Boundary-fixed Homeomorphisms of 2-Manifolds with Boundary

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Abstract - Let $X$ be a closed, orientable 2-manifold and let $X_n$ denote the bounded manifold obtained by removing the interiors of $n$ disjoint closed disks from $X$. Let $H(X_n)$ denote the group of isotopy classes (rel boundary of $X_n$) of homeomorphisms of $X_n$ which are the identity on the boundary of $X_n$. $H(X_n)$ has been determined for all $n$ when $X$ is the 2-sphere (see [8] and [10]). This paper investigates the structure of $H(X_n)$ for $X$ not equal to the 2-sphere. In particular, a relationship between $H(X_n)$ and the homeotopy group (mapping class group) of $X$ (see [4],[5] and [11]) is developed.
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I. INTRODUCTION.

Let \( X \) be a closed, orientable 2-manifold and let \( D_1, ..., D_n \) be disjoint closed disks in \( X \) with \( p_k \) a point in the interior of \( D_k \) for \( 1 \leq k \leq n \). Let \( X_n = X - \bigcup_{k=1}^{n} \text{Int}(D_k) \) and \( F_n = \{ p_1, ..., p_n \} \). This paper is concerned with the group \( H(X_n) \) consisting of all isotopy classes (rel \( \partial X_n \)) of homeomorphisms of \( X_n \) which are the identity on the boundary of \( X_n \). Note that in order for two boundary-fixed homeomorphisms of \( X_n \) to represent the same element in \( H(X_n) \) not only must these homeomorphisms be isotopic, but also the isotopy between them must be the identity on the boundary of \( X_n \) for all values \( t, 0 \leq t \leq 1 \). Presentations of \( H(X_n) \) for all \( n \) in the case that \( X \) is the 2-sphere are given in [8]. In this paper it will be shown that if \( X \) is not the 2-sphere, then \( H(X_n) \) can be obtained as part of a short exact sequence involving the free abelian group on \( n \) generators, denoted \( \mathbb{Z}^n \), and the subhomeotopy group of \( X \) consisting of all isotopy classes (rel \( F_n \)) of orientation preserving homeomorphisms of \( X \) which are the identity on \( F_n \), denoted \( H(X, F_n) \). This short exact sequence will then be used to relate \( H(x_n) \) to the homeotopy group of \( X \).

II. BOUNDARY-FIXED HOMEOMORPHISMS

Let \( f : H(X_n) \to H(X, F_n) \) be the function which sends the isotopy class (rel \( \partial X_n \)) of a boundary-fixed homeomorphism \( h \) of \( X_n \) onto the isotopy class (rel \( F_n \)) of the homeomorphism of \( X \) which is obtained by extending \( h \) by the identity over each disk \( D_k \), \( 1 \leq k \leq n \). The function \( f \) is clearly well-defined since any isotopy (rel \( \partial X_n \)) of \( X_n \) can be extended by the identity on \( \bigcup_{k=1}^{n} D_k \) to an isotopy (rel \( F_n \)) of \( X \). In fact, \( f \) is an epimorphism since any orientation preserving homeomorphism of \( X \) which is the identity on \( F_n \) can be isotope (rel \( F_n \)) to a homeomorphism which is the identity on \( \bigcup_{k=1}^{n} D_k \) (for details see Part 3c of Lemma 3 of [7]). Moreover using Part 4 of Lemma 3 of [7] we have that a homeomorphism of \( X_n \) is in the kernel of \( f \) if and only if it is isotopic to the identity. That is, the nontrivial elements of the kernel of \( f \) are represented by boundary-fixed homeomorphisms that are isotopic to the identity, but not by an isotopy that keeps the boundary of \( X_n \) fixed. The next two lemmas are concerned with finding representatives of the isotopy classes (rel \( \partial X_n \)) of such homeomorphisms.

Let \( K(X_n) \) denote the kernel of \( f \) and let \( A_k \) be a collar neighborhood of \( \partial D_k \) for \( 1 \leq k \leq n \), with \( A_i \cap A_j = \emptyset \) if \( i \neq j \).

Lemma 1: Every element of \( K(X_n) \) can be represented by a homeomorphism that is the identity on \( X - \bigcup_{k=1}^{n} \text{Int}(A_k) \).
Proof: Let $h$ be a homeomorphism that represents an element of $K(X_n)$ and let $h_i$ be an isotopy that takes $h$ to the identity. Using the “unwinding” technique of Proposition 3.22 of [6] it is possible to extend $h_i^{-1}$ to an isotopy $g_i$ of $X_n$ that takes the identity to a homeomorphism that is the identity on $X - \bigcup_{k=1}^n \text{Int}(A_k)$. The isotopy $g_i h_i$ is then an isotopy (rel $\partial X_n$) that takes $h$ to a homeomorphism of the type given in the statement of the lemma.

For each $k$, $1 \leq k \leq n$, let $s_k : \partial A_k \rightarrow \partial A_k$ be the homeomorphism given in which spins one component of $\partial A_k$ $r$-times while holding the other boundary component fixed and by letting $s_k$ restricted to $X_n - A_k$ be the identity. $s_k$ will be referred to as a “spin homeomorphism” of $X_n$.

Lemma 2: If $X$ is not the 2-sphere, then every element of $K(X_n)$ has a unique representation as a product of spin homeomorphisms.

Proof: A consequence of Theorem 7.2 of [3] is that every homeomorphism of $A_k$ which is the identity on $\partial A_k$ is isotopic (rel $\partial A_k$) to $s_k / A_k$ for some $r$. Since $A_i$ is disjoint from $A_j$ for $i \neq j$, this means that any homeomorphism of $X_n$ which is the identity on $X_n - \bigcup_{k=1}^n \text{Int}(A_k)$ is isotopic (rel $\partial X_n$) to a product of homeomorphisms of the form $s_1 \cdots s_n$. Thus by Lemma 1, every element of $K(X_n)$ can be represented by a product of spin homeomorphisms.

To show that the representation if unique it suffices to show that if $s_1 \cdots s_n$ is isotopic (rel $\partial X_n$) to the identity, then $r_i = 0$ for $1 \leq i \leq n$. On the contrary, assume this product is isotopic (rel $\partial X_n$) to the identity, but $r_k \neq 0$ for some $k$. Let $\alpha$ be a curve which represents a generator of $\pi_1(X_n, q)$ where $q$ is in $\partial D_k$ and $\alpha$ is chosen so that $\alpha \cap A_j = \emptyset$ for $i \neq j$.

Let $\beta_k$ be a curve based at $q$ which wraps once around $\partial D_k$ in the direction of the spin corresponding to $s_k$. In the free group $\pi_1(X_n, q)$, the spin homeomorphism $s_k$ represents the same element as $\beta_k^{-r_k} \alpha \beta_k^{r_k}$. However, since $s_1 \cdots s_n$ is isotopic (rel $\partial X_n$) to the identity and $\alpha$ is outside the support of $s_k$, for $i \neq k$, we have that $s_k^{r_k}(\alpha)$ must also represent the same element as $\alpha$ in the free group $\pi_1(X_n, q)$. This contradiction establishes the lemma.

It should be noted that Lemma 2 is false in the case that $X$ is the 2-sphere and $n < 3$. For example, when $X$ is the 2-sphere and $n = 1$, then every spin homeomorphism of $X_1$ is isotopic (rel $\partial X_1$) to the identity (see [8]).

The next theorem is an immediate consequence of the fact that since the elements of $K(X_n)$ can be represented uniquely as products of spin homeomorphisms and these spin homeomorphisms all commute, the kernel of the epimorphism $f$ is the free group on $n$ generators.

Theorem: If $X$ is not the 2-sphere, then the following sequence is exact:

$1 \rightarrow \mathbb{Z}^n \rightarrow H(X_n) \rightarrow H(X, F_n) \rightarrow 1$ where the function from $H(X_n)$ to $H(X, F_n)$ is given by $f$ as defined at the beginning of this section.

The above theorem shows that $H(X_n)$ can be obtained as an extension of $\mathbb{Z}^n$ by $H(X, F_n)$. In turn $H(X, F_n)$ is part of the short exact sequence

$1 \rightarrow \pi_1(X - F_n^{-1}, p_n) \rightarrow H(X, F_n) \rightarrow H(X, F_n^{-1}) \rightarrow 1$ where the function from $H(X, F_n)$ to $H(X, F_n^{-1})$ sends the isotopy class (rel $F_n$) of a homeomorphism of $X$ to the isotopy class (rel $F_n^{-1}$) of this homeomorphism. If we denote this function $\mathcal{D}$, then the representation of an element in the kernel of $\mathcal{D}$ as an element in $\pi_1(X - F_n^{-1}, p_n)$ is obtained by taking the curve formed by tracing the path of $p_n$ during the isotopy (rel $F_n^{-1}$) which takes a representative homeomorphism of an element in the kernel of $\mathcal{D}$ to the identity.

Thus, we can build up $H(X, F_n)$ from the homeotopy group of $X$, $H(X)$, by repeatedly extending $\pi_1(X - F_k)$ by $H(X, F_k)$ for $k = 1, \cdots, n - 1$ (see [1] and [2]).

References Références Referencias


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