Uniformly Starlike and Uniformly Convexity Properties Pertaining to Certain Special Functions

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Abstract - The aim of this paper is to establish sufficient conditions for the function $z \{ p \overline{\psi}_q(z) \}$ to be in the classes of uniformly starlike and uniformly convex functions associated with the parabolic region $\Re \{ \omega \} > | \omega - 1 |$. Further, convolution of the functions which are analytic in the open unit disk with negative coefficients have been investigated. Finally, similar results using an integral operator have also been obtained.

Keywords : Univalent functions, Parabolic Starlike functions, Uniformly Convex functions, Integral operator, Generalized Fox-Wright function.

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Uniformly Starlike and Uniformly Convexity Properties Pertaining to Certain Special Functions

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Abstract - The aim of this paper is to establish sufficient conditions for the function \( f(z) = q(z) \) to be in the classes of uniformly starlike and uniformly convex functions associated with the parabolic region \( \Re \{ \omega \} > |\omega - 1| \). Further, convolution of the functions which are analytic in the open unit disk with negative coefficients have been investigated. Finally, similar results using an integral operator have also been obtained.

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1. INTRODUCTION

Let \( A \) denote the class of functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad ... \ (1.1) \]

that are analytic in the open unit disk \( \Delta = \{ z : |z| < 1 \} \).

Also let \( S \) denote the subclass of \( A \) consisting of all functions \( f(z) \) of the form

\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \quad ... (1.2) \]

A function \( f \in A \) is said to be uniformly convex in \( \Delta \) if \( f \) is a univalent convex function having the property that for every circular arc \( \gamma \) contained in \( \Delta \) with centre also in \( \Delta \), the image curve \( f(\gamma) \) is a convex arc. Denoting the class of all uniformly convex functions by \( UCV \), it was shown in [10, 13] that

\[ f \in UCV \iff \left| \frac{zf''(z)}{f'(z)} \right| < \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right), \quad (z \in \Delta). \quad ... (1.3) \]

To give the geometric interpretation of (1.3), let

\[ \Omega_p = \{ \omega : \omega = u + i\nu, \Re(\omega) > \omega - 1 \} \]

which is the interior of the parabola \( \nu^2 = 2u - 1 \).

Then \( f \in UCV \iff 1 + \frac{zf''(z)}{f'(z)} \in \Omega_p \quad ... (1.4) \)

A class closely related to the class \( UCV \) is the class of parabolic starlike functions denoted by \( US_p \) [15]. This class \( US_p \) and the class \( UC_p \) of uniformly convex functions have been studied in [1, 6, 7, 11, 15, 16, 17]. A survey of these functions can the works of [12]. Let \( f \in S \), \( 0 \leq \lambda < \infty \), and \( 0 \leq \mu < 1 \), then [2, 13]

\[ f \in UC_p(\lambda, \mu) \iff \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \lambda \left| \frac{zf''(z)}{f'(z)} \right| + \mu \quad ... (1.5) \]

and

\[ f \in US_p(\lambda, \mu) \iff \Re \left( \frac{zf'(z)}{f(z)} \right) \geq \lambda \left| \frac{zf'(z)}{f(z)} \right| - 1 + \mu \quad ... (1.6) \]

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We recall here that the hadamard product (or convolution) of $f(z)$ of the form (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, is defined as

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \Delta.$$ \hspace{1cm} (1.7)

The generalized Fox-Wright function [8] appearing in the present paper is defined by

$$p\overline{\Psi}_q(z) = \overline{p\Psi}_q\left[\begin{array}{l} \{a_j, \alpha_j; A_j\}, p; \\ \{b_j, \beta_j; B_j\}, q; \\ z \end{array}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \{\Gamma(a_j + \alpha_j n)\} A_j z^n}{\prod_{j=1}^{q} \{\Gamma(b_j + \beta_j n)\} B_j n!}, \quad (1.8)$$

where $\alpha_j (j = 1, \ldots, p)$ and $\beta_j (j = 1, \ldots, q)$ are real and positive, $\sum_{j=1}^{q} \beta_j > \sum_{j=1}^{p} \alpha_j$, and $A_j (j = 1, \ldots, p)$ and $B_j (j = 1, \ldots, q)$ can take non-integer values.

For (1.8), we have

$$z \{p\overline{\Psi}_q(z)\} = \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} \{\Gamma[a_j + \alpha_j (n-1)]\} A_j z^n}{\prod_{j=1}^{q} \{\Gamma[b_j + \beta_j (n-1)]\} B_j (n-1)!}$$

or

$$z p\overline{\Omega}_q = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{q} \{\Gamma[b_j]\} B_j}{\prod_{j=1}^{p} \{\Gamma[a_j]\} A_j} \frac{\prod_{j=1}^{p} \{\Gamma[a_j + \alpha_j (n-1)]\} A_j z^n}{\prod_{j=1}^{q} \{\Gamma[b_j + \beta_j (n-1)]\} B_j (n-1)!}$$ \hspace{1cm} (1.9)

where

$$p\overline{\Omega}_q = \frac{\prod_{j=1}^{q} \{\Gamma[b_j]\} b_j}{\prod_{j=1}^{p} \{\Gamma[a_j]\} A_j} p\overline{\Psi}_q. \hspace{1cm} (1.10)$$

Now, we define a linear operator $\overline{G}_q^p : S \rightarrow S$ as follows:

$$\overline{G}_q^p f(z) = z p\overline{\Omega}_q * f(z)$$
Corresponding to the operator defined in (1.11), we let \( \mathcal{G}_q^p \) denote the subclass of functions \( f \in S \) satisfying the inequality:

\[
\left| \frac{z^q (\mathcal{G}_q^p f(z))^{(1+\lambda)}}{\mathcal{G}_q^p f(z)} - \gamma \right| \geq \left| \frac{z^q (\mathcal{G}_q^p f(z))^{(1+\mu)}}{\mathcal{G}_q^p f(z)} - \gamma \right|
\]

...(1.13)

In the present paper we shall use the following Lemmas [2, 18] to establish our main results:

**Lemma 1.** A function \( f(z) \) of the form (1.1) is in the class \( \text{US}_p (\lambda, \mu) \) if

\[
\sum_{n=2}^{\infty} \frac{n(1+\lambda) - (\lambda + \mu)}{M_1} a_n \leq (1 - \mu) M_1,
\]

...(1.14)

where \( M_1 > 0 \) is a suitable constant.

**Lemma 2.** A function \( f(z) \) of the form (1.1) is in the class \( \text{UC}_p (\lambda, \mu) \) if

\[
\sum_{n=2}^{\infty} \frac{n(1+\lambda) - (\lambda + \mu)}{M_2} a_n \leq (1 - \mu) M_2,
\]

...(1.15)

where \( M_2 > 0 \) is a suitable constant.

**II. MAIN RESULTS**

**Theorem 2.1.** If \( \sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j + 1 \), and \( \sum_{j=1}^{q} b_j > \sum_{j=1}^{p} A_j a_j \),

then a sufficient condition for the function \( z \{ q \mathcal{G}_q^p \} \) to be in the class \( \text{US}_p (\lambda, \mu) \), \( 0 \leq \lambda < \infty \) and \( 0 \leq \mu < 1 \), is

\[
\left( \frac{1+\lambda}{1-\mu} \right) \mathcal{G}_q^p \left[ (a_j \alpha_j, \alpha_j; A_j)_{1,p}; 1 \right] + \mathcal{G}_q^p \left[ (a_j \alpha_j; A_j)_{1,p}; 1 \right] \leq M_1 + \sum_{j=1}^{p} \frac{\{ \Gamma a_j \} A_j}{\{ \Gamma b_j \} B_j}.
\]

...(2.1)

**Proof.** Since
so by virtue of Lemma 1.1, we need only to show that

\[
\sum_{n=2}^{\infty} \left[ (1+\lambda)n - (\lambda + \mu) \right] \left[ \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j (n-1))}{\Gamma(a_j + \alpha_j (n-1))} \right]^{A_j} \sum_{j=1}^{q} \frac{\Gamma(b_j + \beta_j (n-1))}{\Gamma(b_j + \beta_j (n-1))}^{B_j (n-1)!} \leq (1-\mu)M_1
\]

Now, we have

\[
\sum_{n=0}^{\infty} \left[ (1+\lambda)(n+2)\alpha - (\lambda + \mu) \right] \left[ \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j (n+1))}{\Gamma(a_j + \alpha_j (n+1))} \right]^{A_j} \sum_{j=1}^{q} \frac{\Gamma(b_j + \beta_j (n+1))}{\Gamma(b_j + \beta_j (n+1))}^{B_j (n+1)!} = (1+\lambda) \sum_{n=0}^{\infty} \left[ \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j + n \alpha_j)}{\Gamma(a_j + \alpha_j + n \alpha_j)} \right]^{A_j} \sum_{j=1}^{q} \frac{\Gamma(b_j + \beta_j + n \beta_j)}{\Gamma(b_j + \beta_j + n \beta_j)}^{B_j n!} + (1-\mu) \sum_{n=0}^{\infty} \left[ \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j n)}{\Gamma(a_j + \alpha_j n)} \right]^{A_j} \sum_{j=1}^{q} \frac{\Gamma(b_j + \beta_j n)}{\Gamma(b_j + \beta_j n)}^{B_j n!} - \frac{\prod_{j=1}^{p} \Gamma(a_j \alpha_j)}{\prod_{j=1}^{q} \Gamma(b_j \beta_j)}
\]

\[
= (1+\lambda) \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j + n \alpha_j)}{\Gamma(a_j + \alpha_j + n \alpha_j)}^{A_j} \sum_{j=1}^{q} \frac{\Gamma(b_j + \beta_j + n \beta_j)}{\Gamma(b_j + \beta_j + n \beta_j)}^{B_j n!} + (1-\mu) \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j n)}{\Gamma(a_j + \alpha_j n)}^{A_j} \sum_{j=1}^{q} \frac{\Gamma(b_j + \beta_j n)}{\Gamma(b_j + \beta_j n)}^{B_j n!} - \frac{\prod_{j=1}^{p} \Gamma(a_j \alpha_j)}{\prod_{j=1}^{q} \Gamma(b_j \beta_j)}
\]

\[
= (1+\lambda) \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j + n \alpha_j)}{\Gamma(a_j + \alpha_j + n \alpha_j)}^{A_j} \prod_{j=1}^{q} \frac{\Gamma(b_j + \beta_j + n \beta_j)}{\Gamma(b_j + \beta_j + n \beta_j)}^{B_j n!} + (1-\mu) \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j n)}{\Gamma(a_j + \alpha_j n)}^{A_j} \prod_{j=1}^{q} \frac{\Gamma(b_j + \beta_j n)}{\Gamma(b_j + \beta_j n)}^{B_j n!} - \frac{\prod_{j=1}^{p} \Gamma(a_j \alpha_j)}{\prod_{j=1}^{q} \Gamma(b_j \beta_j)}
\]

\[
= (1+\lambda) \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j + n \alpha_j)}{\Gamma(a_j + \alpha_j + n \alpha_j)}^{A_j} \prod_{j=1}^{q} \frac{\Gamma(b_j + \beta_j + n \beta_j)}{\Gamma(b_j + \beta_j + n \beta_j)}^{B_j n!} + (1-\mu) \prod_{j=1}^{p} \frac{\Gamma(a_j + \alpha_j n)}{\Gamma(a_j + \alpha_j n)}^{A_j} \prod_{j=1}^{q} \frac{\Gamma(b_j + \beta_j n)}{\Gamma(b_j + \beta_j n)}^{B_j n!} - \frac{\prod_{j=1}^{p} \Gamma(a_j \alpha_j)}{\prod_{j=1}^{q} \Gamma(b_j \beta_j)}
\]
Hence the theorem.

**Theorem 2.2** Let \( f(z) \) be given by (1.1) and \(-1 < \gamma < 1\), then \( f(z) \in S^p_q(\gamma) \) if

\[
\sum_{n=2}^{\infty} (2n-1-\gamma) \overline{B}(a_j,b_j,A_j,B_j,n) |a_n| \leq 1-\gamma
\]

... (2.3)

where \( \overline{B}(a_j,b_j,A_j,B_j,n) \) is given by (1.12).

**Proof.** By virtue of (1.13), it is sufficient to show that

\[
\text{Re} \left\{ \frac{zG_q^n f(z)'}{G_q^n f(z)} - \gamma \right\} \geq \left| \frac{zG_q^n f(z)'}{G_q^n f(z)} - 1 \right|
\]

or

\[
2 \left| \frac{zG_q^n f(z)'}{G_q^n f(z)} - 1 \right| \leq 1-\gamma
\]

or

\[
\left| \frac{z\left(1- \sum_{n=2}^{\infty} \overline{B}(a_j,b_j,A_j,B_j,n) a_n n z^{n-1}\right)}{z - \sum_{n=2}^{\infty} \overline{B}(a_j,b_j,A_j,B_j,n) a_n z^n} - 1 \right| \leq 1-\gamma
\]

or

\[
2 \sum_{n=2}^{\infty} (n-1) \overline{B}(a_j,b_j,A_j,B_j,n) |a_n| \leq (1-\gamma) \left\{ 1 - \sum_{n=2}^{\infty} \overline{B}(a_j,b_j,A_j,B_j,n) |a_n| \right\}
\]

or

\[
\sum_{n=2}^{\infty} (2n-1-\gamma) \overline{B}(a_j,b_j,A_j,B_j,n) |a_n| \leq 1-\gamma
\]

Hence the theorem.

**III. AN INTEGRAL OPERATOR**

In this section we obtain sufficient conditions for the function

\[
\frac{1}{p} \sum_{j=1}^{p} \frac{(a_j,b_j;A_j,A_j)_p}{(b_j,b_j;B_j,B_j)_q} \left[ \frac{\Gamma(a_j)}{\Gamma(b_j)} \right] = \int_0^z p q^p_\gamma(x) dx
\]

to be in the classes \( US_p(\lambda,\mu) \) and \( UC_p(\lambda,\mu) \).

**Theorem 3.1.** If \( \sum_{j=1}^{p} b_j > \sum_{j=1}^{p} a_j + 1, \) and \( \sum_{j=1}^{p} B_j b_j > \sum_{j=1}^{p} A_j a_j, \) then a sufficient condition for the function

\[
p q^p_\gamma(z) = \int_0^z p q^p_\gamma(x) dx
\]

to be in the class \( US_p(\lambda,\mu) \), \( 0 \leq \lambda < \infty \) and \( 0 \leq \mu < 1, \) is
Now, we have
\[
\sum_{n=2}^{\infty} \frac{[n(1+\lambda)-(\lambda+\mu)]}{(1+\lambda)} - \frac{[n(1+\lambda)-(\lambda+\mu)]}{(\lambda+\mu)} \]

Proof, Since
\[
\prod_{j=1}^{p} \frac{\Gamma(a_j-\alpha_j)}{\Gamma(b_j-\beta_j)} A_j = \frac{\prod_{j=1}^{p} \{\Gamma(a_j-\alpha_j)\}^{A_j}}{\prod_{j=1}^{q} \{\Gamma(b_j-\beta_j)\}^{B_j}} \leq M_1 \]

... (3.1)
which in view of Lemma 1.1, leads to the result (3.1).

Theorem 3.2 If \[ \sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j + 1 \text{ and } 1+ \sum_{j=1}^{q} B_j \beta_j > \sum_{j=1}^{p} A_j \alpha_j, \]
then a sufficient condition for the function \[ p \bar{v}_q(x) = \int_{0}^{x} p \bar{v}_q(t) \, dt \] to be in the class \( UC_{p} (\lambda, \mu), 0 \leq \lambda < \infty \text{ and } 0 \leq \mu < 1, \) is

**Proof.** The result follows as direct consequence of the Theorem 3.1, keeping Lemma 1.2 in view.

### IV. PARTICULAR CASES

4.1 For \( \lambda = 2 \) and \( \mu = 0, \) Theorem 2.1 and Theorem 3.1 corresponds to the results recently obtained by Chaurasia and Kumawat [3] for \( \alpha = 0. \)

4.2 For \( A_j = 1(j = 0, 1, \ldots, p); B_j = 1(j = 0, 1, \ldots, q) \), the Theorems established in the present paper readily yield the results due to Bapna and Jain [2].

4.3 The results due to Chaurasia and Srivastava [4], Dixit and Verma [5] and Shanmugam et al. [14] also follow as particular cases of our main results.

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### REFERENCES


