A Note on Semilattice Decompositions of Epigroups

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1. Introduction and Preliminaries

The relation $\rightarrow$ introduced by M. S. Putcha in [1] and T. Tamura in [2], plays a crucial role in semilattice decompositions of semigroups. General properties of the graphs that correspond to these relations were studied by M. S. Putcha in [3] and the structure of semigroups in which the minimal paths in the graph corresponding to $\rightarrow$ are bounded was described by M. Cirić and S. Bogdanović in [4]. The latter semigroups have also been studied by S. Bogdanović, M. Ćirić and Ž. Popović in [5]. Further, semilattice decompositions are especially interesting when they are considered for epigroups. A characterization of the least semilattice congruence on such semigroups was given by M. S. Putcha in [6], and by L. N. Shevrin (See the survey paper [7]). In [8], Ž. Popović, S. Bogdanović and M. Ćirić study epigroups admitting a decomposition into a semilattice of $\sigma_n$-simple semigroups and described them in terms of properties of their idempotents. In this paper we will give a note on semilattice decompositions of epigroups by using some relations, ideals on/of $S$ and certain special elements in $S$.

Now we give precise definitions of the notions used above and the ones that will be used in the further text. $\mathbb{N}$ will be used in the sequel to denote the set of all positive integers. Let $S$ be a semigroup. For a subset $A$ of $S$, we define

$$\sqrt{A} = \{x \in S | (\exists n \in \mathbb{N}) x^n \in A\}.$$ 

A subset $A$ of $S$ is completely semiprime if for any $x \in S, x^2 \in A$ implies $x \in A$. If $A$ is an ideal of $S$, then it is completely semiprime if and only if $\sqrt{A} \subseteq A$. The division relation $|$ and the relation $\rightarrow$ on $S$ are defined by

$$a|b \iff (\exists x, y \in S^1) b = xay, \quad a \rightarrow b \iff (\exists k \in \mathbb{N}) a|b^k.$$
For \( n \in \mathbb{N} \), \( n \geq 2 \), the relation \( \rightarrow^n \) on \( S \) is defined by
\[
a \rightarrow^n b \iff (\exists x \in S) a \rightarrow^{n-1} x \rightarrow b,
\]
and for \( n = 1 \), \( \rightarrow^1 \) is \( \rightarrow \). In other words, \( \rightarrow^n \) is the \( n \)-th power of \( \rightarrow \) in the semigroup of binary relations on \( S \). The transitive closure of \( \rightarrow \) is denoted by \( \rightarrow^\infty \). For \( n \in \mathbb{N} \) and \( a \in S \), the sets \( \Sigma_n(a) \) and \( \Sigma(a) \) are defined by
\[
\Sigma_n(a) = \{ x \in S \mid a \rightarrow^n x \}; \quad \Sigma(a) = \{ x \in S \mid a \rightarrow^\infty x \},
\]
and the equivalence relations \( \sigma_n \) and \( \sigma \) on \( S \) are defined by
\[
(a, b) \in \sigma_n \iff \Sigma_n(a) = \Sigma_n(b); \quad (a, b) \in \sigma \iff \Sigma(a) = \Sigma(b).
\]
In other words,
\[
\Sigma_1(a) = \sqrt{aSa}, \quad \Sigma_{n+1}(a) = \sqrt{a \Sigma_n(a)S} \supseteq \Sigma_n(a); \quad \text{and} \quad \Sigma(a) = \bigcup_{n \in \mathbb{N}} \Sigma_n(a).
\]
As it was proved by M. Ćirić and S. Bogdanović in \([4]\), \( \sigma \) is the least semilattice congruence on \( S \) and \( \Sigma(a) \) is the least completely semiprime ideal of \( S \) containing \( a \), called the principal radical of \( S \) generated by \( a \). The set \( \Sigma_n(a) \) is called the \( n \)-radical generated by \( a \). Let \( A \) be a nonempty subset of a semigroup \( S \). Then
\[
\Sigma(A) \overset{def}{=} \bigcup_{a \in A} \Sigma(a)
\]
is the least completely semiprime ideal of \( S \) containing \( A \). A semigroup \( S \) is \( \sigma_n \)-simple if \( \sigma_n \) coincides with the universal relation on \( S \), and \( \sigma_1 \)-simple semigroups are also called archimedean semigroups. The set of all idempotents of a semigroup \( S \) is denoted by \( E(S) \). If \( e \in E(S) \), then
\[
G_e = \{ x \in S \mid x \in eS \cap Se, \ e \in xS \cap Sx \}
\]
is the largest subgroup of \( S \) having \( e \) as its identity, called the maximal subgroup of \( S \) determined by \( e \), and the set \( K_e \) is defined by \( K_e = \sqrt{G_e} \). An element \( a \) of \( S \) is group-bound if at least one of its powers lies in some subgroup of \( S \). There is exactly one such subgroup, and its identity is denoted by \( a^\omega \). A semigroup \( S \) is called an epigroup if every element \( a \) of \( S \) is group-bound. Any epigroup \( S \) is partitioned into the subsets \( K_e \) called unipotency classes. The idempotent of the unipotency class to which an element \( a \) belongs will be denoted by \( e_a \) (here \( e_a = a^\omega \)). The element \( \overline{a} = (ae_a)^{-1} \) is the inverse of \( ae_a \) in the group \( G_{e_a} \). This element is called the pseudo-inverse of \( a \). The following equalities hold:
\[
\overline{a}a = a\overline{a} = e_a, \ e_a\overline{a} = \overline{a}, \ a^m e_a = a^m \text{ for some } m \in \mathbb{N}.
\]
We will denote by \( \mathcal{K} \) the equivalence relation on an epigroup \( S \) corresponding to the partition of the given epigroup \( S \) into its unipotency classes and \( \mathcal{H}, \mathcal{D} \) and \( \mathcal{J} \) are the well known Green relations.

For undefined notions and notations we refer to the book \([10]\).
II. THE MAIN RESULT

We start this section by recalling some results obtained from the paper [4] by M. Ćirić and S. Bogdanović, the paper [9] by Ž. Popović, S. Bogdanović and M. Ćirić.

**Theorem 2.1**[4] Let \( n \in \mathbb{N} \). Then the following conditions on a semigroup \( S \) are equivalent:

(i) \( S \) is a semilattice of \( \sigma_n \)-simple semigroups;
(ii) every \( \sigma_n \)-class of \( S \) is a subsemigroup;
(iii) for every \( a \in S \), \( \Sigma_n(a) \) is an ideal of \( S \);
(iv) \((\forall a, b \in S) \Sigma_n(ab) = \Sigma_n(a) \cap \Sigma_n(b)\);
(v) \((\forall a, b, c \in S) a \rightarrow^n b \land b \rightarrow^n c \implies a \rightarrow^n c\); 
(vi) \( \sigma_n = \rightarrow^n \cap (\rightarrow^n)^{-1} \) on \( S \).

**Theorem 2.2**[9] Let \( S \) be an epigroup and \( n \in \mathbb{N} \). Then \( S \) is a semilattice of \( \sigma_n \)-simple semigroups if and only if for every \( a \) of \( S \) \( a\sigma_n a\omega \).

Next we prove some auxiliary lemmas.

**Lemma 2.1** Let \( a \) be a group-bound element of a semigroup \( S \). Then for every \( b \in S \) and every \( n \in \mathbb{N} \), \( a \rightarrow^n b \) implies \( a \rightarrow^n b \). In other words, for every \( n \in \mathbb{N} \), 

\[ \Sigma_n(a) \subseteq \Sigma_n(b). \]

**Proof** Since \( a = a^2a \in SaS \), we have \( S\sigma S \subseteq SaS \). It follows that 

\[ \Sigma_1(a) = \sqrt{S\sigma S} \subseteq \sqrt{SaS} \subseteq \Sigma_1(a). \]

Now, by induction we easily verify that \( \Sigma_n(a) \subseteq \Sigma_n(a) \), for every \( n \in \mathbb{N} \).

**Lemma 2.2** Let \( a \) be some element of an epigroup \( S \). Then for every \( b \in S \) and every \( n \in \mathbb{N} \), 

\[ \pi \rightarrow^n b \text{ if and only if } a^\omega \rightarrow^n b. \]

In other words, for every \( n \in \mathbb{N} \), 

\[ \Sigma_n(\pi) = \Sigma_n(a^\omega). \]

**Proof** Since \( \pi H a^\omega \), then \( \pi D a^\omega \). This together with Lemma 5 in [4], and the known fact \( D = J \) for any epigroup, \( D \subset \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_n \subset \cdots \), we have \( \pi \sigma_n a^\omega \).

**Lemma 2.3** Let \( b \) be a group-bound element of a semigroup \( S \). Then for every \( a \in S \) and every \( n \in \mathbb{N} \), 

\[ a \rightarrow^n b \text{ if and only if } a \rightarrow^n b. \]

**Proof** Let \( m \in \mathbb{N} \) such that \( b^m \in G_{cb} \). Consider an arbitrary \( a \in S \). Suppose that \( a \rightarrow b \). Then \( b^k = uav \), for some \( u, v \in S \), \( k \in \mathbb{N} \), and thus 

\[ b^k = (uvb)^k = b^kb^k = b^k uavb^k \in SaS. \]

Hence we obtain \( a \rightarrow b \).
Conversely suppose that \(a \rightarrow \bar{b}\). Then \(b^k = uav\), for some \(u, v \in S\), \(k \in \mathbb{N}\), and hence for some \(m \in \mathbb{N}\)

\[
b^{mk} = b^{mk}b^ω = b^{mk}(b^ω)^k = b^{mk}(\bar{b})^k = b^{mk}\bar{b}^k = b^{mk}uavb^k \in SaS.
\]

Thus we get \(a \rightarrow b\). Therefore, we have proved that our assertion holds for \(n = 1\). By induction we easily verify that this assertion holds for every \(n \in \mathbb{N}\).

**Lemma 2.4** For any epigroup, we have \(K ∨ D = (\rightarrow ∩ \rightarrow^{-1})^∞\).

**Proof** It is easy to verify that \(K \subseteq (\rightarrow ∩ \rightarrow^{-1})\), \(D \subseteq (\rightarrow ∩ \rightarrow^{-1})\). Since the join \(K ∨ D\) is the smallest equivalence containing \(K\) and \(D\) and \((\rightarrow ∩ \rightarrow^{-1})^∞\) is an equivalence, it follows that \(K ∨ D \subseteq (\rightarrow ∩ \rightarrow^{-1})^∞\).

Conversely, by virtue of Corollary 3 in [6], \((\rightarrow ∩ \rightarrow^{-1})^∞\) is the transitive closure of \(\sim ∩ D\) on \(S\), where \(\sim\) is the Schwartz’s equivalence \((a \sim b\) if and only if \(a^i = b^j\) for some \(i, j \in \mathbb{N}\)). But then as \(\sim \subseteq K\), we have \((\rightarrow ∩ \rightarrow^{-1})^∞ \subseteq K ∨ D\).

For any ideal \(I\) of \(S\), we set

\[
Q_I = \sqrt{I}, \quad Q_{I_{n+1}} = \sqrt{SQ_{I_n}S} \supseteq Q_{I_n}, \quad n \in \mathbb{N}.
\]

Now we are prepared for the main result of the paper.

**Theorem 2.3** Let \(S\) be an epigroup and \(n \in \mathbb{N}\). Then the following conditions are equivalent:

(i) \(S\) is a semilattice of \(σ_n\)-simple semigroups;
(ii) \((∀a \in S)\ aσ_n\bar{a}\);
(iii) Every \(σ_n\)-class of \(S\) is a subepigroup;
(iv) \(√σ_n \subseteq σ_n\);
(v) \(√D \subseteq σ_n\);
(vi) \(K \subseteq σ_n\);
(vii) \(K ∨ D \subseteq σ_n\);
(viii) For any ideal \(I\) of \(S\), the set \(Q_{I_n}\) is ideal.

**Proof** (i)\(⇒\)(ii) For any element \(a\) of \(S\), \(a^i \in G_{e_a}\) and \(a^i = a^i\bar{a}\), for some \(i \in \mathbb{N}\). Then \(\bar{a} \rightarrow a\) and \(a[\bar{a}\) and if (i) holds, then by (vi) of Theorem 1 it follows \(aσ_n\bar{a}\).

(ii)\(⇒\)(iii) By Lemma 2.3, for every \(a \in S\), \(aσ_n\bar{a}σ_n a\bar{a}\). Notice that \(σ_n\) is an equivalence relation on \(S\). Again by Theorem 2.1, \(S\) is a semilattice of \(σ_n\)-simple semigroups and Every \(σ_n\)-class of \(S\) is a subsemigroup. This together with the assumption of (ii), every \(σ_n\)-class of \(S\) is a subepigroup, since a subsemigroup of an epigroup that is closed under pseudo-inversion is a subepigroup.

(iii)\(⇒\)(iv) Let \(a√σ_n b\). Then \(a^mσb^n\) for some \(m, n \in \mathbb{N}\). By hypothesis and Theorem 2.1 we have \(aσ_n a^mσb^n\bar{a}\). Thus \(aσ_n b\), which was to be proved.

(iv)\(⇒\)(v) By Lemma 5 in [4] (see the proof of Lemma 2.2) we have \(D \subseteq σ_n\) and thus \(√D \subseteq √σ_n\). Therefor (v) holds.

(v)\(⇒\)(vi) It is known that in epigroup \(K \subseteq √H \subseteq √D\). So (vi) holds.

(vi)\(⇒\)(vii) By (vi) we have \(K ∨ D \subseteq σ_n\), since \(D \subseteq σ_n\) always holds and these relation are all equivalence relations on \(S\).
(vii)$\Rightarrow$(viii) Notice that $a(K \vee D)a^\omega$ holds such that $a\sigma_n a^\omega$ by assumption, hence $S$ is a semilattice of $\sigma_n$-simple semigroups by Theorem 2.2 and thus, $\longrightarrow^n=\longrightarrow^\infty$, which implies $\Sigma_n(a)=\Sigma(a)$. Hence for any nonempty subset $A$ of $S$,

$$\Sigma_n(A) = \bigcup_{a \in A} \Sigma_n(a) = \bigcup_{a \in A} \Sigma(a) = \Sigma(A)$$

is the smallest completely semiprime ideal of $S$ containing $A$. Let $I$ be an ideal of $S$ and $a \in I$. Then

$$SaS \subseteq I, \Sigma_1(a) \subseteq Q_{I_1}, \ldots, \Sigma_n(a) \subseteq Q_{I_n}$$

and thus $\Sigma_n(I) = \Sigma(I) \subseteq Q_{I_n}$. On the other hand, for any $b \in Q_{I_n}$, that is,

$$a \longrightarrow x_1 \longrightarrow \ldots \longrightarrow x_{n-1} \longrightarrow b,$$

where $a \in I, x_i \in Q_{I_n}, 1 \leq i < n, i \in \mathbb{N}$. It follows that $b \in \Sigma_n(a)$ and thus $Q_{I_n} \subseteq \Sigma(I)$.

(viii)$\Rightarrow$(i) For any $a \in S$, Let $I = S^1aS^1$. Obviously $I$ is an ideal of $S$. Then by the hypothesis of (viii), together with Lemma 1 in [4] and Theorem 2.1, $\Sigma_n(a)$ is an ideal. Again by Theorem 2.1, $S$ is a semilattice of $\sigma_n$-simple semigroups.

**Remark** Notice that in the proof ((vii)$\Rightarrow$(viii)) of Theorem 2.3, $a(K \vee D)a^\omega$ always holds such that $a\sigma_n a^\omega$ by assumption and hence $\sigma_n$ is a semilattice congruence on $S$. Therefor by Lemma 2.4, $K \vee D = (\longrightarrow \cap \longrightarrow^{-1})^\infty \subseteq \sigma_n$ prevails, since $(\longrightarrow \cap \longrightarrow^{-1})^\infty$ is the smallest semilattice congruence on $S$.

**REFERENCES**