The Collatz 3n+1 Conjecture is Unprovable

By Craig Alan Feinstein

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In 2005, the famous mathematician Freeman Dyson was asked, “What do you believe is true even though you cannot prove it?” He answered:

“Since I am a mathematician, I give a precise answer to this question. Thanks to Kurt Gödel, we know that there are true mathematical statements that cannot be proved. But I want a little more than this. I want a statement that is true, unprovable, and simple enough to be understood by people who are not mathematicians. Here it is.

“Numbers that are exact powers of two are 2, 4, 8, 16, 32, 64, 128 and so on. Numbers that are exact powers of five are 5, 25, 125, 625 and so on. Given any number such as 131072 (which happens to be a power of two), the reverse of it is 270131, with the same digits taken in the opposite order. Now my statement is: it never happens that the reverse of a power of two is a power of five.

“The digits in a big power of two seem to occur in a random way without any regular pattern. If it ever happened that the reverse of a power of two was a power of five, this would be an unlikely accident, and the chance of it happening grows rapidly smaller as the numbers grow bigger. If we assume that the digits occur at random, then the chance of the accident happening for any power of two greater than a billion is less than one in a billion. It is easy to check that it does not happen for powers of two smaller than a billion. So the chance that it ever happens at all is less than one in a billion. That is why I believe the statement is true.

“But the assumption that digits in a big power of two occur at random also implies that the statement is unprovable. Any proof of the statement would have to be based on some non-random property of the digits. The assumption of randomness means that the statement is true just because the odds are in its favor. It cannot be proved because there is no deep mathematical reason why it has to be true. (Note for experts: this argument does not work if we use powers of three instead of powers of five. In that case the statement is easy to prove because the reverse of a number divisible by three is also divisible by three. Divisibility by three happens to be a non-random property of the digits).

“It is easy to find other examples of statements that are likely to be true but unprovable. The essential trick is to find an infinite sequence of events, each of which might happen by accident, but with a small total probability for even one of them happening. Then the statement that none of the events ever happens is probably true but cannot be proved.” [1]

In the spirit of Dyson’s observation, we shall give an example of a statement that is likely to be true and then take things one step further by presenting a formal proof that the statement is unprovable. Consider the following function:
**Definition 1:** Let $T : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $T(n) = \frac{3n+1}{2}$ if $n$ is odd and $T(n) = \frac{n}{2}$ if $n$ is even.

The Collatz $3n + 1$ Conjecture states that for each $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $T^{(k)}(n) = 1$, where $T^{(k)}(n)$ is the function $T$ iteratively applied $k$ times to $n$ [2]. As of May 10, 2011, this conjecture has been verified for all positive integers up to about $2^{60}$ [3]. Furthermore, one can give a heuristic probabilistic argument [4] that since every iterate of the function $T$ decreases on average by a multiplicative factor of about $(\frac{3}{2})^{1/2} (\frac{1}{2})^{1/2} = (\frac{3}{4})^{1/2}$, all iterates will eventually converge into the infinite cycle $\{1, 2, 1, 2, \ldots\}$, assuming that each $T^{(k)}$ sufficiently mixes up $n$ as if each $T^{(k)}(n)$ (mod 2) were drawn at random from the set $\{0, 1\}$.

However, the Collatz $3n+1$ Conjecture has never been formally proven. We shall prove that the Collatz $3n+1$ Conjecture can, in fact, never be formally proven, even though there is a lot of evidence for its truth. The underlying assumption in our argument is that any proof of a theorem can be written in a computer text-file, which is composed of bits (zeros and ones). First, let us present a definition of “random”.

**Definition 2:** We shall say that vector $x \in \{0,1\}^k$ is random if $x$ cannot be specified in less than $k$ bits in a computer text-file [5].

For example, the vector of one million concatenations of the vector $(0, 1)$ is not random, since we can specify it in less than two million bits in a computer text-file by just writing, “the vector of one million concatenations of the vector $(0, 1)$” in the text-file. However, the vector of outcomes of one million coin-tosses has a good chance of fitting our definition of “random”, since much of the time the most compact way of specifying such a vector is to simply make a list of the results of each coin-toss, in which one million bits are necessary. We now prove three theorems.

**Theorem 1:** For any vector $x \in \{0, 1\}^k$, there exists an $n \in \mathbb{N}$ such that $x = (n, T(n), \ldots, T^{(k-1)}(n))$ (mod 2).

**Proof:** A proof of this can be found in “The $3x + 1$ problem and its generalizations” [2].

**Theorem 2:** If $k, n \in \mathbb{N}$ and $T^{(k)}(n) = 1$, then in order to prove that $T^{(k)}(n) = 1$, it is necessary to specify the values of $(n, T(n), \ldots, T^{(k-1)}(n))$ (mod 2) in the proof.

**Proof:** Let the vector $(x_0(n), x_1(n), \ldots, x_{k-1}(n))$ equal $(n, T(n), \ldots, T^{(k-1)}(n))$ (mod 2). Then notice that the formula, $T^{(k)}(n) = \lambda_k(n)n + \rho_k(n)$ [2], where

$$\lambda_k(n) = \frac{3x_{0}(n)+\ldots+x_{k-1}(n)}{2^k}$$

and

$$\rho_k(n) = \sum_{i=0}^{k-1} x_i(n) \frac{3x_{i+1}(n)+\ldots+x_{k-1}(n)}{2^{k-i}}.$$

is determined by the values of $(n, T(n), \ldots, T^{(k-1)}(n))$ (mod 2) and there is a one-to-one correspondence between all of the possible formulas for $T^{(k)}(n)$ and all of the possible values of $(n, T(n), \ldots, T^{(k-1)}(n))$ (mod 2); therefore, in order to prove that $T^{(k)}(n) = 1$, it
is necessary to specify the values of \((n, T(n), \ldots, T^{(k-1)}(n)) \pmod{2}\) in the proof, since in order to prove that \(T^{(k)}(n) = 1\), it is necessary to specify the formula for \(T^{(k)}(n)\) in the proof.

\textbf{Theorem 3:} It is impossible to prove the Collatz 3n+1 Conjecture.

\textit{Proof:} Suppose that there exists a proof of the Collatz 3n+1 Conjecture, and let \(L\) be the number of bits in such a proof. Now, let \(\mathbf{x} \in \{0, 1\}^{L+1}\) be a random vector, as defined above. (It is not difficult to prove that at least half of all vectors in \(\{0, 1\}^{L+1}\) are random \cite{5}.) By Theorem 1, there exists an \(n \in \mathbb{N}\) such that \(\mathbf{x} = (n, T(n), \ldots, T^{(L)}(n)) \pmod{2}\) and \(T^{(L+1)}(n) = T^{(L)}(n) \pmod{2}\). Then \(T^{(L)}(n) > 2\), so if \(T^{(k)}(n) = 1\), then \(k > L\). Hence, by Theorem 2 it is necessary to specify the values of \((n, T(n), \ldots, T^{(L)}(n)) \pmod{2}\) in order to prove that there exists a \(k \in \mathbb{N}\) such that \(T^{(k)}(n) = 1\). But since \((n, T(n), \ldots, T^{(L)}(n)) \pmod{2}\) is a random vector, at least \(L + 1\) bits are necessary to specify \((n, T(n), \ldots, T^{(L)}(n)) \pmod{2}\), contradicting our assumption that the proof contains only \(L\) bits; therefore, a formal proof of the Collatz 3n+1 Conjecture cannot exist.

\textbf{References Références Referencias}

1. Dyson, F., “What do you believe is true even though you cannot prove it?”, http://www.edge.org/q2005/q05_9.html
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