Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 21 Issue 1 Version 1.0 Year 2021
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Generic Rank-2 Perturbation of Hamiltonian Systems with Periodic Coefficients

By Mouhamadou DOSSO
Université Félix Houphouët-Boigny

Abstract- In this paper, it is about a theory of double rank-one perturbation of a Hamiltonian system with periodic coefficients. Some reminders of the rank-one perturbation and an adaptation of a theorem given in [C. Mehl, et al., Linear Algebra Appl. J., 435(2011), 687-716] to the cases of symplectic matrices have been made.

Keywords: *hamiltonian system, rank-one perturbation, symplectic matrix.*

GJSFR-F Classification: *MSC 2010: MSC 2010: 15A21, 47A55, 93B10, 93C73*

GENERICRANK2PERTURBATIONOFHAMILTONIANSYSTEMSWITHPERIODICCOEFFICIENTS

Strictly as per the compliance and regulations of:

Notes

Generic Rank-2 Perturbation of Hamiltonian Systems with Periodic Coefficients

Mouhamadou DOSSO

Abstract- In this paper, it is about a theory of double rank-one perturbation of a Hamiltonian system with periodic coefficients. Some reminders of the rank-one perturbation and an adaptation of a theorem given in [C. Mehl, et al., Linear Algebra Appl. J., 435(2011), 687-716] to the cases of symplectic matrices have been made.

Keywords: *hamiltonian system, rank-one perturbation, symplectic matrix.*

I. Introduction

Let $J \in \mathbb{R}^{2n\times 2n}$ (n is a fixed non-zero positive integer) be skew-symmetric and nonsingular (i.e. $J^T = -J$) and τ be a positive real. Consider the following Hamiltonian system with τ-periodic coefficients

$$\begin{cases} J\dfrac{dX(t)}{dt} &= H(t)X(t) \\ X(0) &= I \end{cases} \tag{1.1}$$

where $t \mapsto H(t) \in \mathbb{R}^{2n\times 2n}$ is a piecewise continuous matrix function on $[0, \tau]$ such that

$$H(t+\tau) = H(t) = (H(t))^T, \ \ \forall t \in \mathbb{R}.$$

Throughout this paper, the identity and zero matrices of order k are denoted by I_k and 0_k or just I and 0 whenever the order is clear from the context.

The solution of the system (1.1) is called the fundamental solution of the Hamiltonian system with τ-periodic coefficients $J\dfrac{dX(t)}{dt} = H(t)X(t), \ \forall t \in \mathbb{R}$. It verifies $\forall t \in \mathbb{R}, \ \ X(t)^T JX(t) = J$ and satisfies the following relationship

$$X(t + p\tau) = X(t)X^p(\tau) \neq X^p(\tau)X(t), \ \ \forall (t, p) \in \mathbb{R} \times \mathbb{N}.$$

We say that the solution of (1.1) has a symplectic structure. Recall that a matrix $W \in \mathbb{R}^{2n\times 2n}$ has a J-symplectic structure or W is J-symplectic (or J-orthogonal) if it verifies $W^T JW = J$.

The symplectic matrices come very often from Hamiltonian differential systems with periodic coefficients (see [14, Chapter 3]). Besides many problems in physics and engineering lead to systems of linear differential equations with periodic coefficients consequently to Hamiltonian systems with periodic coefficients. This gives an important place to the study of these systems ; particularly to the study of the stability of Hamiltonian systems which is closely related to the analysis of their perturbations. Regarding stability (strong stability) of system (1.1), we have the following definition

Author: *Université Félix Houphouët-Boigny de cocody-Abidjan. Laboratoire de Mathématiques fondamentales et Application. UFR de Mathématiques et Informatique, 22 BP 582 Abidjan 22, Côte d'Ivoire. e-mail: mouhamadou.dosso@univ-fhb.edu.ci*

Definition 1.1 1. *System (1.1) is stable if its solution $X(t)$ remains bounded for all $t \in \mathbb{R}$.*

2. *System (1.1) is strongly stable if any Hamiltonian system with τ-periodic coefficients sufficiently close to (1.1) is stable.*

Specifically, system (1.1) is strongly stable if there exists $\varepsilon > 0$ such that any Hamiltonian system with τ-periodic coefficients of the form $J\frac{\widetilde{X}(t)}{dt} = \widetilde{H}(t)\widetilde{X}(t)$ and satisfying $\|H - \widetilde{H}\| \equiv \int_0^\tau \|H(t) - \widetilde{H}(t)\|dt < \varepsilon$, is stable. Therefore, we focus our study in this paper to study of a type of perturbation of Hamiltonian system with periodic coefficients called rank-one pertubation studied by Mehl, et al. in [11, 12] but within the framework of a structured matrix such as a symplectic matrix. In some of our work, we have defined from the work of Mehl, et al. the rank-one perturbation of a Hamiltonian system with τ-periodic coefficients [1, 2, 5].

In this paper, we consider the case of generic structure-preserving rank-2 perturbation of system (1.1). Let us recall the meta-conjecture resulting from a numerical experiment with random perturbations [4].

Meta -Conjecture 1 *Let $W \in \mathbb{R}^{p\times p}$ be a structured matrix with respect to some indefinite inner product and $E \in \mathbb{R}^{p\times p}$ be a matrix of rank k so that $W + E$ is of the same structure class as W. Then generically the Jordan structure and sign characteristic of $W + E$ are the same that one would obtain by performing a sequence of k generic structure-preserving rank-one perturbations on W.*

Let us give some reminders on generic sets [3, 11]

Definition 1.2 1. *A set $\Omega \subseteq \mathbb{R}^{2n}$ is said to be algebraic if there exists a finite set of polynomials $p_1(x_1, ..., s_{2n}), ..., p_k(x_1, ..., x_{2n})$ with real coefficients such that $(\alpha_1, \alpha_2, ..., \alpha_{2n})^T \in \Omega$ if and only if $p_j(\alpha_1, ..., \alpha_{2n}) = 0, \ \forall j = 1, ..., k$.*

2. *An algebraic set $\Omega \subset \mathbb{R}^{2n}$ is said non-trivial if $\Omega \neq \mathbb{R}^{2n}$.*

3. *A non-trivial set $\Omega \in \mathbb{R}^{2n}$ is said to be generic if Ω is not empty and $\mathbb{R}^{2n} \setminus \Omega$ is contained in a finite union of non-trivial algebraic sets.*

Recall a result of [12] to the case of unstructured generic rank-one perturbation theory

Theorem 1.1 *Let $W \in \mathbb{C}^{\ell\times\ell}$ be a matrix having the pairwise distinct eigenvalues $\lambda_1, ..., \lambda_p$ with geometric multiplicities $r_1, ..., r_p$ and having the Jordan canonical form*

$$\bigoplus_{k=1}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{r_p} \mathcal{J}_{\ell_{p,k}}(\lambda_p),$$

where $l_{j,1} \geq \cdots \geq l_{j,r_j}$, $j = 1, ..., p$. Consider the rank one matrix $E = uv^T$, with $u, v \in \mathbb{C}^\ell$. The generically (with respect to the entries of u and v) the Jordan blocks of $W + E$ with eigenvalue λ_j are just the $r_j - 1$ smallest Jordan blocks of W with eigenvalue λ_j, and all other eigenvalues of $W + E$ are simple ; if $r_j = 1$, then generically λ_j is not an eigenvalue $W + E$.
More precisely, there is a generic set $\Omega \subseteq \mathbb{C}^\ell \times \mathbb{C}^\ell$ such that for every $(u, v) \in \Omega$, the Jordan structure of $W + uv^T$ is described in (a) and (b) bellow :

(a) the Jordan structure of $W + uv^T$ for the eigenvalues $\lambda_1, ..., \lambda_p$ is given by

$$\bigoplus_{k=2}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=2}^{r_p} \mathcal{J}_{\ell_{p,k}}(\lambda_p) ;$$

(b) the eigenvalues of $W + uv^T$ that are different from any of $\lambda_1, ..., \lambda_p$, are all simple.

In the rest of the paper, we will recall, in section 2, the rank-one perturbation of symplectic matrices and the Hamiltonian system with periodic coefficients. In this part, an adaptation of Theorem 1.1 to the case of symplectic matrices will be given. As for section 3, it defines the rank-2 perturbation as a double rank-one perturbation of a Hamiltinian system with periodic coefficients.

Global Journal of Science Frontier Research (F) Volume XXI Issue I Version I Year 2021

Ref
4. L. Batzke, C. Mehl, A. C.M. Ran and L. Rodman, Generic rank-k Perturbations of Structured Matrices. Operator Theory, Function Spaces, and Applications Birkhuser, Cham. (2016), p. 27-48.

II. Generality on a Rank-One Perturbation Theory

a) Generic rank-one perturbation of a symplectic matrix

Consider a symplectic matrix $W \in \mathbb{R}^{2n\times 2n}$ and an anti-symmetric matrix $J \in \mathbb{R}^{2n\times 2n}$. Recall that the spectrum of any symplectic matrix of order $2n$ is divided into three groups of eigenvalues : n_0 eigenvalues inside the unit circle, $n_\infty = n_0$ eigenvalues outside the unit circle and symmetrically placed with respect to the first group, and $n_1 = 2(n - n_0)$ eigenvalues on the unit circle [8, 9, 10]. These types of matrices which belong to a group of so-called structured matrices, have simple and useful spectral properties that we recall in the following theorem ([8])

Theorem 2.1 Let $W \in \mathbb{R}^{2n\times 2n}$ be a symplectique matrice. Then any eigenvalue of W verifies : .for all eigenvalue λ of W,

1. *if $\lambda \in \mathbb{C}^*$, with $|\lambda| \neq 1$, then $\bar{\lambda}$, $1/\lambda$ and $1/\bar{\lambda}$ are eigenvalues of W;*
2. *if $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ then $\bar{\lambda}$ is an eigenvalue of W;*
3. *if $\lambda \in \mathbb{R}^*$, $1/\lambda$ is an eigenvalue of W.*

Regarding a rank-one perturbation treated by Mel, et al. in [11], we have the following lemma :

Lemma 2.1 If W and $\widetilde{W} \in \mathbb{R}^{2n\times 2n}$ are J-symplectiques such that

$$rg\left(\widetilde{W} - W\right) = 1,$$

then there exists a vector $u \in \mathbb{R}^{2n}$ verifying

$$\widetilde{W} = (I + cuu^T J)W, \tag{2.1}$$

where $c = \pm 1$. Moreover for all $u \in \mathbb{R}^{2n}$, the matrix $\widetilde{W}$ is J-symplectic.

Proof
The hypothesis

$$rg\left(\widetilde{W} - W\right) = 1,$$

implies that there exists two non-zero vectors $\widehat{u}$ and $v \in \mathbb{R}^{2n}$ such that

$$\widetilde{W} = W + \widehat{u}v^T.$$

Thus, $\widetilde{W}$ J-symplectic implies $\widetilde{W}^T J\widetilde{W} = J$ and we have

$$(W + \widehat{u}v^T)^T J(W + \widehat{u}v^T) = J$$

$$\underbrace{W^T JW}_{=J} + v\widehat{u}^T JW + W^T J\widehat{u}v^T + v\underbrace{\widehat{u}^T J\widehat{u}}_{=0}v^T = J$$

this implies

$$v\widehat{u}^T JW + W^T J\widehat{u}v^T = 0, \tag{2.2}$$

which gives, by multiplying on the right by v,

$$v\widehat{u}^T JWv + W^T J\widehat{u}v^T v = 0.$$

Ref

8. M. Dosso, Sur quelques algorithms d'analyse de stabilit'e forte de matrices symplectiques, PHD Thesis (September 2006), Université de Bretagne Occidentale. Ecole Doctorale SMIS, Laboratoire de Mathématiques, UFR Sciences et Techniques.

We deduce

$$W^T J\widehat{u} = -v\frac{\widehat{u}^T JWv}{v^T v}, \quad (\text{since} \ \ v \neq 0)$$

$$= -v\frac{\widehat{u}^T w}{v^T v}, \quad \text{where} \ \ w = JWv.$$

By setting

$$n = \frac{\widehat{u}^T w}{v^T v},$$

We get $W^T J\widehat{u} = -nv$ which shows that v and $W^T J\widehat{u}$ are collinear. Thus, there is a non-zero real constant α such as $v = -\alpha W^T J\widehat{u}$. Therefore

$$\widetilde{W} = \left(I + \alpha\widehat{u}\widehat{u}^T J\right) W, \quad \text{where} \ \ \alpha = \pm 1.$$

Since the matrix $\alpha\widehat{u}\widehat{u}^T J$ is J-Hamiltonian $\left(\left(J\alpha\widehat{u}\widehat{u}^T J\right)^T = J\alpha\widehat{u}\widehat{u}^T\right)$, according to point 3) of Lemma 2.3 of [4], there is a vector $u \in \mathbb{R}^{2n}$ and a constant $c = \pm 1$ such that $\alpha\widehat{u}\widehat{u}^T J = cuu^T J$. Therefore

$$\widehat{W} = \left(I + c\, uu^T J\right) W, \quad \text{where} \ \ c = \pm 1.$$

Moreover, for any $u \in \mathbb{R}^{2n}$, we easily show that $\widetilde{W} J \widetilde{W} = J$.

We have the following general definitions

Definition 2.1 *Let $W \in \mathbb{R}^{2n\times 2n}$ be a symplectic matrix. We call rank-one perturbation of W, any symplectic matrix $\widetilde{W}$ of the form*

$$\widetilde{W} = (I + c\, uu^T)W, \tag{2.3}$$

where $c = \pm 1$ and $u \in \mathbb{R}^{2n}$.

In Theorem (1.1), if we consider a J-symplectic matrix $W \in \mathbb{R}^{2n\times 2n}$ and take $v = W^T J^T u$, we get the following theorem

Theorem 2.2 *Let $W \in \mathbb{R}^{2n\times 2n}$ be a matrix having the pairwise distinct eigenvalues $\lambda_1, ..., \lambda_{2p}$ with geometric multiplicities $r_1, ..., r_{2p}$ and having the Jordan canonical form*

$$\bigoplus_{k=1}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{r_{2p}} \mathcal{J}_{\ell_{2p,k}}(\lambda_{2p}), \tag{2.4}$$

where $l_{j,1} \geq \cdots \geq l_{j,r_j}$, $j = 1, ..., 2p$. Consider the rank one matrix $E = uu^T JW$, with $u \in \mathbb{R}^{2n}$. Then generically (with respect to the entries of u) the Jordan blocks of $W + E$ with eigenvalue λ_j are just the $r_j - 1$ smallest Jordan blocks of W with eigenvalue λ_j, and all other eigenvalues of $W + E$ are simple ; if $r_j = 1$, then generically λ_j is not an eigenvalue $W + E$.
More precisely, there is a generic set $\Omega \subseteq \mathbb{C}^{2n}$ such that for every $u \in \Omega$, the Jordan structure of $\left(I + uu^T J\right) W$ is described in (a) and (b) bellow :

(a) the Jordan structure of $\left(I + uu^T J\right) W$ for the eigenvalues $\lambda_1, ..., \lambda_{2p}$ is given by

$$\bigoplus_{k=2}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{r_{2p}} \mathcal{J}_{\ell_{2p,k}}(\lambda_{2p})$$

(b) the eigenvalues of $\left(I + uu^T J\right) W$ that are different from any of $\lambda_1, ..., \lambda_{2p}$, are all simple.

Ref

4. L. Batzke, C. Mehl, A. C.M. Ran and L. Rodman, Generic rank-k Perturbations of Structured Matrices. Operator Theory, Function Spaces, and Applications Birkhuser, Cham. (2016), p. 27-48.

Notes

Proof
Note that, λ being an eigenvalue of W, $1/\lambda$, $\bar{\lambda}$ et $1/\bar{\lambda}$ are also eigenvalues of W. So the number of eigenvalues W is even. Thus, W will have the Jordan canonical form (2.4).

According to (a) of Theorem (1.1), the structure of W by the eigenvalues λ_1, λ_2,...,λ_{2p} is given by

$$\bigoplus_{k=2}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{r_{2p}} \mathcal{J}_{\ell_{2p,k}}(\lambda_{2p})\ ;$$

and from point (b) of the same Theorem, the eigenvalues of $(I + uu^T J)W$ which are different from $\lambda_1, ..., \lambda_{2p}$ are all simple.

b) Generic rank-one perturbation of the Hamiltonian system with periodic coefficients

Let u be a vector of a generic set $\Omega \subset \mathbb{R}^{2n}$. Consider the Hamiltonian systems with τ-periodic coefficients

$$J\frac{d\widetilde{X}(t)}{dt} = [H(t) + E(t)]\,\widetilde{X}(t), \tag{2.5}$$

where $t \mapsto H(t)$ and $t \mapsto E(t)$ are piecewise continuous matrix functions on $[0, \tau]$ such that for all $t \in \mathbb{R}$ and $\tau > 0$,

$$H(t+\tau) = H(t) = H^T(t) \in \mathbb{R}^{2n\times 2n} \text{ and } E(t+\tau) = E(t) = E^T(t) \in \mathbb{R}^{2n\times 2n}.$$

Definition 2.2 *We call a rank-one perturbation of the fundamental solution $(X(t))_{t\in\mathbb{R}}$ of (1.1) any matrix function of the form*

$$\widetilde{X}(t) = (I + cuu^T J)X(t), \forall t \in \mathbb{R} \tag{2.6}$$

where $c = \pm 1$.

The rank-one perturbations of fundamental solution of (1.1) are J-symplectic [2, 5]. Therefore, we collect some properties of Hamiltonian systems with periodic coefficients of the type (2.5) in Proposition 2.1

Proposition 2.1
1. *Let $t \in \mathbb{R}$ and $(X(t))_{t\in\mathbb{R}}$ be the fundamental solution of (1.1). If a solution $\left(\widetilde{X}(t)\right)_{t\in\mathbb{R}}$ of (2.5) is of the form*

$$\widetilde{X}(t) = (I + c(t)u(t)u^T(t)J)X(t), \tag{2.7}$$

where $t \mapsto u(t) \in \mathbb{R}^{2n}$ is a vector function and $t \mapsto c(t)$ is a function with value in $\{-1, +1\}$. Then there exists a constant vector u such that $u(t) = u$, $c(t) = c = \pm 1$ is a real constant and $E(t)$ is of the form

$$E(t) = (cJuu^T H(t))^T + cJuu^T H(t) + c^2(uu^T)^T H(t)(uu^T J). \tag{2.8}$$

2. *Let u be a non-zero vector of $\mathbb{R}^{2n}$. Consider the perturbed Hamiltonian equation of (1.1)*

$$J\frac{d\widetilde{X}(t)}{dt} = [H(t) + E(t)]\,\widetilde{X}(t) \tag{2.9}$$

where $t \mapsto H(t)$ is piecewise continuous and

$$E(t) = (cJuu^T H(t))^T + cJuu^T H(t) + c^2(uu^T)^T H(t)(uu^T J).$$

Then $\widetilde{X}(t) = (I + cuu^T)X(t)$ is a solution of (2.9)

Global Journal of Science Frontier Research (F) Volume XXI Issue I Version I Year 2021

3. System (2.9) can be put in the form

$$\left\{\begin{array}{rcl} J\dfrac{\widetilde{X}(t)}{dt} & = & (I-cuu^T)^T H(t)(I-cuu^T)\widetilde{X}(t), \forall t\in\mathbb{R} \\ \widetilde{X}(0) & = & I+cuu^T J. \end{array}\right. \tag{2.10}$$

Proof

1. Suppose $u(t)$ is not constant. Then $u(t)u(t)^T$ is not also constant. We have

$$J\frac{d\widetilde{X}(t)}{dt} = J(I+c(t)u(t)u^T(t)J)\frac{X(t)}{dt} + J\left[\frac{d(c(t)u(t)u^T(t))}{dt}\right]JX(t)$$

$$= J(I+c(t)u(t)u^T(t)J)J^{-1}H(t)X(t) + J\left[\frac{d(c(t)u(t)u^T(t))}{dt}\right]JX(t)$$

$$= \big[(I-c(t)u(t)u^T(t)J)^T H(t)(I-c(t)u(t)u^T(t)J) +$$

$$J\frac{d(c(t)u(t)u^T(t))}{dt}J(I-c(t)u(t)u^T(t)J)\bigg]\widetilde{X}(t) \quad \text{with } \widetilde{X}(t)=(I+c(t)u(t)u^T(t)J)X(t)$$

$$= [H(t) +$$

$$\underbrace{(c(t)Ju(t)u^{(}t)H(t))^T + c(t)Ju(t)u^T(t)H(t) + c(t)^2(u(t)u^T(t)J)^T H(t)(u(t)u^T(t)J)}_{E(t)}\Bigg]\widetilde{X}(t)$$

$$+\left[\underbrace{J\frac{d(c(t)u(t)u^T(t))}{dt}J(I-c(t)u(t)u^T(t)J)}_{F(t)}\right]\widetilde{X}(t)$$

$$= [H(t)+E(t)+F(t)]\,\widetilde{X}(t),$$

We note that $E(t)+F(t)$ is not symmetric because $E(t)$ is symmetric and $F(t)$ is not symmetric. Which gives us a contradiction. To have $H(t)+E(t)+F(t)$ symmetric, we must have $F(t)=0,\ \forall t\in\mathbb{R}$. Then $c(t)u(t)u^T(t)$ is constant. In particular, $c(t)u(t)u^T(t)=c(0)u(0)u(0)^T \quad \forall t\in\mathbb{R}$. We deduce that there is a constant vector $u=u(0)$ and a real constant $c=c(0)\in\{-1,+1\}$ such that $E(t)$ is of the form (2.8).

2. By deriving $\widetilde{X}(t)$, we get

$$\begin{aligned} J\frac{d\widetilde{X}(t)}{dt} &= J(I+cuu^TJ)J^{-1}J\frac{dX(t)}{dt} \\ &= J(I+cuu^TJ)J^{-1}H(t)X(t), \quad \text{from (1.1)} \\ &= \left[H(t)+cJuu^TH(t)\right]X(t) \\ &= \left[H(t)+cJuu^TH(t)\right](I-cuu^TJ)\widetilde{X}(t) \end{aligned}$$

because the matrix $(I-cuu^TJ)$ is the inverse of ($I+cuu^TJ$) see [14]

$$= \left[H(t)+\underbrace{(cJuu^TH(t))^T + cJuu^TH(t) + c^2(uu^TJ)^TH(t)(uu^TJ)}_{E(t)}\right]\widetilde{X}(t)$$

Ref

14. YAN Qing-you. The properties of a kind of random symplectic matrices. Applied mathematics and Mechanics. Vol 23, No 5, May 2002.

We then obtain equation (2.9) with

$$E(t) = c(Juu^T H(t))^T + cJuu^T H(t) + c^2(uu^T J)^T H(t)(uu^T J).$$

This shows that $\widetilde{X}(t) = (I + cuu^T J)X(t)$ is a solution of (2.9).

3. Indeed, it suffices to develop $(I - uu^T J)^T H(t)(I - uu^t J)$ to obtain

$$(I - cuu^T J)^T H(t)(I - cuu^t J) = H(t) + \underbrace{(cJ^T uu^T H(t))^T + cJ^T uu^T H(t) + c^2(uu^T J)^T H(t)(uu^T J)}_{E(t)}$$

III. Generic Double Rank-One Perturbation of Hamiltonian System with Periodic Coefficients

In this section, we consider two vectors u_1 and u_2 taken in a generic set Ω of $\mathbb{R}^{2n}$ such that $u_1^T J u_2 = 0$. Note that this holds if $u_1 = u_2$. On the other hand, we can consider generic vectors belonging to isotropic (or Lagrangian) subspaces. A subspace $\mathcal{X} \subseteq \mathbb{R}^{2n}$ is called isotropic if $\mathcal{X} \perp J\mathcal{X}$. The maximum isotropic subspaces containing $\mathcal{X}$ are of dimension n [4]. Hence we have this definition

Definition 3.1 *A subspace $\mathcal{L}$ of $\mathbb{R}^{2n}$ is called a Lagrangian subspace if it is of the dimension n and*

$$x^T Jy = 0, \qquad \forall x, y \in \mathcal{L}.$$

We first consider the rank-one pertubation

$$X_1(t) = (I + c_1 u_1 u_1^T J)X(t), \ \ \forall t \in \mathbb{R} \quad \text{and} \quad c = \pm 1$$

of the solution $(X(t))_{t\in\mathbb{R}}$ of system (1.1) using the vector u. Then we perturb the solution a second time using the second vector u_2. We get

$$X_2(t) = (I + c_2 u_2 u_2^T J)(I + c_1 u_1 u_1^T J)X(t), \forall t \in \mathbb{R} \quad \text{and} \quad c_1, c_2 \in \{-1, +1\} \tag{3.1}$$

We have the following Proposition

Proposition 3.1 *The double rank-one perturbation of the solution of (1.1) is the solution of the following system*

$$\begin{cases} J\dfrac{\widehat{X}(t)}{dt} &= (I - c_2 u_2 u_2^T J)^T (I - c_1 u_1 u_1^T J)^T H(t)(I - c_1 u_1 u_1^T J)(I - c_2 u_2 u_2^T J)\widehat{X}(t) \\ \widehat{X}(0) &= (I + c_2 u_2 u_2^T J)(I + c_1 u_1 u_1^T J) \end{cases} \tag{3.2}$$

Proof
The double rank-one perturbation of the solution of (1.1) is given by (3.1). Then $\forall t \in \mathbb{R}$ and $c_1, c_2 \in \{-1, +1\}$, we have

$$\begin{aligned} \frac{dX_2(t)}{dt} &= (I + c_2 u_2 u_2^T J)(I + c_1 u_1 u_1^T J)\frac{X(t)}{dt}, \\ &= (I + c_2 u_2 u_2^T J)(I + c_1 u_1 u_1^T J)J^{-1} H(t)X(t), \\ &= \left[(I + c_2 u_2 u_2^T J)(J^{-1} + c_1 u_1 u_1^T)H(t)(I + c_1 u_1 u_1^T J)^{-1}(I + c_2 u_2 u_2^T J)^{-1}\right] \times \end{aligned}$$

Ref

4. L. Batzke, C. Mehl, A. C.M. Ran and L. Rodman, Generic rank-k Perturbations of Structured Matrices. Operator Theory, Function Spaces, and Applications Birkhuser, Cham. (2016), p. 27-48.

$$\underbrace{(I + c_2u_2u_2^TJ)(I + c_1u_1u_1^TJ)X(t)}_{X_2(t)},$$

$$= \left[(I + c_2u_2u_2^TJ)J^{-1}(I - c_1u_1u_1^TJ)^TH(t)(I + c_1u_1u_1^TJ)^{-1}(I + c_2u_2u_2^TJ)^{-1}\right]X_2(t)$$

$$= J^{-1}\left[(I - c_2u_2u_2^TJ)^T(I - c_1u_1u_1^TJ)^TH(t)(I - c_1u_1u_1^TJ)(I - c_2u_2u_2^TJ)\right]X_2(t),$$

because for all vector u and $c \in \{-1, +1\}$, we have $(I + cuu^TJ)^{-1} = (I - cuu^TJ)$. Moreover $X_2(0) = (I + c_2u_2u_2^TJ)(I + c_1u_1u_1^TJ)X(0)$.

Remark 3.1 The double rank-one perturbation $X_2(t)$ (or $\widehat{X}(t)$) of $X(t)$ can be put in the form

$$X_2(t) = X(t) + c_2u_2u_2^TJX(t) + c_1u_1u_1JX(t), \quad \forall t \in \mathbb{R} \;\; c_1, c_2 \in \{-1, +1\}; \tag{3.3}$$

or by putting the vectors u_1 and u_2 as column of a matrix $U = [u_1\ u_2] \in \mathbb{R}^{2n\times 2}$

$$X_2(t) = (I + U\Sigma_2U^TJ)X(t) = X(t) + U\Sigma_2U^TJX(t) \tag{3.4}$$

where $\Sigma_1 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ is a diagonal matrix with $c_1, c_2 \in \{-1, +1\}$.

We have the following Definition

Definition 3.2 We call a generic rank-2 perturbation of system (1.1), any system given by (3.2).

From (3.2), the following corollary gives another writing of system (3.2)

Corollary 3.1 System (3.2) can be put in the form below

$$\begin{cases} J\dfrac{\widehat{X}(t)}{dt} &= (I - U\Sigma_2U^TJ)^TH(t)(I - U\Sigma_2U^TJ)\widehat{X}(t) \\ \widehat{X}(0) &= (I + U\Sigma_2U^TJ) \end{cases} \tag{3.5}$$

or in a following simple form

$$\begin{cases} J\dfrac{\widehat{X}(t)}{dt} &= (H(t) + E(t))\,\widehat{X}(t) \\ \widehat{X}(0) &= I + U\Sigma_2U^TJ \end{cases} \tag{3.6}$$

where

$$E(t) = JU\Sigma_2U^TH(t) + (JU\Sigma_2U^TH(t))^T + (U\Sigma_2U^TJ)^TH(t)(U\Sigma_2U^TJ)$$

Proof

To have (3.5), It suffices to notice that

$$(I - c_1u_1u_1^TJ)(I - c_2u_2u_2J) = I - c_1u_1u_1^TJ - c_2u_2u_2^TJ$$

$$= I - [u_1\ u_2]\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}[u_1\ u_2]^TJ$$

$$= I - U\Sigma_2U^TJ$$

Notes

Similarly, we have $(I + c_1 u_1 u_1^T J)(I + c_2 u_2 u_2 J) = I + U\Sigma_2 U^T J$.

Next, by developing $(I - U\Sigma_2 U^T J)^T H(t)(I - U\Sigma_2 U^T J)$ in (3.5), we get

$$(I - U\Sigma_2 U^T J)^T H(t)(I - U\Sigma_2 U^T J) = H(t) + E(t)$$

where $E(t) = JU\Sigma_2 U^T H(t) + (JU\Sigma_2 U^T H(t))^T + (U\Sigma_2 U^T J)^T H(t)(U\Sigma_2 U^T J)$. Hence we have (3.6).

Notes

Acknowledgment

Many thanks to the referees for the helpful remarks and suggestions.

References Références Referencias

1. T. G. Y. Arouna, M. Dosso, J.-C. Koua Brou, Application of a Rank-One Perturbation to Pendulum Systems. Journal of Mathematics Research; Vol. 12, No. 5; October 2020.
2. T. G. Y. Arouna, M. Dosso, & J. C. Koua Brou, On a pertubation theory of Hamiltonian systems with periodic coefficients. International Journal of Numerical Methods and Applications, 17(2018), 47-89.
3. L. Batzke, Generic rank-one perturbations of structured regular matrix pencils. Linear Algebra and its Applications, vol. 458 (2014), p. 638-670.
4. L. Batzke, C. Mehl, A. C.M. Ran and L. Rodman, Generic rank-k Perturbations of Structured Matrices. Operator Theory, Function Spaces, and Applications Birkhuser, Cham. (2016), p. 27-48.
5. M. Dosso, T. G. Y. Arouna, & J. C. Koua Brou, On rank one perturbation of Hamiltonian system with periodic coefficients. Wseas Translations on Mathematics, 17(2018), 377-384.
6. M. Dosso, & M. Sadkane, On the strong stability of symplectic matrices. Numerical Linear Algebra with Applications, 20(2013), 234-249.
7. M. Dosso, & N. Coulibaly, Symplectic matrices and strong stability of Hamiltonian systems with periodic coefficients. Journal of Mathematical Sciences: Advances and Applications, 28(2014), 15-38.
8. M. Dosso, Sur quelques algorithms d'analyse de stabilit˙e forte de matrices symplectiques, PHD Thesis (September 2006), Université de Bretagne Occidentale. Ecole Doctorale SMIS, Laboratoire de Mathématiques, UFR Sciences et Techniques.
9. S. K. Godunov & M. Sadkane, (2001). Numerical determination of a canonical form of a symplectic matrix. Siberian Mathematical Journal, 42(2001), 629-647.
10. S. K. Godunov & M. Sadkane, Spectral analysis of symplectic matrices with application to the theory of parametric resonance. SIAM Journal on Matrix Analysis and Applications, 28(2006), 1083-1096.
11. C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Eigenvalue perturbation theory under generic rank one perturbations: Symplectic, orthogonal, and unitary matrices. BIT, 54(2014), 219-255.
12. C. Mehl, V. Mehrmann, A.C.M. Ran and L. Rodman. Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations. Linear Algebra Appl., 435(2011), 687-716.

13. V. A. Yakubovich, V. M. Starzhinskii, Linear differential equations with periodic coefficients, Vol. 1 & 2., Wiley, New York (1975).
14. YAN Qing-you. The properties of a kind of random symplectic matrices. Applied mathematics and Mechanics. Vol 23, No 5, May 2002.

Notes