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A New Construction of the Degree of Maximal Monotone Maps

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A New Construction of the Degree of Maximal Monotone Maps

Mohammad Niksirat

Abstract- The inclusion equations of the type $f \in T(x)$ where $T: X \to 2^{X^*}$ is a maximal monotone map, are extensively studied in nonlinear analysis. In this paper, we present a new construction of the degree of maximal monotone maps of the form $T: Y \to 2^{X^*}$, where $Y \subset X$ is a locally uniformly convex and separable Banach space continuously embedded in X The advantage of the new construction lies in the remarkable simplicity it offers for calculation of degree in comparison with the classical one suggested by F. Browder. We prove a few classical theorems in convex analysis through the suggested degree.

Keywords: degree theory, finite rank approximation, maximal monotone maps, multivalued maps.

I. Introduction

Assume that X and Y are separable and reflexive Banach spaces equipped with uniformly convex norms, and $i : Y \to X$ is the continuous embedding. Furthermore, assume that $T : Y \to 2^{X^*}$ is a maximal monotone map with the effective domain $D(T) = Y$ in the following sense. A pair $(\tilde{y}, \tilde{x}^*)$ is in the graph of T if the condition $\langle x^* - \tilde{x}^*, i(y - \tilde{y})\rangle \geq 0$ holds for all $(y, x^*) \in \text{graph}(T)$. We construct a degree for T. The construction generalizes the F. Browder's classical degree of maximal monotone maps [1]. The construction of the Browder's degree is as follows. Assume that X is a separable reflexive Banach space equipped with a uniformly convex norm, and $T : X \to 2^{X^*}$ a maximal monotone map. The map $T_\epsilon = T + \epsilon J$, where $\epsilon > 0$ and $J : X \to X^*$ is the duality mapping possesses the following properties:

(1) T_ϵ is a map of class $(S)_+$, that is, if $x_n \rightharpoonup x$ in X, there is $x_n^* \in T(x_n)$ such that

$$\limsup_{n\to\infty} \langle x_n^* + \epsilon J(x_n), x_n - x\rangle \leq 0,$$

then $x_n \to x$.

(2) The map T_ϵ is onto X^*,

(3) if $x_1 \neq x_2$, the sets $T(x_1), T(x_2)$ are disjoint, that is,

$$T(x_1) \cap T(x_2) = \emptyset.$$

(4) The map $T_\epsilon^{-1} : X^* \times (0, \infty) \to X$ is well defined and continuous.

Note that J is single valued, bijective and bi-continuous if X is uniformly convex. It is shown that the map $(T_\epsilon^{-1} + \epsilon J^{-1})^{-1} : X \to X^*$ is demi-continuous and $(S)_+$ for which a degree theory has been developed by F. Browder. The degree of T in the open bounded set $D \subset X$ at 0 is defined by the following relation

$$\deg(T, D, 0) = \lim_{\epsilon\to 0} \deg((T_\epsilon^{-1} + \epsilon J^{-1})^{-1}, D, 0). \tag{1.1}$$

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The degree suggested in this article generalizes the Browder's degree. In particular if $Y = X$, two degrees are the same. Another advantage of the suggested degree is the direct use of finite rank approximation we employed in our previous work [2]. This method make the calculations much easier than the formula (1.1). It should be noted that the suggested degree is different from the degree of the map $i^* \circ T : Y \to 2^{Y^*}$. The difference between two formulations is discussed in [2] for single valued maps.

Definition 1.1. Assume X_1 and X_2 are Banach spaces. A map $A : X_1 \to 2^{X_2}$ is called upper semi-continuous at $x \in X_1$ if for every neighborhood V of $A(x)$, there exists an open neighborhood U of x such that $T(U) \subset V$.

We have the following theorem for the upper semi-continuous multi-valued mappings; see for example [3, 4].

Theorem 1.2. (**ϵ-continuous subgraph**) *Assume that X_1 and X_2 are Banach spaces, and the map $A : X_1 \to 2^{X_2}$ is upper semi-continuous. If $A(x)$ is closed and convex for all $x \in X_1$, then for any $\epsilon > 0$, there exists a continuous single valued function $A_\epsilon : X_1 \to X_2$ such that for any $x \in X_1$, there exists $z_1 \in X_1$ and $\tilde{z}_2 \in A(z)$ such that $\|x - z_1\| < \epsilon$ and $\|A_\epsilon(x) - \tilde{z}_2\| < \epsilon$.*

Proposition 1.4. Let X, Y be Banach spaces, $i : Y \to X$ a continuous embedding and $T : Y \to 2^{X^*}$ a maximal monotone map with the effective domain Y. Then $T(y)$ is closed and convex for all $y \in Y$, and T is norm to weak-star upper semi-continuous in the following sense. For arbitrary $y \in Y$, and arbitrary sequence (y_n) converges to y in norm, there is a weakly limit point x^* of $\cup_n T(y_n)$ such that $x^* \in T(y)$.

The proof is completely similar to one for the map $T : X \to 2^{X^*}$. For a proof of the standard version see for example [5].

If Y is a separable and reflexive Banach space equipped with a uniformly convex norm, a theorem by Browder and Ton [6] guarantees the existence of a separable Hilbert space H such that the embedding $j : H \hookrightarrow Y$ is dense and compact. Choosing an orthogonal basis $\{h_k\}_{k=1}^{\infty}$ for H, we obtaine the basis $\mathcal{Y} = \{y^1, y^2, \cdots, y^n, \cdots\}$ for Y where $y^k = j(h_k)$, and accordingly, the filtration $\mathbb{Y} = \{Y_n\}$, where $Y_n = \text{span}\{y^1, \ldots, y^n\}$. The following proposition is simply verified.

Proposition 1.4. For any $y \in Y$, there is a sequence $(y_n), y_n \in Y_n$ such that $y_n \to y$.

The pairing in Y_n is denoted by $(,)$ and is defined by the relation $(y^i, y^j) = \delta_{ij}$ for all $y^i, y^j \in \mathcal{Y}$. We define the maximal monotone operato $T : Y \to 2^{X^*}$ in the following sense.

Definition 1.5. Suppose X and Y are separable and reflexive Banach spaces equipped with uniformly convex norm, and assume that $T : Y \to 2^{X^*}$ is a maximal monotone map. The finite rank approximation of arbitrary $x^* \in T(y)$ in $Y_n \in \mathbb{Y}$, is defined by $\hat{x}_n = \sum_{k=1}^{n} \langle x^*, i(y^k) \rangle y^k$. Accordingly, the finite rank map $T_n : Y \to 2^{Y_n}$ is defined by the relation

$$T_n(y) = \bigcup_{x^* \in T(y)} \hat{x}_n. \tag{1.2}$$

For any $x^* \in X^*$ and $y \in Y_n$, we have the property

$$(\hat{x}_n, y) = \langle x^*, i(y) \rangle,$$

where $\langle , \rangle$ is the pairing between X^*, X. In fact, if $x^* \in X^*$, then for $\hat{x}_n = \sum_{k=1}^{n} \langle x^*, y^k \rangle y^k$, we have

Ref

2. M. Niksirat, *A new generalization of the Browder's degree of (S)+ maps*, appear soon.

$$\sum_{k=1}^{n} \langle x^*, i(y^k)\rangle (y^k, y) = \sum_{k=1}^{n} \langle x^*, i((y^k, y)y^k)\rangle = \left\langle x^*, i \quad \sum_{k=1}^{n} (y^k, y) y^k \right) \right\rangle,$$

and thus $\langle x^*, i(y)\rangle = (\hat{x}_n, y)$.

Lemma 1.6. The finite rank approximation T_n is upper semi-continuous and for every $x \in Y$, the set $T_n(x)$ is closed and convex.

Proof. Fix n and $\epsilon > 0$. If T_n is not upper semi-continuous at $x \in Y$, there is a sequence $(\delta_m), \delta_m \to 0$ and $x_m \in B_{\delta_m}(x)$ such that for some $\hat{x}_{n,m} \in T_n(x_m)$, we have $\hat{x}_{n,m} \notin V_\epsilon(T_n(x))$. T is maximal monotone, and thus locally bounded. Therefore, there is a subsequence (shown for the sake of simplicity again by $\hat{x}_{n,m}$) such that $\hat{x}_{n,m} \to \hat{x}$. We show $\hat{x} \in T_n(x)$. Since $\hat{x}_{n,m} \in T_n(x_m)$, there is $x_m^* \in T(x_m)$ such that $\hat{x}_{n,m}$ are the finite rank approximation in Y_n of x_m^*, that is,

$$\hat{x}_{n,m} = \sum_{k=1}^{n} \langle x_m^*, i(y^k)\rangle y^k.$$

Since T is norm to weak-star upper semi-continuous, and $x_m \to x$, we have $x_m^* \rightharpoonup x^*$ for some $x^* \in T(x)$. Thus $\langle x_m^*, i(y^k)\rangle \to \langle x^*, i(y^k)\rangle$ for all $1 \le k \le n$. Therefore

$$\hat{x}_{n,m} = \sum_{k=1}^{n} \langle x_m^*, i(y^k)\rangle y^k \to \sum_{k=1}^{n} \langle x^*, i(y^k)\rangle y^k \in T_n(x),$$

and therefore $\hat{x} \in T_n(x)$. Now we show that $T_n(x)$ is closed for all $x \in Y$. Consider an arbitrary sequence $\hat{x}_m \in T_n(x)$, and $\hat{x}_m \to \hat{x}$. Let $x_m^* \in T(y)$ be the sequence such that $\hat{x}_m = \sum_{k=1}^{n} \langle x_m^*, i(y^k)\rangle y^k$. Since $T(y)$ is bounded and convex, the sequence x_m^* converges weakly (in a subsequence) to some $x^* \in T(x)$ and thus

$$\hat{x}_m = \sum_{k=1}^{n} \langle x_m^*, i(y^k)\rangle y^k \to \sum_{k=1}^{n} \langle x^*, i(y^k)\rangle y^k,$$

and thus

$$\hat{x} = \sum_{k=1}^{n} \langle x^*, i(y^k)\rangle y^k \in T_n(x).$$

That $T_n(x)$ is convex follows simply from the convexity of $T(x)$.

By the Lemma (1.6) and Theorem (1.2), the ϵ-continuous selection $T_{n,\epsilon}$ of T_n is well defined. The single valued map $T_{n,\epsilon}$ is continuous and for any $x \in Y_n$, there is $z \in Y_n$ and $\hat{z} \in T_n(z)$ such that $\|z - x\| \in \epsilon$ and $\|\hat{z} - T_{n,\epsilon}(x)\| < \epsilon$.

II. Degree Definition

Let (ϵ_n) be a positive sequence such that $\epsilon_n \to 0$. Fix $\epsilon > 0$. Consider the function $\tilde{T}_{n,\epsilon_n} : Y_n \to Y_n$ defined by the relation

$$\tilde{T}_{n,\epsilon_n} = T_{n,\epsilon_n} + \epsilon J_n, \tag{2.1}$$

where T_{n,ϵ_n} is the ϵ_n-continuous selection of T_n and J_n is the finite rank approximation of $J \circ i : Y \to X^*$ in Y_n where $J : X \to X^*$ is the bi-continuous duality map.

Lemma 2.1. Let $D \subset Y$ be an open bounded set and assume that for some $\epsilon > 0$, we have $0 \notin \mathrm{cl}T^\epsilon(\partial D)$, where $T^\epsilon = T + \epsilon J \circ i$. Then there is $N > 0$ such that $0 \notin \tilde{T}_{n,\epsilon_n}(\partial D_n)$ for all $n \ge N$ where $D_n = D \cap Y_n$.

Proof. Otherwise, there is a sequence $z_n \in \partial D_n$ such that $\tilde{T}_{n,\epsilon_n}(z_n) = 0$ for all $n \ge 1$. Since ∂D is bounded, there is a subsequence (we show again by z_n) that weakly converges to z. We first show that z_n converges strongly to z. Choose a sequence $\zeta_n \in Y_n$ that converges to z in norm. Since $\tilde{T}_{n,\epsilon_n}(z_n) = 0$ on Y_n, we have

$$(T_{n,\epsilon_n}(z_n), z_n - \zeta_n) + \epsilon(J_n(z_n), z_n - \zeta_n) = 0,$$

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because $z_n - \zeta_n \in Y_n$. By the relation

$$(J_n(z_n), z_n - \zeta_n) = \langle J \circ i(z_n), i(z_n - \zeta_n)\rangle,$$

we can write

$$(T_{n,\epsilon_n}(z_n), z_n - \zeta_n) + \epsilon\langle J \circ i(z_n), i(z_n - \zeta_n)\rangle = 0. \tag{2.2}$$

On the other hand, for each z_n, there is $x_n \in Y_n$ and $\hat{x}_n \in T_n(x_n)$ such that

$$\|x_n - z_n\| < \epsilon_n, \|\hat{x}_n - T_{n,\epsilon_n}(z_n)\| < \epsilon_n.$$

Therefore, we have

$$(T_{n,\epsilon_n}(z_n), z_n - \zeta_n) = (T_{n,\epsilon_n}(z_n) - \hat{x}_n, z_n - \zeta_n) + (\hat{x}_n, z_n - \zeta_n),$$

and by the relation $\|T_{n,\epsilon_n}(z_n) - \hat{x}_n\| < \epsilon_n$, we obtain

$$(T_{n,\epsilon_n}(z_n), z_n - \zeta_n) \geq -\epsilon_n\|z_n - \zeta_n\| + (\hat{x}_n, z_n - \zeta_n)$$

By the relation $\|x_n - z_n\| < \epsilon_n$, we have

$$(\hat{x}_n, z_n - \zeta_n) \geq -\epsilon_n\|\hat{x}_n\| + (\hat{x}_n, x_n - \zeta_n).$$

Since $\hat{x}_n \in T_n(x)$, there are $x_n^* \in T(x_n)$ such that $\hat{x}_n$ are the finite rank approximations of x_n^* in Y_n. Thus, we can write

$$(\hat{x}_n, x_n - \zeta_n) = \langle x_n^*, i(x_n - \zeta_n)\rangle.$$

Also, for some $C > 0$, we can write

$$\langle x_n^*, i(x_n - \zeta_n)\rangle \geq -C\|x_n^*\|\|z - \zeta_n\| + \langle x_n^*, i(x_n - z)\rangle.$$

Choose an arbitrary $z^* \in T(z)$. We have

$$\langle x_n^*, i(x_n - z)\rangle \geq \langle x_n^* - z^*, i(x_n - z)\rangle + \langle z^*, i(x_n - z)\rangle \geq \langle z^*, i(x_n - z)\rangle.$$

Since $z_n \rightharpoonup z$ and $\|x_n - z_n\| \to 0$, we conclude

$$\lim_{n\to\infty} (T_{n,\epsilon_n}(z_n), z_n - \zeta_n) \geq 0.$$

Thus, the relation (2.2) implies

$$\limsup_{n\to\infty}\langle J \circ i(z_n), i(z_n - \zeta_n)\rangle \leq 0.$$

By the relation $\zeta_n \to z$, we conclude

$$\limsup_{n\to\infty}\langle J \circ i(z_n), i(z_n - \zeta_n)\rangle \leq 0,$$

and since J is a map of class $(S)_+$, we obtain $z_n \to z \in \partial D$. Now we show $0 \in \text{cl}T^\epsilon(z)$. Choose arbitrary $y \in Y$ and sequence $y_n \in Y_n$, $y_n \to y$. We have

$$(T_{n,\epsilon_n}(z_n), y_n) + \epsilon\langle J \circ i(z_n), i(y_n)\rangle = 0.$$

Since J is continuous, we have $\langle J \circ i(z_n), i(y_n)\rangle \to \langle J \circ i(z), y\rangle$. Choose $x_n \in Y_n$ and $\hat{x}_n \in T_n(x_n)$ such that

$$\|x_n - z_n\| < \epsilon_n, \|\hat{x}_n - T_{n,\epsilon_n}(z_n)\| < \epsilon_n.$$

We have

$$\lim |(T_{n,\epsilon_n}(z_n), y_n) - (\hat{x}_n, y_n)| = 0.$$

For $x_n^* \in T(x_n)$, and by the relation $y_n \to y$, we obtain

$$\lim |(T_{n,\epsilon_n}(z_n), y_n) - \langle x_n^*, i(y)\rangle| = 0.$$

Since T is norm to weak-star upper semi-continuous, we conclude $\langle x_n^*, i(y)\rangle \to \langle x^*, y\rangle$ for some $x^* \in T(z)$. This implies that $x^* + \epsilon J \circ i(z)$=0 and thus $0 \in \text{cl}T^\epsilon(z)$ that contradicts the condition $0 \notin \text{cl}T^\epsilon(\partial D)$.

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Proposition 2.2. Assume that $0 \notin \mathrm{cl}T(\partial D)$. *Then there is* $\epsilon > 0$ *such that* $0 \notin \mathrm{cl}T^\epsilon(\partial D)$.

Proof. By the assumption, there is $r > 0$ such that $\mathrm{dist}(0, \mathrm{cl}T(\partial D)) = r$. Let $z \in \partial D$ is arbitrary. Take arbitrary $z^* \in T(z)$. We have

$$\|z^* + \epsilon J(z)\| \geq \|z^*\| - \epsilon\|z\| \geq r - \epsilon\|z\|.$$

Therefore $0 \notin \mathrm{cl}T^\epsilon(\partial D)$ if

$$0 < \epsilon < \frac{r}{\max_{z\in\partial D}\|z\|}. \tag{2.3}$$

The boundedness of ∂D guarantees the existence of $\epsilon > 0$.

Definition 2.3. Assume that X and Y are separable and reflexive Banach spaces equipped with uniformly convex norms, $D \subset Y$ is an open bounded set and $T : Y \to 2^{X^*}$ is a maximal monotone map such that $0 \notin \mathrm{cl}T(\partial D)$. Choose $\epsilon > 0$ satisfying (2.3) and consider the map $\tilde{T}_{n,\epsilon_n}$ defined in (2.1). The degree of T in D with respect to 0 is defined by the following formula

$$\deg(T, D, 0) = \lim_{n\to\infty} \deg_B(\tilde{T}_{n,\epsilon_n}, D_n, 0), \tag{2.4}$$

where $\deg_B$ is the usual Brouwer's degree of the map $\tilde{T}_{n,\epsilon_n}$.

The degree defined in the relation (2.4) *is stable with respect to* n.

Proof. Consider the sequence of mappings $(\tilde{T}_{k,\epsilon_k})$ such that for sufficiently large n the condition $0 \notin \mathrm{cl}\tilde{T}_{k,\epsilon_k}(\partial D_k)$ is satisfied for $k \geq n-1$. First note that there is $\epsilon_0 > 0$ such that for $0 < \epsilon_1, \epsilon_2 < \epsilon_0$, the following relation holds

$$\deg_B(\tilde{T}_{n,\epsilon_1}, D_n, 0) = \deg_B(\tilde{T}_{n,\epsilon_2}, D_n, 0), \tag{2.5}$$

In fact, for any $x \in Y_n$, there is $z_1, z_2 \in Y_n$ and $\hat{z}_1 \in T_n(z_1)$, $\hat{z}_2 \in T_n(z_2)$ such that

$$\|z_1 - x\| < \epsilon_1, \|\hat{z}_1 - T_{n,\epsilon_1}(x)\| < \epsilon_1, \|z_2 - x\| < \epsilon_2, \|\hat{z}_2 - T_{n,\epsilon_2}(x)\| < \epsilon_2.$$

The continuity of T_{n,ϵ_1}, T_{n,ϵ_2} implies that $\|\tilde{T}_{n,\epsilon_1} - \tilde{T}_{n,\epsilon_2}\|$ can be controlled and thus (2.5) holds. Let us write $\tilde{T}_{n,\epsilon_n}$ as

$$\tilde{T}_{n,\epsilon_n} = \tilde{T}^1_{n,\epsilon_n} + \tilde{T}^2_{n,\epsilon_n},$$

where $\tilde{T}^1_{n,\epsilon_n}$ is the projection of $\tilde{T}_{n,\epsilon_n}$ into Y_{n-1} and $\tilde{T}^2_{n,\epsilon_n}$ is the projection into $\{y^n\}$. Define the map $S_{n,\epsilon_n} : Y_n \to 2^{Y_n}$ as

$$S_{n,\epsilon_n}(x) = \tilde{T}^1_{n,\epsilon_n}(x) + (x, y^n)y^n$$

Obviously, we have

$$\deg_B(S_{n,\epsilon_n}, D_n, 0) = \deg_B(\tilde{T}^1_{n,\epsilon_n}, D_{n-1}, 0). \tag{2.6}$$

First we show

$$\deg_B(\tilde{T}^1_{n,\epsilon_n}, D_{n-1}, 0) = \deg_B(\tilde{T}_{n-1,\epsilon_n}, D_{n-1}, 0) = \deg_B(\tilde{T}_{n-1,\epsilon_{n-1}}, D_{n-1}, 0). \tag{2.7}$$

The last equality follows from (2.5). In order to prove the first equality, we note that if T_{n,ϵ_n} is an ϵ_n-continuous selection of T_n, then T^1_{n,ϵ_1} is also an ϵ_n-continuous selection of T_{n-1,ϵ_n}. In fact, let $x \in Y_{n-1}$ be arbitrary, then there is $z \in Y_n$ and $\hat{z} \in T_n(y)$ such that

$$\|\hat{z}^1 - T^1_{n,\epsilon_n}(x) + \hat{z}^2 - T^2 n, \epsilon_n\|^2 < \epsilon_n^2,$$

where $\hat{z}^1 \in Y_{n-1}$ and $\hat{z}^2 \in \{y^n\}$. This implies

$$\|\hat{z}^1 - T^1_{n,\epsilon_n}(x)\| < \epsilon_n, \|z^1 - y\| < \epsilon_n.$$

Again it follows that $\|T_{n-1,\epsilon_n} - T^1_{n,\epsilon_n}\|$ can be controlled and thus the first equality in (2.7) is proved. Now, we show

$$\deg_B(\tilde{T}_{n,\epsilon_n}, D_n, 0) = \deg_B(S_{n,\epsilon_n}, D_n, 0). \tag{2.8}$$

Consider the convex homotopy

$$h_n(t) = (1-t)\tilde{T}_{n,\epsilon_n} + tS_{n,\epsilon_n}.$$

It is enough to show $0 \notin h_n(t)(\partial D_n)$ for $t \in [0,1]$. Clearly, $0 \notin h_n(t)(\partial D_n)$ for $t = 0, 1$. For $t \in (0,1)$ assume that there exists a sequence $t_n \in (0,1)$ and (z_n), $z_n \in \partial D_n$ such that $h_n(t_n)(z_n) = 0$. According to the construction of $h_n(t)$ we have

$$0 = h_n(t_n)(z_n) = \tilde{T}^1_{n,\epsilon_n}(z_n) + (1-t_n)\tilde{T}^2_{n,\epsilon_n}(z_n) + t_n(z_n, y^n)y^n.$$

The above relation implies $\tilde{T}^1_{n,\epsilon_n}(z_n) = 0$ and

$$\tilde{T}^2_{n,\epsilon_n}(z_n) = -\frac{t_n}{1-t_n}(z_n, y^n)y^n.$$

Since ∂D is bounded then $z_n \rightharpoonup z$ in a subsequence. Choose the sequence $(\zeta_n), \zeta_n \in Y_n$ and $\zeta_n \to z$ and obtain

$$(\tilde{T}_{n,\epsilon_n}(z_n), z_n - \zeta_n) = -\frac{t_n}{1-t_n}|(z_n, y^n)|^2 + \frac{t_n}{1-t_n}(z_n, y^n)(\zeta_n, y^n).$$

On the other hand since $\zeta_n \to z$, we have $(\zeta_n, y^n) \to 0$. Since there exists sequence $\hat{z}_n \in T_n(z_n)$ such that $\|\hat{z}_n - T_{n,\epsilon_n}(z_n)\| < \epsilon_n$ we can write for some $z^*_n \in T(z_n)$

$$\limsup_{n\to\infty}\langle z^*_n + \epsilon J \circ i(z_n), i(z_n - z)\rangle = \limsup_{n\to\infty}(\tilde{T}_{n,\epsilon_n}(z_n), z_n - \zeta_n).$$

Therefore we obtain

$$\limsup_{n\to\infty}\langle z^*_n + \epsilon J \circ i(z_n), i(z_n - z)\rangle \le 0.$$

Since

$$\lim_{n\to\infty}\langle z^*_n, i(z_n - z)\rangle \ge 0,$$

we obtain

$$\limsup_{n\to\infty}\langle J \circ i(z_n), i(z_n - z)\rangle \le 0,$$

and thus $z_n \to z$. This is impossible because $0 \notin \mathrm{cl}T^\epsilon(\partial D)$.

Now, we show that the definition (2.4) satisfies the classical properties of a topological degree including the solvability and the homotopy invariance.

Theorem 2.5. Let $D \subset Y$ be an open bounded set and assume that $T : Y \to 2^{X^*}$ is maximal monotone and $0 \notin \mathrm{cl}T(\partial D)$. If

$$\deg(T, D, 0) \neq 0,$$

then there is $y \in D$ such that $0 \in T(y)$.

Proof. Assume $\deg(T, D, 0) \neq 0$, then there exists a sequence $z_n \in D$ such that $\tilde{T}_{n,\epsilon_n}(z_n) = 0$ for sufficiently small $\epsilon_n > 0$. This implies that there is the sequence $\hat{z}_n \in T_n(z_n)$ such that $\|\hat{z}_n + \epsilon J_n(z_n)\| < \epsilon_n$. Since D is bounded then z_n converges weakly (in a subsequence) to some z. Since T is monotone, we conclude

$$\limsup_{n\to\infty}\langle J \circ i(z_n), i(z_n - z)\rangle \le 0,$$

and thus z_n converges strongly to $z \in \mathrm{cl}(D)$. Let $\hat{z}_n$ be the n-approximation of $z^*_n \in T(z_n)$. Since T is norm to weak-star upper semi-continuous, the sequence z^*_n converges weakly in a subsequence to some $z^* \in T(z)$. Let $v \in Y$ be arbitrary. Consider the sequence $(v_n), v_n \in Y_n$ and $v_n \to y$. Then we have

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$$\langle z_n^*, i(v_n)\rangle = -\epsilon\langle J \circ i(z_n), i(v_n)\rangle - \epsilon(T_{n,\epsilon_n}(z_n) - \hat{z}_n, v_n\rangle \to 0,$$

On the other hand, $\langle z_n^*, i(v_n)\rangle \to \langle z^*, i(y)\rangle$, and thus $z^* = 0$ or equivalently $0 \in T(z)$. Since $0 \notin \mathrm{cl}T(\partial D)$, we conclude $z \in D$.

Definition 2.6. Let $D \subset X$ be an open bounded set, and assume $h : [0,1] \times Y \to 2^{X^*}$ be a continuous homotopy with respect to t such that for any $t \in [0,1]$, the map $h(t) : Y \to 2^{X^*}$ is maximal monotone. Furthermore assume that $0 \notin \mathrm{cl}h([0,1] \times \partial D)$. The map h is called an admissible homotopy for maximal monotone maps.

Proposition 2.7. *The degree defined in* (2.4) *is stable under the admissible homotopy of maximal monotone maps.*

Proof. According to the definition of the admissible homotopy, the degree

$$\deg_B(\tilde{h}_{n,\epsilon_n}(t), D_n, 0)$$

is independent of t due to the fact $0 \notin \tilde{h}_{n,\epsilon_n}(t)(\partial D_n)$ for $t \in [0,1]$ and the homotopy invariance of the Brouwer's degree. Now, the stability of the defined degree (2.4) with respect to n implies that the degree $\deg(h(t), D, 0)$ is independent of t.

III. Dergree Theoretic Proofs

We give degree theoretic proofs of some theorems in convex analysis. The first theorem is due to D. DeFigueirdo [7].

Theorem 3.1. *Assume that X is a separable uniformly convex Banach space, $T : X \to 2^{X^*}$ is a maximal monotone map such that $0 \notin (T + \lambda J)(S_r)$, where S_r is the sphere of radius r and $\lambda > 0$ is arbitrary. Then there exists $u \in \mathrm{cl}(B_r)$ such that $0 \in T(u)$, where B_r is the ball of radius r in X.*

Proof. Assume that $0 \notin T(\mathrm{cl}(B_r))$. We show first that $0 \notin \mathrm{cl}T(S_r)$. Otherwise there exist a sequence $u_n \in S_r$ and $u_n^* \in T(u_n)$ such that $u_n^* \to 0$. The sequence (u_n) converges weakly in a subsequence (that we show again by u_n) to some $u \in \mathrm{cl}(B_r)$. Claim: $[u, 0] \in \mathrm{graph}(T)$. For any $[x, x^*] \in \mathrm{graph}(T)$ we have the inequality

$$\langle x^*, x - u\rangle = \lim\langle x^* - u_n^*, x - u_n\rangle \geq 0.$$

Since T is maximal monotone, the above inequality implies $[x, 0] \in \mathrm{graph}(T)$ or equivalently $0 \in T(u)$. This contradicts the assumption $0 \notin T(\mathrm{cl}(B_r))$. It is also apparent that $0 \notin \mathrm{cl}(J(S_r))$. Next, we show $0 \notin \mathrm{cl}((1-t)T + tJ)(S_r))$ for $t \in (0,1)$. Otherwise there exist $t_n \in (0,1)$, $u_n \in S_r$ and $u_n^* \in T(u_n)$ such that

$$(1 - t_n)u_n^* + t_n J(u_n) \to 0.$$

Again for $u_n \rightharpoonup u$ and $t_n \to t$ we obtain by the monotonicity property of T the following inequality

$$\limsup_{n\to\infty}\langle J(u_n), u_n - u\rangle \leq 0,$$

that implies $u_n \to u \in S_r$. Claim: $[u, \frac{-t}{1-t}J(u)] \in \mathrm{graph}(T)$. For any $[x, x^*] \in \mathrm{graph}(T)$ we obtain by the fact $\frac{-t}{1-t}J(u_n) \in T(u_n)$ the following relation

$$\langle x^* + \frac{t}{1-t}J(u), x - u\rangle = \lim\langle x^* + \frac{t}{1-t}J(u_n), x - u_n\rangle \geq 0,$$

that proves the claim. Now by degree theoretic argument we have

$$\deg(T, B_r, 0) = \deg((1-t)T + tJ, B_r, 0) = \deg(J, B_r, 0) = 1.$$

The above calculation guarantees the existence of $u \in B_r$ such that $0 \in T(u)$ and this contradicts the assumption $0 \notin T(\mathrm{cl}(B_r))$. Therefore the assumption $0 \notin T(\mathrm{cl}(B_r))$ is wrong and thus $0 \in T(\mathrm{cl}(B_r))$.

The next theorem is again from DeFigueirdo [7].

Proposition 3.2. *Let X be a separable uniformly convex Banach space and assume that $f : X \to X^*$ is a pseudo-monotone map. Then* $\mathrm{Rang}(\partial N_r + f) = X^*$ *where N_r is the map*

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7. D. de Figuerido, *An existence theorem for pseudo-monotone operator equation in Banach spaces* J. Math. Anal. Appl., 34:151156, 1971.

$$N_r(x) = \begin{cases} 0 & \text{if } x \in B_r \\ 1 & \text{if } x \in S_r \end{cases},$$

and ∂N_r is the set of the sub-gradients of N_r.

Proof. Apparently, we have

$$\partial N_r(x) = \begin{cases} 0 & \text{if } x \in B_r \\ \{\lambda J(x), \lambda \geq 0\} & \text{if } x \in S_R \end{cases}. \tag{3.1}$$

Claim: for every $f_0 \in X^*$, we have

$$\deg(\partial N_r + f - f_0, B_r, 0) \neq 0. \tag{3.2}$$

First we show if $0 \notin (\partial N_r + f - f_0)(\mathrm{cl}(B_r))$ then

$$0 \notin \mathrm{cl}(\partial N_r + f - f_0)(S_r). \tag{3.3}$$

Otherwise, there is a sequence $u_n \in S_r$ and $u_n^* \in \partial N_r(u_n)$ such that $u_n^* + f(u_n) - f_0 \to 0$. But $u_n \rightharpoonup u \in \mathrm{cl}(B_r)$ in a sub-sequence. We prove that $[u, f_0 - f(u)] \in \mathrm{graph}(\partial N_r)$. Let $f_0 = u_n^* + f(u_n) + \epsilon(n)$ where $\epsilon(n) \in X^*$ and $\epsilon(n) \to 0$. For any arbitrary $[x, x^*] \in \mathrm{graph}(\partial N_r)$, we have

$$\langle x^* - f_0 + f(u), x - u\rangle = \lim\langle x^* + f(u) - u_n^* - f(u_n), x - u_n\rangle \geq \lim\langle f(u) - f(u_n), x - u\rangle.$$

But

$$0 = \lim \langle u_n^* + f(u_n) - f_0, u_n - u\rangle \geq \limsup\langle f(u_n), u_n - u\rangle \tag{3.4}$$

Since f is pseudo-monotone we obtain $f(u_n) \rightharpoonup f(u)$ and therefore

$$\langle x^* - f_0 + f(u), x - u\rangle \geq 0.$$

This implies that $[u, f_0 - f(u)] \in \mathrm{graph}(\partial N_r)$ and thus $0 \in (\partial N_r + f - f_0)(\mathrm{cl}(B_r))$ which is impossible by the assumption. Now consider the affine homotopy

$$h(t) = (1-t)(\partial N_r + f - f_0) + tJ. \tag{3.5}$$

for $t \in (0, 1]$. We show

$$0 \notin \mathrm{cl}((1-t)(\partial N_r + f - f_0) + tJ)(S_r).$$

Otherwise, there is a sequence $u_n \in S_r$, $u_n^* \in \partial N_r(u_n)$ and $t_n \to t$ such that

$$(1 - t_n)(u_n^* + f(u_n) - w) + t_n J u_n \to 0.$$

But $u_n \rightharpoonup u \in \mathrm{cl}(B_r)$ in a subsequence. W show

$$[u, f_0 - f(u) - \frac{t}{1-t} J(u)] \in \mathrm{graph}(\partial N_r).$$

For any $[x, x^*] \in \mathrm{graph}(\partial N_r)$ we have

$$\langle x^* - f_0 + f(u) + \frac{t}{1-t} J(u), x - u\rangle =$$

$$\lim\langle x^* + f(u) + \frac{t}{1-t} J(u) - u_n^* - f(u_n) - \frac{t_n}{1-t_n} J(u_n), x - u\rangle \geq$$

$$\geq \limsup\langle f(u) - f(u_n), x - u\rangle + \liminf\langle \frac{t}{1-t} J(u) - \frac{t_n}{1-t_n} J(u_n), x - u\rangle$$

We conclude $u_n \to u \in S_r$ and $f(u_n) \rightharpoonup f(u)$ because f is pseudo-monotone. Therefore we obtain again

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$$0 \in (\partial N_r + f + \frac{t}{1-t}J - f_0)(\mathrm{cl}(B_r)).$$

But $(\partial N_r + f + \frac{t}{1-t}J - f_0)(\mathrm{cl}(B_r)) = (\partial N_r + f - f_0)(\mathrm{cl}(B_r))$ and then $0 \in (\partial N_R + f - f_0)(\mathrm{cl}(B_r))$ that is impossible. Finally we use the homotopy invariance property of degree and write

$$\deg(\partial N_R + f - f_0, B_r, 0) = \deg(h(t), B_R, 0) = \deg(J, B_r, 0) = 1.$$

Therefore there exist $u \in \mathrm{cl}(B_r)$ such that $f_0 \in \partial N_r(u) + f(u)$.

The following theorem is due to F. Browder [8] for the surjectivity of the monotone maps with locally bounded inverse.

Theorem 3.3. Assume $A : X \to X^*$ is a demi-continuous monotone map such that A^{-1} is locally bounded, that is, for every $f \in X^*$ there is a bounded $V_f \ni f$ such that $A^{-1}(V_f)$ is bounded. Then A is onto.

Proof. For any $f \in X^*$, we show that there is sufficiently large $r = r(f)$ such that:

$$\deg(A, B_r, f) \neq 0.$$

Choose $r > 0$ such that for a neighborhood $V_f \ni f$ the following condition is satisfied

$$S_r \cap A^{-1}(V_f) = \emptyset,$$

or equivalently $f \notin \mathrm{cl}A(S_r)$. Since there is $\epsilon > 0$ such that

$$\deg(A, B_r, f) = \deg(A + \epsilon J, B_r, f),$$

it is enough to show

$$\deg(A + \epsilon J, B_r, f) \neq 0, \tag{3.6}$$

for sufficiently large r and sufficiently small $\epsilon > 0$. First, we show

$$\deg(A + \epsilon J, B_r, 0) \neq 0.$$

In fact, if $(A + \epsilon J)(z) = 0$ for $z \in \partial B_r$, then

$$\langle A(z) - A(0), z\rangle + \epsilon\|z\|^2 + \langle A(0), z\rangle = 0.$$

Since A is monotone, the inequality $\epsilon\|z\|^2 + \langle A(0), z\rangle \leq 0$ implies $\epsilon\|z\| \leq \|A(0)\|$, that is impossible for sufficiently large r. Since $A + \epsilon J$ is a map of class $(S)_+$, define the homotopy $h(t) = tA + \epsilon J$. It is simply seen that $0 \notin h(t)(\partial B_r)$ and then

$$\deg(A + \epsilon J, B_r, 0) = \deg(h(t), B_r, 0) = \deg(J, B_r, 0) \neq 0.$$

The proof of (3.6) is completely similar to one presented above.

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