}(\xi)=0$. It is suitably indented in order to avoid the singularities of the gamma functions and to keep those singularities on appropriate sides. Again, for $A_j(j=1,...,N)$ not an integer, the poles of the gamma functions of the numerator in (1.2) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j-\beta_j\xi)$ (j = 1,…,M) and $\Gamma(1-a_j-\alpha_j\xi)$ (j = 1,…,N) pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

Evidently, when the exponents A_j and B_j all take an integer values, the $\bar{H}$-function reduces to the well known Fox's H-function [4].

The following sufficient conditions for the absolute convergence of the defining integral for

$\bar{H}$-function given by equation (1.1) have been given by Buschman and Srivastava[2].

$$\theta=\sum_{j=1}^{M}|\beta_j|+\sum_{j=1}^{N}|A_j\alpha_j|-\sum_{j=M+1}^{Q}|B_j\beta_j|-\sum_{j=N+1}^{P}|\alpha_j|>0, \tag{1.3}$$

and
$$|\arg z|<\frac{1}{2}\theta\pi\,. \tag{1.4}$$

where θ is given by (1.3).

The behaviour of the $\bar{H}$-function for small values of $|z|$ follows easily from a result recently given by Rathie [13, p. 306, eq. (6.9)], we have

$$\bar{H}_{P,Q}^{M,N}[z]=o\left(|z|^{\alpha}\right),\ \alpha=\underset{1\leq j\leq M}{Min.}\left\{\mathrm{Re}\left(b_j/\beta_j\right)\right\}\text{ for small }|z|. \tag{1.5}$$

Investigations of the convergence conditions, all possible types of contours, type of critical points of the integrand of (1.1), etc. can be made by an interested reader by following analogous techniques given in the well known works of Braaksma [1], Hai and Yakubovich [8]. We however omit the details.

Srivastava ([14], P.185,eq.(7)) has defined and introduced the generalized polynomials (multivariable)

Ref.

4. Fox C., The G and H functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc., 98(1961), 395-429.

Notes

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[x_1,\dots,x_r\right]=$$
$$\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\dots;n_r,k_r]x^{k_1}\dots,x^{k_r} \tag{1.6}$$

where $n_i = 0,1,2,\dots(I = 1,\dots,r)$, $m_1,\dots,m_r$ are an arbitrary positive integers and the coefficients $A[n_1,k_1;\dots;n_r,k_r]$ are arbitrary constants, real or complex .

II. Integrals Required

The following integrals will be required in our results

$$\int_0^b x^{\lambda-1}(b-x)^{\eta-1}\,dx=b^{\lambda+\eta-1}B(\lambda,\eta)\quad;\quad \mathrm{Re}(\lambda)>0,\ \mathrm{Re}(\eta)>0 \tag{2.1}$$

$$\int_0^u x^{\mu-1}(u-x)^{\upsilon-1}e^{\alpha x}\,dx= B(\upsilon,\mu)u^{\mu+\upsilon-1}\,{}_1F_1(\mu;\mu+\upsilon;\alpha u)\quad; \qquad \mathrm{Re}(\mu)>0,\ \mathrm{Re}(\upsilon)>0 \tag{2.2}$$

$$\int_0^u x^{-\mu-1}(u-x)^{\mu-1}e^{-\alpha/x}\,dx=\alpha^{-\mu}u^{\mu-1}\Gamma(\mu)e^{-\alpha/u}\quad;\quad \mathrm{Re}(\mu)>0,\ u>0 \tag{2.3}$$

III. Main Integrals

a) First Integral

We shall establish the following integral formulas :

$$\int_0^b x^{\rho-1}(b-x)^{\sigma-1}\bar{H}_{P,Q}^{M,N}\left[zx^u(b-x)^v\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\times$$

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[z_1x^{\lambda_1}(b-x)^{\mu_1},\dots,z_rx^{\lambda r}(b-x)^{\mu_r}\right]\mathrm{dx}$$

$$= b^{\rho+\sigma+\sum_{i=1}^{r}(\lambda_i+\mu_i)k_i-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\dots;n_r,k_r]\prod_{i=1}^{r}z_i^{k_i}$$

$$\bar{H}_{P+2,Q+1}^{M,N+2}\left[zb^{u+v}\left|\begin{matrix}\left(1-\rho-\sum_{\mp 1}^{r}\lambda_ik_i,u;1\right),\left(1-\sigma-\sum_{\mp 1}^{r}\mu_ik_i,v;1\right),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\left(1-\rho-\sigma-\sum_{\mp 1}^{r}(\lambda_i+\mu_i)k_i,u+v;1\right)\end{matrix}\right.\right] \tag{3.1}$$

valid under the conditions

(i) $u \geq 0, v \geq 0$ (not both zero simultaneously

(ii) $\mathrm{Re}(\rho)+\sum_{i=1}^{r}\lambda_i\left(\frac{n_i}{m_i}\right)+u\min_{1\leq j\leq M}\left[\mathrm{Re}(b_j/\beta_j)\right]>0$

$$\mathrm{Re}(\sigma)+\sum_{i=1}^{r}\mu_i\left(\frac{n_i}{m_i}\right)+v\min_{1\leq j\leq M}\left[\mathrm{Re}(b_j/\beta_j)\right]>0$$

(iii) The $\bar{H}$ - function occurring in (3.1) satisfy conditions corresponding appropriately to those given by (1.3) and (1.4).

b) Second Integral

$$\int_0^b x^{\rho-1}(b-x)^{\sigma-1}e^{\alpha x}\bar{H}_{P,Q}^{M,N}\left[zx^u(b-x)^v e^{\delta x}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\times$$

$$S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[z_1x^{\lambda_1}(b-x)^{\mu_1},\ldots,z_rx^{\lambda r}(b-x)^{\mu_r}\right]\mathrm{dx}$$

$$=b^{\rho+\sigma+\sum_{i=1}^{r}(\lambda_i+\mu_i)k_i-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}\frac{b^t}{t!}A[n_1,k_1;\ldots;n_r,k_r]\prod_{i=1}^{r}z_i^{k_i}$$

$$\bar{H}_{P+3,Q+2}^{M,N+3}\left[zb^{u+v}\left|\begin{matrix}\left(1-\rho-\sum_{i=1}^{r}\lambda_ik_i-t,u;1\right),\left(1-\sigma-\sum_{i=1}^{r}\mu_ik_i,v;1\right),(-\alpha,\delta;r),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\alpha,\delta;r),\left(1-\rho-\sigma-\sum_{i=1}^{r}(\lambda_i+\mu_i)k_i-t,u+v;1\right)\end{matrix}\right.\right]$$

(3.2)

where the $\bar{H}$-function occurring in the left hand side of (3.2) stands for the new generalized H-function defined by (1.1) and $s_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[x_1,\ldots,x_r\right]$ stands for the generalized polynomials given in(1.6).

The above integral holds true under the following conditions:-

(i) Re(ρ,σ)>0, u,v ≥ 0,

(ii) when min(μ_i,λ_i) ≥ 0 for all I = 1,…,r (not all zero simulteneously).

I $\mathrm{Re}(\rho)+\sum_{i=1}^{r}\lambda_i\left[\frac{n_i}{m_i}\right]+u\min_{1\leq j\leq M}\mathrm{Re}(b_j/B_j)>0$

Notes

II $\quad \mathrm{Re}(\sigma)+\sum_{i=1}^{r}\mu_i\left[\frac{n_i}{m_i}\right]+v\min_{1\le j\le M}\mathrm{Re}(b_j/B_j)>0$

(iii) when $\max(\mu_i,\lambda_i)<0$ for all I = 1,…,r (not all zero simulteneously).

I $\quad \mathrm{Re}(\rho)+\sum_{i=1}^{r}\lambda_i\left[\frac{n_i}{m_i}\right]+u\min_{1\le j\le M}\mathrm{Re}(b_j/B_j)>0$

II $\quad \mathrm{Re}(\sigma)+\sum_{i=1}^{r}\mu_i\left[\frac{n_i}{m_i}\right]+v\min_{1\le j\le M}\mathrm{Re}(b_j/B_j)>0$

(iv) when $\lambda_i\ge 0$ and $\mu_i<0$ inequalities I and IV are satisfied.

(v) when $\lambda_i<0$ and $\mu_i\ge 0$ inequalities II and III are satisfied.

Notes

c) Third Integral

$$\int_0^b x^{-\rho-1}(b-x)^{\rho-1}e^{-\alpha/x}\bar{H}_{P,Q}^{M,N}\left[ze^{-\delta/x}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\end{matrix}\right.\right]\times$$

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[z_1x^{-\lambda_1}(b-x)^{\lambda_1},\dots,z_rx^{-\lambda r}(b-x)^{\lambda_r}\right]\mathrm{dx}$$

$$= b^{\rho+\sum_{i=1}^{r}\lambda_ik_i-1}e^{-\alpha/b}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}A[n_1,k_1;\dots;n_r,k_r]\prod_{i=1}^{r}z_i^{k_i}\Gamma(\rho+\sum_{i=1}^{r}\lambda_ik_i)\times$$

$$\bar{H}_{P+1,Q+1}^{M,N+1}\left[zb^{u+v}\left|\begin{matrix}\left(1-\alpha,\delta;\rho+\sum_{i=1}^{r}\lambda_ik_i\right),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\alpha,\delta;\rho+\sum_{i=1}^{r}\lambda_ik_i)\end{matrix}\right.\right] \qquad (3.3)$$

The above result is valid under the following conditions :-

(i) $\quad \mathrm{Re}(\alpha)>0,\ \delta>0$

(ii) when $\lambda_i>0,\ \rho>0$

when $\lambda_i<0,\ \rho+\sum_{i=1}^{r}\lambda_i\left[\frac{n_i}{m_i}\right]>0$

PROOF :- To establish the integral (3.1), we express the generalized polynomials occurring in the left hand side in the series form given by (1.6) and the $\bar{H}$-function in terms of Mellin-Barnes contour integral given by (1.1) and then interchanging the order of summation and integration (which is permissible under the conditions stated with (3.1)) so that the left hand side of (3.1) (say Δ) assume the following after little simplification

$$\Delta = \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1k_1}}{k_1!} \cdots \frac{(-n_r)_{m_rk_r}}{k_r!} A[n_1,k_1;\ldots;n_r,k_r] \prod_{i=1}^{r} z_i^{k_i} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \theta(s) z^s$$

$$\left\{ \int_0^b x^{\rho+\sum_{i=1}^{r}\lambda_i k_i + us-1} (b-x)^{\sigma+\sum_{i=1}^{r}\mu_i k_i+vs-1} dx \right\} ds \tag{3.4}$$

On evaluating the inner integral occurring in (3.4) by using Eulerian integral (2.1) and on reinterpreting the Mellin-Barnes contour integral in terms of the $\bar{H}$- function given by (1.1), we easily arrive at the desired result (3.1).

Similarly the integrals (3.2) and (3.3) can also be established in the same manner by using the integral (2.2) and the integral (2.3) respectively.

IV. Special Case

(i) If we take $A(n_1,k_1;\ldots;n_r,k_r) = \prod_{i=1}^{r} A(n_i,k_i)$ in the definition of generalized polynomials occurring in the left hand side of the integral (3.1), we get

$$\int_0^b x^{\rho-1}(b-x)^{\sigma-1} \bar{H}_{P,Q}^{M,N}\left[z x^u (b-x)^v \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right] \times \prod_{i=1}^{r} S_{n_i}^{m_i}\left[z_i x^{\lambda_i}(b-x)^{\mu_i} \right] \mathrm{d}x$$

$$= b^{\rho+\sigma+\sum_{i=1}^{r}(\lambda_i+\mu_i)k_i-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1k_1}}{k_1!} \cdots \frac{(-n_r)_{m_rk_r}}{k_r!} \prod_{i=1}^{r} A(n_i,k_i) \prod_{i=1}^{r} z_i^{k_i}$$

$$\bar{H}_{P+2,Q+1}^{M,N+2}\left[z b^{u+v} \left| \begin{matrix} \left(1-\rho-\sum_{i=1}^{r}\lambda_i k_i, u; 1\right), \left(1-\sigma-\sum_{i=1}^{r}\mu_i k_i, v; 1\right), (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}, \left(1-\rho-\sigma-\sum_{i=1}^{r}(\lambda_i+\mu_i)k_i, u+v; 1\right) \end{matrix} \right. \right] \tag{4.1}$$

(a) Taking i = 2 in our result (4.1), we obtain the result discussed by Gupta and Soni [7, p.100, eq.(2.1)].

Ref.

7. Gupta K.C. and Soni R.C. New Properties of a generalization of Hypergeometric Series Associated with Feynman Integrals, KYUNGPOOK Math. J. 41(2001), 97-104.

$$\int_0^b x^{\rho-1}(b-x)^{\sigma-1}\bar{H}_{P,Q}^{M,N}\left[zx^u(b-x)^v\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\times S_{n_1}^{m_1}\left[z_1x^{\lambda_1}(b-x)^{\mu_1}\right]S_{n_2}^{m_2}\left[z_2x^{\lambda_2}(b-x)^{\mu_2}\right]dx$$

$$=b^{\rho+\sigma-1}\sum_{k_1=0}^{[n_1/m_1]}\sum_{k_2=0}^{[n_2/m_2]}\frac{(-n_1)_{m_1k_1}}{k_1!}\frac{(-n_2)_{m_2k_2}}{k_2!}A[n_1,k_1;n_2,k_2]z_1^{k_1}z_2^{k_2}b^{(\lambda_1+\mu_1)k_1+(\lambda_2+\mu_2)k_2}$$

$$\bar{H}_{P+2,Q+1}^{M,N+2}\left[zb^{u+v}\left|\begin{matrix}(1-\rho-\lambda_1k_1-\lambda_2k_2,u;1),(1-\sigma-\mu_1k_1-\mu_2k_2,v;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\rho-\sigma-(\lambda_1+\mu_1)k_1-(\lambda_2+\mu_2)k_2,u+v;1)\end{matrix}\right.\right]$$

(4.1.1)

(b) Taking i = 1 in the result (4.1), we obtain the result discussed by Gupta and Soni [7, p.101, eq.(3.1)].

$$\int_0^b x^{\rho-1}(b-x)^{\sigma-1}\bar{H}_{P,Q}^{M,N}\left[zx^u(b-x)^v\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\times S_{n_1}^{m_1}\left[z_1x^{\lambda_1}(b-x)^{\mu_1}\right]dx$$

$$=b^{\rho+\sigma-1}\sum_{k_1=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k_1}}{k_1!}A[n_1,k_1]z_1^{k_1}b^{(\lambda_1+\mu_1)k_1}$$

$$\bar{H}_{P+2,Q+1}^{M,N+2}\left[zb^{u+v}\left|\begin{matrix}(1-\rho-\lambda_1k_1,u;1),(1-\sigma-\mu_1k_1,v;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\rho-\sigma-(\lambda_1+\mu_1)k_1,u+v;1)\end{matrix}\right.\right]$$

(4.1.2)

In the similar manner if we put i = 2 and i = 1 in both the integrals (3.2) and (3.3), we obtain the known results given by Mishra Rupakshi [11, p.42, eq.(1.3.1)] and [11, p.43, eq. (1.3.2)].

(iv) Taking the exponents $A_j=B_j=1$ in the $\bar{H}$- function occurring in the left hand side of the integrals (3.1), (3.2) and (3.3) we get the results in terms of well known Fox's H-function.

The importance of the main integral of the present paper lies in its many fold generality. Again several integrals obtained by various authors and lying scattered in the literature also follow as simple special cases of our findings. Thus, if we reduce the $\bar{H}$-

Ref.

7. Gupta K.C. and Soni R.C. New Properties of a generalization of Hypergeometric Series Associated with Feynman Integrals, KYUNGPOOK Math. J. 41(2001), 97-104.

function occurring on the left hand side of (4.1.1) to the Fox's H function and the generalized polynomials $s_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x_1,\ldots,x_r]$ occurring therein to unity, we get a known integral [5,p.202].

V. Applications

We shall define the Rieman – Liouville fractional derivative of a function f(x) of order σ (or, alternatively, $-\sigma^{\text{th}}$ order fractional integral) [3,p.181;12,p.49] by (5.1)

$$ {}_aD_x^{\sigma}\{f(x)\} = \begin{cases} \dfrac{1}{\Gamma(-\sigma)}\displaystyle\int_a^x (x-t)^{-\sigma-1} f(t)\,dt, \operatorname{Re}(\sigma)<0, \\ \dfrac{d^q}{dx^q} D_x^{\sigma-q}\{f(x)\}, (q-1)\le \operatorname{Re}(\sigma)<q, \end{cases} \tag{5.1}$$

where q is a positive integer and the integral exists.

For simplicity the special case of the fractional derivative operator ${}_aD_x^{\sigma}$ when a = 0 will be written as D_x^{σ}. Thus we have

$$D_x^{\sigma} \equiv {}_0D_x^{\sigma} \tag{5.2}$$

Now by setting b = x and x = t in the main integral (3.1), it can be written as the following fractional integral formula :

$$D_x^{-\sigma}\left\{ t^{\rho-1}\bar{H}_{P,Q}^{M,N}\left[zt^u(x-t)^v \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}, \end{matrix} \right. \right] \times S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[z_1t^{\lambda_1}(x-t)^{\mu_1}\ldots z_rt^{\lambda_r}(x-t)^{\mu_r} \right] \right\}$$

$$= \frac{x^{\rho+\sigma-1}}{\Gamma(\sigma)} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k'=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1k_1}\ldots(-n_r)_{m'k'}}{k_1!\ldots k_r!} A[n_1,k_1;\ldots;n_r,k_r]\prod_{i=1}^{r} z_i^{k_i} x^{(\lambda_i+\mu_i)k_i} \times$$

$$\bar{H}_{P+2,Q+1}^{M,N+2}\left[zx^{u+v} \left| \begin{matrix} \left(1-\rho-\sum_{i=1}^{r}\lambda_ik_i,u;1\right),\left(1-\sigma-\sum_{i=1}^{r}\mu_ik_i,v;1\right),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\left(1-\rho-\sigma-\sum_{i=1}^{r}(\lambda_i+\mu_i)k_i,u+v;1\right) \end{matrix} \right. \right] \tag{5.3}$$

where Re(σ) >0 and all the conditions of validity mentioned with (3.1) are satisfied.

The fractional integral formula given by (5.3) is also quite general in nature and can easily yield Riemann-Liouville fractional integrals of a large number of simpler functions polynomials merely by specializing the parameters of $\bar{H}$-function and $S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x_1,\ldots,x_r]$, occurring in it which may find applications in electromagnetic theory and probability.

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Notes

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