On \((LCS)_n\) -Manifolds Satisfying Certain Conditions on D-Conformal Curvature Tensor

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I. INTRODUCTION

An \(n\)-dimensional Lorentzian manifold \(M\) is smooth connected para contact Hausdorff manifold with Lorentzian metric \(g\), i.e., \(M\) admits a smooth symmetric tensor field \(g\) of type \((0,2)\) such that for each point \(p \in M\), the tensor \(g_{p}: T_{p}M \times T_{p}M \rightarrow \mathbb{R}\) is a non degenerate inner product of signature \((-+,+,...,+)\) where \(T_{p}M\) denotes the tangent space of \(M\) at \(p\) and \(\mathbb{R}\) is the real number space. A non-zero vector \(\nu \in (T_{p}M)\) is said to be time like (res., non-space like, null, space like) if it satisfies \(g_{p}(\nu,\nu) < 0\) (resp., \(\leq 0, = 0, > 0\)) (see [2]).

Definition 1.1. In a Lorentzian manifold \((M, g)\) a vector field \(P\) defined by

\[ g(X,P) = A(X) \]

for any vector field \(X \in \chi(M)\) is said to be concircular vector field if

\[ (\nabla_{X} A)(Y) = \alpha[g(X,Y) + \omega(X)A(Y)] \]

where \(\alpha\) is a non zero scalar function, \(A\) is a 1-form and \(\omega\) is a closed 1-form.

Let \(M^{n}\) be a Lorentzian manifold admitting a unit time like concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have

\[ g(\xi,\xi) = -1 \]
Since $\xi$ is the unit concircular vector field, there exist a non zero 1-form $\eta$ such that

$$g(X, \xi) = \eta(X)$$

(1.2)

the equation (1.2) of the following form holds

$$\nabla_X \eta(Y) = \alpha [g(X, Y) + \eta(X)\eta(Y)] \quad (\alpha \neq 0)$$

(1.3)

for all vector field $X, Y$, where $\nabla$ denotes the operator of covariant differentiation with respect to Lorentzian metric $g$ and $\alpha$ is a non zero scalar function satisfying

$$\nabla X \alpha = (X \alpha) = \rho \eta(X),$$

(1.4)

where $\rho$ being a scalar function. If we put

$$\phi X = \frac{1}{\alpha} \nabla X \xi,$$

(1.5)

then from (1.3) and (1.5), we have

$$\phi^2 X = X + \eta(X)\xi,$$

(1.6)

from which it follows that $\phi$ is a symmetric (1,1) -tensor. Thus the Lorentzian manifold $M^n$ together with unit time like concircular vector field $\xi$, its associate 1-form $\eta$ and (1,1)–tensor field $\phi$ is said to be $(LCS)n$-manifold. Especially, if we take $\alpha = 1$, then the manifold becomes LP-Sasakian structure of Matsumoto (see [3]).

The $D$-conformal curvature tensor $B$ (see [4]), projective curvature tensor $P$, concircular curvature tensor $C$ (see [5]) on a Riemannian manifold $(M^n, g)$, $(n > 4)$ are defined as

$$B(X, Y)Z = R(X, Y)Z + \frac{1}{n-3} \left[ S(X, Z)Y - S(Y, Z)X + g(X, Z)QY - g(Y, Z)QX - S(X, Z)\eta(Y)\xi \right]$$

(1.7)

$$- \frac{(k-2)}{(n-3)} \left\{ g(X, Z)Y - g(Y, Z)X \right\} + \frac{k}{(n-3)} \left\{ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \right\}$$

$$+ \left( \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \right)$$

(1.8)

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)} \left\{ S(Y, Z)X - S(X, Z)Y \right\}$$

(1.9)

respectively, where $r$ is the scalar curvature, $Q$ is the Ricci tensor and $k = \frac{(r+2)(n-1)}{(n-2)}$. 

A differentiable manifold $M$ of dimension $n$ is called $(LCS)_n$-manifold if it admits a $(1,1)$-tensor $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ which satisfy the following.

(2.1) $\eta(\xi) = -1$

(2.2) $\phi^2 = I + \eta \otimes \xi$

(2.3) $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$

(2.4) $g(X, \xi) = \eta(X)$

(2.5) $\phi \xi = 0$, $\eta(\phi X) = 0$

for all $X, Y \in TM$. Also in a $(LCS)_n$-manifold the following relations are satisfied (see[4]).

(2.6) $\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]$

(2.7) $R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y]$

(2.8) $R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X]$

(2.9) $R(\xi, X)\xi = (\alpha^2 - \rho)[\eta(X)\xi + X]$

(2.10) $(\nabla X \phi)(Y) = \alpha [g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]$

(2.11) $S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X)$

(2.12) $S(\phi X, \phi Y) = S(X, Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y)$

(2.13) $(X \rho) = d\rho(X) = \beta \eta(X)$

**Definition 2.1.** A Lorentzian concircular structure manifold is said to be $\eta$-Einstein if the Ricci operator $Q$ satisfies

$$Q = ald + b\eta \otimes \xi,$$

where $a$ and $b$ are smooth functions on the manifolds, In particular if $b = 0$, then $M$ is an Einstein manifold.

**III. Main Results**

**Theorem 3.1.** There is no $(LCS)_n$-manifold that satisfying $B(X, Y)Z = 0$.

**Proof.** Assume that in a $(LCS)_n$-manifold

(3.1) $B(X, Y)Z = 0$. 
Then it is follows from (1.7) and (3.1) that

\[ R(X,Y)Z = -\frac{1}{(n-3)} \begin{bmatrix} S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX \\ -S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY \\ +\eta(Y)\eta(Z)QX \end{bmatrix} \]

(3.2)

\[ + \frac{(k-2)}{(n-3)} \left\{ g(X,Z)Y - g(Y,Z)X \right\} - \frac{k}{(n-3)} \left\{ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \right\} + \eta(Y)\eta(Z)X \]

It can also written as

\[ g(R(X,Y)Z,U) = -\frac{1}{(n-3)} \begin{bmatrix} S(X,Z)g(Y,U) - S(Y,Z)g(X,U) + g(X,Z)S(Y,U) \\ -g(Y,Z)S(X,U) - S(X,Z)\eta(Y)\eta(U) + S(Y,Z)\eta(X)\eta(U) \\ -\eta(X)\eta(Z)S(Y,U) + \eta(Y)\eta(Z)S(X,U) \end{bmatrix} \]

(3.3)

\[ + \frac{(k-2)}{(n-3)} \left\{ g(X,Z)g(Y,U) - g(Y,Z)g(X,U) \right\} - \frac{k}{(n-3)} \left\{ g(X,Z)\eta(Y)\eta(U) - g(Y,Z)\eta(X)\eta(U) \right\} + \eta(Y)\eta(Z)g(X,U) \]

Taking \( X = U = \xi \) in (3.3) and using (2.1) (2.4) and (2.11), it becomes

(3.4)

\[ \left\{ (\rho - \alpha^2)(5n+3)+2(k-1) \right\} \left\{ g(Y,Z)+\eta(Y)\eta(Z) \right\} = 0 \]

Then (3.4) implies that

(3.5)

\[ g(Y,Z)+\eta(Y)\eta(Z) = 0. \]

From (3.5) and (2.3) it is seen that \( g(\phi Y, \phi Z) = 0 \), however, as this is not possible.

This proves the theorem 3.1.

**Theorem 3.2.** A Ricci \( D \)-conformal semi-symmetric \((LCS)_n\)-manifold is an Einstein manifold with scalar curvature \( r = 2n^2(\alpha^2 - \rho) \).

**Proof.** From (1.7) by virtue of (2.6) and (2.11), we obtain

(3.6) \[ \eta(B(X,Y)Z) = \left[ (\alpha^2 - \rho) + \frac{(k-2)}{(n-3)} \right] \left\{ g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \right\} \]

From (3.6), it follows that

(3.7) \[ \eta(R(X,Y)\xi) = 0. \]
and

\[(3.8) \quad \eta(B(\xi,Y)Z) = \left(\alpha^2 - \rho + \frac{(k-2)}{(n-3)}\right) \{ -g(Y,Z) - \eta(Y) \eta(Z) \}\]

Assume that \(M^n\) is a Lorentzian concircular manifold satisfies the condition

\[(3.9) \quad B(X,Y)S(Z,W) = 0.\]

From (3.9), it is obtained that

\[(3.10) \quad S(B(X,Y)Z,W) + S(Z,B(X,Y)W) = 0\]

Taking \(X = W = \xi\) in (3.10) and using (3.6) (3.7) (3.8) and (2.11), we get

\[(3.11) \quad S(Y,Z) = 2n(\alpha^2 - \rho) g(Y,Z)\]

This proves the theorem3.2.

**Definition 3.1.** A Riemannian manifold \((M^n, g)\) is termed as Ricci \(D\)-conformal semi-symmetric if \(B(X,Y)S = 0\).

**Theorem 3.3.** There is no \((LCS)_n\)-manifold that satisfying \(R(X,Y)B = 0\).

**Proof.** Assume that in a \((LCS)_n\)-manifold satisfies the conditions \(R(\xi,Y)B = 0\), then it is expressed as


for all vector field \(X,Y,Z,V\) and \(W\) on \(M^n\).

For \(X = \xi\), it is follows from (2.8) and (3.12) that

\[(3.13) \quad (\alpha^2 - \rho) \left[ B(Z,V,W,Y) - \eta(B(Z,V)W)\eta(Y) - g(Y,Z)\eta(B(\xi,V)W) + \eta(Z)\eta(B(Y,V)W) \right] = 0\]

In fact \(Y = Z\) in (3.13) and by use of (3.6) (3.7) and (3.8) we have

\[(3.14) \quad (\alpha^2 - \rho) \left[ B(Z,V,W,Y) - g(Z,Z)\eta(B(\xi,V)W) - g(Z,W)\eta(B(Z,V)\xi)W + \eta(W)\eta(B(Z,V)Z) \right] = 0\]

From (3.14), by contracting we get

\[\left[ -2(n-3)(\rho - \alpha^2)^2 - 2(\alpha^2 - \rho)(k-1) \right] \{ g(V,W) + \eta(V)\eta(W) \} = 0\]

This implies that \(g(V,W) = -\eta(V)\eta(W)\). Then from (2.3) we get \(g(\phi V, \phi W) = 0\), however, as this is not possible. This proves the theorem3.3.
Theorem 3.4. A $(LC S)_n$-manifold is projectively Ricci symmetric if and only if the manifold in an Einstein manifold.

Proof. Assume that in $(LC S)_n$-manifold the condition $P(X, Y) \cdot S(Z, W) = 0$ are satisfies, and then it can be expressed as

\begin{equation}
S(P(X, Y)Z, W) + S(Z, P(X, Y)W) = 0
\end{equation}

From (1.8) and (2.11) we get

\begin{equation}
P(\xi, Y)Z = (\alpha^2 - \rho)\left[g(Y, Z)\xi - \eta(Z)Y\right] - \frac{1}{(n-1)}S(Y, Z)\xi - (n-1)(\alpha^2 - \rho)\eta(Z)Y
\end{equation}

Taking $X = \xi$ in (3.15) by virtue of (2.11) and (3.16) we obtain

\begin{equation}
S(Y, Z) = (n-1)(\alpha^2 - \rho) g(Y, Z)
\end{equation}

This proves the theorem 3.4.

Theorem 3.5. A $(LC S)_n$-manifold is concircularly Ricci symmetric if and only if either scalar curvature $r = n(n-1)(\alpha^2 - \rho)$ or the manifold in an Einstein manifold.

Proof. Assume that in $(LC S)_n$-manifold satisfies the condition $C(X, Y) \cdot S(Z, W) = 0$, and then it can be expressed as

\begin{equation}
S(C(X, Y)Z, W) + S(Z, C(X, Y)W) = 0
\end{equation}

From (1.9) and (2.11), we have

\begin{equation}
C(\xi, Y)Z = \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right]\{g(Y, Z)\xi - \eta(Z)Y\}
\end{equation}

Taking $X = \xi$ in (3.18) by virtue of (3.19) and (2.11), we get

\begin{equation}
\left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right]\{S(Y, Z) - 2n(\alpha^2 - \rho)g(Y, Z)\} = 0
\end{equation}

This implies that either $r = n(n-1)(\alpha^2 - \rho)$ or $S(Y, Z) = 2n(\alpha^2 - \rho)g(Y, Z)$

This proves the theorem 3.5

Theorem 3.6. A $(LC S)_n$-manifold satisfies the condition $P(\xi, X) \cdot S = 0$ if and only if the $M^n$ is an Einstein manifold with scalar curvature $r = 2n^2(\alpha^2 - \rho)$.
Proof. The condition \( P(\xi, X) \cdot S = 0 \) implies
\[
(3.21) \quad S(P(\xi, X) Y, \xi) + S(Y, P(\xi, X) \xi) = 0
\]

By virtue of (2.8) and (2.11), equation (1.8) reduces that
\[
(3.22) \quad S(P(\xi, X) Y, \xi) = -2n(\alpha^2 - \rho)^2 \left\{ g(X,Y) + \eta(X)\eta(Y) \right\} \\
+ \frac{1}{(n-1)} 2n(\alpha^2 - \rho) \left\{ S(X,Y) + 2n(\alpha^2 - \rho)\eta(X)\eta(Y) \right\}
\]
and
\[
(3.23) \quad S(P(\xi, X) \xi, \xi) = (\alpha^2 - \rho)^2 \left\{ S(X,Y) + 2n(\alpha^2 - \rho)\eta(X)\eta(Y) \right\} \\
- \frac{1}{(n-1)} 2n(\alpha^2 - \rho) \left\{ S(X,Y) + 2n(\alpha^2 - \rho)\eta(X)\eta(Y) \right\}
\]

Using (3.22)(3.23) in (3.21), we get
\[
S(X,Y) = 2n(\alpha^2 - \rho) g(X,Y)
\]
This proves the theorem 3.6

**Corollary 1.** In \((LCS)_n\)-manifold the D-conformal curvature tensor \( B \) satisfies
\[
(3.24) \quad B(X,Y) \xi = \lambda \left\{ \eta(Y)X - \eta(X)Y \right\}
\]
where
\[
\lambda = \frac{(n+1)(\rho - \alpha^2) + (k-1)}{(n-3)}
\]

**Proof.** Using (2.7) and (2.11) in (1.7) we get (3.24).

**Definition 3.2.** The rotational motion (curl) of D-conformal curvature tensor \( B \) on a Riemannian manifold is given by
\[
\]

By virtue of second Bianchi identity
\[
(3.26) \quad (\nabla_U B)(X,Y,Z) + (\nabla_X B)(Y,U,Z) + (\nabla_Y B)(U,X,Z) = 0
\]
Equation (3.25) reduces to
\[
Rot B = - (\nabla_Z B)(X,Y)U
\]
If the D-conformal curvature tensor is irrotational then curl \( B = 0 \) and by (3.26), we have
\[
(\nabla_Z B)(X,Y)U = 0
\]
This implies that

\( \nabla_Z (B(X,Y)U) = B(\nabla_Z X,Y)U + B(X,\nabla_Z Y)U + B(X,Y)\nabla_Z U \)  

In view of (3.27) with \( U = \xi \) it is seen that

\( \nabla_Z (B(X,Y)\xi) = B(\nabla_Z X,Y)\xi + B(X,\nabla_Z Y)\xi + B(X,Y)\nabla_Z \xi \)

**Theorem 3.7.** If the \( D \)-conformal curvature tensor in \((LCS)_n\)-manifold is irrotational then the \( D \)-conformal curvature tensor \( B \) is given by (3.30)

**Proof.** Using (3.24) and (1.5) in (3.28), we get

\( B(X,Y)\phi Z = \lambda \left[ (\nabla_Z \eta)(Y)X - (\nabla_Z \eta)(X)Y \right] \)

Replacing \( Z \) by \( \phi Z \) in (3.29) by using (1.3) and (1.6) it is seen that

\( B(X,Y)Z = \lambda \left[ \eta(\phi Z,Y)X - \eta(Y,\phi Z,Y) \right. \eta(Z)X + \eta(X)\eta(Z)Y \}

This proves the theorem 3.7.

**Theorem 3.8.** If the \( D \)-conformal curvature tensor in \((LCS)_n\)-manifold is irrotational then the manifold is an \( \eta \)-Einstein manifold with scalar curvature

\[
\tau = \left[ \frac{n(n-3)+2n[(n-1)(\alpha^2 - \rho) - (k-2)]}{(n-1)} \right]
\]

**Proof.** Using (3.21) in (1.7) the curvature tensor of \( B \) in \((LCS)_n\)-manifold is given by

\[
R(X,Y)Z = \lambda \left[ g(Z,Y)X - g(Z,X)Y - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y \right] - \frac{1}{n-3} \left[ S(Y,Z)Y - S(Y,Z)X + g(X,Z)QY \right]
\]

\[
- g(Y,Z)QX - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi
\]

\[
+ \eta(Y)\eta(Z)QX
\]

Let \( X_i, \quad i=1,2,3,...,n \) be an orthonormal basis of the tangent space at any point. Then the sum for \( 1 \leq i \leq n \) of the relation (3.31) with \( Y = D = X_i \), yields

\[
\sum R(X,X_i)X_i = \lambda \left[ g(X_i,X_i)X - g(X,X_i)X_i \right] - \frac{1}{n-3} \left[ S(X_i,X_i)X_i - S(X,X_i)X_i + g(X,X_i)QX_i \right]
\]

\[
+ \frac{(k-2)}{(n-3)} \left[ g(X_i,X_i)X_i - g(X_i,X_i)X \right] + \frac{k}{(n-3)} \left[ g(X_i,X_i)(X)\xi \right]
\]

(3.32)
The Ricci tensor $S$ is given by

\[ S(X,Y) = \sum g(R(X, X_i)X_i, Y) + g(X, Y) \]

Taking inner product of (3.32) with $Y$ and by virtue of (3.31) and (3.33), we get

\[ S(X,Y) = a g(X,Y) + b\eta(X)\eta(Y) \]

where

\[ a = \left[ \frac{(n-3)+2(n-1)(\alpha^2-\rho)-2(k-2)(n-1)}{n-1} \right], \quad \text{and} \quad b = (\alpha^2 - \rho) \]

This implies that the manifold is an $\eta$–Einstein manifold.

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REFERENCES RÉFÉRENCES REFERENCIAS