Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 2 Version 1.0 Year 2018
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Certain Results on Bicomplex Matrices

By Anjali & Amita
Dr. B. R. Ambedkar University

Abstract- This paper begins the study of bicomplex matrices. In this paper, we have defined bicomplex matrices, determinant of a bicomplex square matrix and singular and non-singular matrices in C_2. We have proved that the set of all bicomplex square matrices of order n is an algebra. We have given some definitions and results regarding adjoint and inverse of a matrix in C_2. We have defined three types of conjugates and three types of tranjugates of a bicomplex matrix. With the help of these conjugates and tranjugates, we have also defined symmetric and skew - symmetric matrices, Hermitian and Skew - Hermitian matrices in C_2.

Keywords: *bicomplex matrices, conjugates matrices, tranjugate matrices, Hermitian matrices, skew Hermitian matrices.*

GJSFR-F Classification: *MSC: 15 B 57, 30 G 35*

CERTAIN RESULTS ON BICOMPLEX MATRICES

Ref

8. Segre, C.: Le Rappresentazioni Reali Delle Forme Complesse e Gli enti Iperalgebrici, Math.Ann. 40 (1892), 413-467.

Certain Results on Bicomplex Matrices

Anjali α & Amita σ

Abstract- This paper begins the study of bicomplex matrices. In this paper, we have defined bicomplex matrices, determinant of a bicomplex square matrix and singular and non-singular matrices in C_2. We have proved that the set of all bicomplex square matrices of order n is an algebra. We have given some definitions and results regarding adjoint and inverse of a matrix in C_2. We have defined three types of conjugates and three types of tranjugates of a bicomplex matrix. With the help of these conjugates and tranjugates, we have also defined symmetric and skew - symmetric matrices, Hermitian and Skew - Hermitian matrices in C_2.

Keywords: bicomplex matrices, conjugates matrices, tranjugate matrices, Hermitian matrices, skew Hermitian matrices.

I. Introduction

In 1892, Corrado Segre (1860-1924) published a paper [8] in which he treated an infinite set of Algebras whose elements he called bicomplex numbers, tricomplex numbers,…., n-complex numbers. A bicomplex number is an element of the form $(x_1+i_1x_2) + i_2 (x_3+i_1x_4)$, where $x_1,\ldots, x_4$ are real numbers, $i_1^2 = i_2^2 = -1$ and $i_1i_2 = i_2i_1$.

Segre showed that every bicomplex number $z_1+i_2z_2$ can be represented as the complex combination

$$(z_1 - i_1 z_2)\left[\frac{1+i_1 i_2}{2}\right] + (z_1 + i_1 z_2)\left[\frac{1-i_1 i_2}{2}\right]$$

Srivastava [9] introduced the notations ${}^1\xi$ and ${}^2\xi$ for the idempotent components of the bicomplex number $\xi = z_1+i_2z_2$, so that

$$\xi = {}^1\xi\frac{1+i_1 i_2}{2} + {}^2\xi\frac{1-i_1 i_2}{2}$$

Michiji Futagawa seems to have been the first to consider the theory of functions of a bicomplex variable [2,3] in 1928 and 1932.

The hyper complex system of Ringleb [7] is more general than the Algebras; he showed in 1933 that Futagawa system is a special case of his own.

In 1953 James D. Riley published a paper [6] entitled "Contributions to theory of functions of a bicomplex variable".

Throughout, the symbols C_2, C_1, C_0 denote the set of all bicomplex, complex and real numbers respectively.

Author α σ: Department of Mathematics, Institute of basic Science, Khandari, Dr. B.R. Ambedkar University, Agra-282002, India.
e-mails: anjalisharma773@gmail.com, amitasharma234@gmail.com

a) Some special subset of C_2

We shall use notation C (i_1), C (i_2) and H for the following sets.

C(i_1) is the set of complex numbers with imaginary unit i_1.i. e.

$$C\ (i_1) = \{a + i_1\ b;\ a,\ b \in C_0\}$$

and C (i_2) is the set of complex numbers with imaginary unit i_2.i. e.

$$C\ (i_2) = \{a + i_2\, b;\ a,\ b \in C_0\}$$

The bicomplex number $\xi = (x_1+i_1x_2) + i_2\,(x_3+i_1x_4)$ for which $x_2 = x_3 = 0$ is called a hyperbolic number.

The set of all hyperbolic numbers is denoted by H and defined as

$$H = \{a + i_1i_2b;\ a,\ b \in C_0\}$$

b) Idempotent elements in C_2

There are exactly four idempotent elements in C_2. Out of these, 0 and 1 are the trivial idempotent elements and two nontrivial idempotent elements denoted by e_1 and e_2 which are defined as

$$e_1 = \frac{1+i_1i_2}{2} \text{ and } e_2 = \frac{1-i_1i_2}{2}$$

Obviously $(e_1)^n = e_1$, $(e_2)^n = e_2$

$$e_1 + e_2 = 1,\ e_1.e_2 = 0$$

C_1 is a field but C_2 is not a field, since C_2 has divisor of zero for example $e_1\ e_2 = 0$ neither e_1 is zero nor e_2 is zero.

Every bicomplex number ξ has unique idempotent representation as complex combination of e_1 and e_2 as follows

$$\xi = z_1 + i_2 z_1 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$$

The complex numbers $(z_1 - i_1 z_2)$ and $(z_1 + i_1 z_2)$ are called idempotent component of ξ and are denoted by ${}^1\xi$ and ${}^2\xi$ respectively (cf. Srivastava [9]).

Thus $\xi = {}^1\xi\ e_1 + {}^2\xi\ e_2$

There are infinite numbers of element in C_2 which do not possess multiplicative inverse. A bicomplex number $\xi = z_1 + i_2 z_1$ is singular if and only if $|z_1^2 + z_2^2| = 0$

The set of all singular elements in C_2 is denoted by O_2.

Evidently a nonzero bicomplex number ξ is singular if and only if either ${}^1\xi = 0$ or ${}^2\xi = 0$ that is if and only if it is a complex multiple of either e_1 or e_2.

c) Algebraic properties of idempotent components

The idempotent representation is perfectly compatible with the algebraic structure of C_2 in the following way

For all ξ, η in C_2

$$\xi \pm \eta = ({}^1\xi e_1 + {}^2\xi e_2) \pm ({}^1\eta e_1 + {}^2\eta e_2)$$

$$= ({}^1\xi \pm {}^1\eta)e_1 + ({}^2\xi \pm {}^2\eta)e_2,$$

Notes

so that $${}^1(\xi \pm \eta) = {}^1\xi \pm {}^1\eta \text{ and } {}^2(\xi \pm \eta) = {}^2\xi \pm {}^2\eta$$

$$\alpha\xi = \alpha \ ({}^1\xi e_1 + {}^2\xi e_2)$$

$$= \alpha\ ({}^1\xi)\ e_1 + \alpha\ ({}^2\xi)\ e_2,\ \ \forall \alpha \in C_1$$

so that $${}^1(\alpha\xi) = \alpha\, {}^1\xi \text{ and } {}^2(\alpha\xi) = \alpha\, {}^2\xi \text{ for } \alpha \in C_1$$

Notes

$$\xi\eta = ({}^1\xi e_1 + {}^2\xi e_2).({}^1\eta e_1 + {}^2\eta e_2)$$

$$= ({}^1\xi\, {}^1\eta)e_1 + ({}^2\xi\, {}^2\eta)e_2,$$

so that $${}^1(\xi\eta) = {}^1\xi\, {}^1\eta \text{ and } {}^2(\xi\eta) = {}^2\xi\, {}^2\eta$$

$$\frac{\xi}{\eta} = \left({}^1\xi e_1 + {}^2\xi e_2\right) \Big/ \left({}^1\eta e_1 + {}^2\eta e_2\right)$$

$$= \left({}^1\xi \big/ {}^1\eta\right)e_1 + \left({}^2\xi \big/ {}^2\eta\right)e_2, \quad \textit{provided}\ \eta \notin O_2$$

so that $${}^1\left(\xi/\eta\right) = {}^1\xi \big/ {}^1\eta \text{ and } {}^2\left(\xi/\eta\right) = {}^2\xi \big/ {}^2\eta$$

d) Norm in C_2[5]

The norm of a bicomplex number

$$\xi = z_1+i_2z_2 = x_1+i_1x_2+i_2x_3+i_1i_2x_4 = {}^1\xi\ e_1 + {}^2\xi\ e_2 \text{ is defined as}$$

$$\|\xi\| = (x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2)^{1/2}$$

$$= (|z_1|^2 + |z_2|^2)^{1/2}$$

$$= \sqrt{\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}}$$

C_2 becomes a modified Banach algebra, in the sense that $\xi, \eta \in C_2$, we have,

In general $\|\xi.\eta\| \le \sqrt{2}\ \|\xi\|\,\|\eta\|$

e) Conjugates of a bicomplex number

Analogous to the concept of conjugate of a complex number, conjugates of a bicomplex number are also defined. As a bicomplex number is four dimensional, different types of conjugate arise.

In bicomplex space C_2, every number ξ possesses three types of conjugates. The i_1 conjugate, i_2 conjugate and i_1i_2 conjugate of $\xi = z_1+i_2z_2 = x_1+i_1x_2+i_2x_3+i_1i_2x_4 = {}^1\xi\ e_1 + {}^2\xi$ e_2 are denoted by $\bar{\xi}$, $\xi^{\sim}$ and $\xi^{\#}$ respectively, therefore

$$\bar{\xi} = (x_1 - i_1x_2) + i_2(x_3 - i_1x_4)$$

$$= (\bar{z}_1 + i_2\bar{z}_2)$$

$$= (\overline{{}^2\xi}\, e_1 + \overline{{}^1\xi}\, e_2)$$

$$\xi^{\sim} = (x_1 + i_1 x_2) - i_2(x_3 + i_1 x_4)$$
$$= (z_1 - i_2 z_2)$$
$$= ({}^{2}\xi e_1 + {}^{1}\xi e_2)$$
$$\xi^{\#} = (x_1 - i_1 x_2) - i_2(x_3 - i_1 x_4)$$
$$= (\bar{z}_1 - i_2 \bar{z}_2)$$
$$= (\overline{{}^{1}\xi} e_1 + \overline{{}^{2}\xi} e_2)$$

Notes

II. Certain Results From Bicomplex Matrices

a) Some Definitions

2.1.1 Bicomplex matrices

A matrix $A = [\xi_{mn}]_{m\times n}$ whose entries belong in C_2, is said to be a bicomplex matrix i.e. we define

$$A = \begin{bmatrix} \xi_{11} & \xi_{12} & -- & \xi_{1n} \\ \xi_{21} & \xi_{22} & -- & \xi_{2n} \\ - & - & -- & - \\ \xi_{m1} & \xi_{m2} & -- & \xi_{mn} \end{bmatrix} \forall\, \xi_{pq} \in C_2$$

Where $1 \le p \le m$ and $1 \le q \le n$

Since every bicomplex number ξ has unique idempotent representation as complex combination of e_1 and e_2 as follows

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$$

Therefore every bicomplex matrices $A = [\xi_{mn}]_{m\times n}$ can be expressed uniquely as ${}^{1}Ae_1 + {}^{2}Ae_2$ such that ${}^{1}A = [z_{mn}]_{m\times n}$ and ${}^{2}A = [w_{mn}]_{m\times n}$ are complex matrices.

2.1.2 Bicomplex square matrices

A bicomplex matrix in which the number of rows is equal to the number of columns is called a bicomplex square matrix. i.e.

$$A = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{bmatrix}; \xi_{pq} \in C_2;\ p, q = 1,2,\ldots,n$$

2.1.3 Bicomplex diagonal matrices

A bicomplex square matrix A is called a diagonal matrix if all its non-diagonal elements are zero i.e.

$$A = \begin{bmatrix} \xi_{11} & 0 & \cdots & 0 \\ 0 & \xi_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \xi_{nn} \end{bmatrix},\quad \xi_{pq} \in C_2$$

Global Journal of Science Frontier Research (F) Volume XVIII Issue II Version I Year 2018

Notes

2.1.4 Determinant of a bicomplex matrix

Let $A = [\xi_{ij}]_{n\times n}$ be a bicomplex square matrix of order n, where n is some positive integer. The determinant of A, is defined as

$$\det A = |A| = \left|[\xi_{ij}]\right|,\ \xi_{ij} \in C_2$$

$$detA = |A| = \sum_{\sigma\in S_n} Sig.(\sigma)\prod_{i=1}^{n}\xi_{i\,\sigma(i)}$$

Where S_n is the group of all permutation on 'n' symbols.

2.1.5 Transpose of a bicomplex matrix

If $A = [\xi_{m\,n}]_{m\times n}$ is any bicomplex matrix then A matrix of order $n\times m$ obtained from 'A' by changing its rows into columns and its columns into rows is called transpose of 'A' and is denoted by A^T.

2.1.6 Cofactor and adjoint matrix of a matrix in C_2

Let $A = [\xi_{ij}]_{n\times n}$ be a bicomplex square matrix of order n then cofactor of the entry ξ_{ij} is defined as $(-1)^{i+j}$ × the determinant obtained by leaving the row and the column(In the matrix A) passing through the entry $\xi_{ij} = \eta_{ij}$ (say).

Then the matrix $[\eta_{ij}]_{n\times n}$ is defined as the cofactor matrix of A and the transpose of cofactor matrix of A is known as adjoint matrix of A. i.e. $AdjA = [\eta_{ij}]^T_{n\times n}$

2.1.7 Bicomplex singular and non-singular matrix

A bicomplex Square matrix is said to be non-singular if $|A| \notin O_2$ (set of all singular element in C_2).

and If $|A| \in O_2$ then it is called singular matrix.

b) Algebraic structure of bicomplex Matrices

Let S be the set of all bicomplex square matrices of order n. Define binary compositions over S called addition "+" , scalar multiplication "." and multiplication "×" as follows:

$$\text{Let}A = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{bmatrix} \text{and}$$

$$B = \begin{bmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1n} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \eta_{n1} & \eta_{n2} & \cdots & \eta_{nn} \end{bmatrix}$$

be the arbitrary member of S and $\alpha \in F$, where F is either field of real numbers or complex numbers.

$$A + B = \begin{bmatrix} \xi_{11}+\eta_{11} & \xi_{12}+\eta_{12} & \ldots\ldots & \xi_{1n}+\eta_{1n} \\ \xi_{21}+\eta_{21} & \xi_{22}+\eta_{22} & \ldots\ldots & \xi_{2n}+\eta_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \xi_{n1}+\eta_{n1} & \xi_{n2}+\eta_{n2} & \ldots\ldots & \xi_{nn}+\eta_{nn} \end{bmatrix}$$

$$\alpha . A = \begin{bmatrix} \alpha\xi_{11} & \alpha\xi_{12} & \ldots\ldots & \alpha\xi_{1n} \\ \alpha\xi_{21} & \alpha\xi_{22} & \ldots\ldots & \alpha\xi_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha\xi_{1n} & \alpha\xi_{2n} & \ldots\ldots & \alpha\xi_{nn} \end{bmatrix}. \text{ and } A \times B = \begin{bmatrix} \xi_{11}\eta_{11}+\ldots\ldots+\xi_{1n}\eta_{n1} & --- & \xi_{11}\eta_{1n}+\ldots\ldots+\xi_{1n}\eta_{nn} \\ \xi_{21}\eta_{11}+\ldots\ldots+\xi_{2n}\eta_{n1} & --- & \xi_{21}\eta_{1n}+\ldots\ldots+\xi_{2n}\eta_{nn} \\ --- & --- & --- \\ \xi_{n1}\eta_{11}+\ldots\ldots+\xi_{nn}\eta_{n1} & --- & \xi_{n1}\eta_{1n}+\ldots\ldots+\xi_{nn}\eta_{nn} \end{bmatrix}$$

2.2.1 Theorem: The set of all bicomplex square matries i.e "S" forms an algebra.

Proof:

a. *Additive abelian group structure*

- *Associativity:*

Let A $=[\xi_{ij}]_{n\times n}$, B = $[\eta_{ij}]_{n\times n}$ and C = $[\varsigma_{ij}]_{n\times n}$ be the member of S and $\alpha, \beta \in F$ then

$$A+(B+C) = \begin{bmatrix} \xi_{11}+(\eta_{11}+\varsigma_{11}) & \ldots\ldots\ldots & \xi_{1n}+(\eta_{1n}+\varsigma_{1n}) \\ \xi_{21}+(\eta_{21}+\varsigma_{21}) & \ldots\ldots\ldots & \xi_{2n}+(\eta_{2n}+\varsigma_{2n}) \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \xi_{n1}+(\eta_{n1}+\varsigma_{n1}) & \ldots\ldots\ldots & \xi_{nn}+(\eta_{nn}+\varsigma_{nn}) \end{bmatrix}$$

$$(A+B)+C = \begin{bmatrix} (\xi_{11}+\eta_{11})+\varsigma_{11} & \ldots\ldots\ldots & (\xi_{1n}+(\eta_{1n})+\varsigma_{1n} \\ (\xi_{21}+\eta_{21})+\varsigma_{21} & \ldots\ldots\ldots & (\xi_{2n}+\eta_{2n})+\varsigma_{2n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ (\xi_{n1}+\eta_{n1})+\varsigma_{n1} & \ldots\ldots\ldots & (\xi_{nn}+\eta_{nn})+\varsigma_{nn} \end{bmatrix}$$

Since C_2 is an algebra

Therefore A + (B +C) = (A + B)+ C

Identity: $\forall$ A $\in$ S $\exists$ null matrix "0"$\in$ S then A + 0 =A i.e

$$\begin{bmatrix} \xi_{11} & \xi_{12} & \ldots\ldots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \ldots\ldots & \xi_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \xi_{n1} & \xi_{n2} & \ldots\ldots & \xi_{nn} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \ldots\ldots & 0 \\ 0 & 0 & \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & \ldots\ldots & 0 \end{bmatrix} = \begin{bmatrix} \xi_{11} & \xi_{12} & \ldots\ldots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \ldots\ldots & \xi_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \xi_{n1} & \xi_{n2} & \ldots\ldots & \xi_{nn} \end{bmatrix}$$

So 0 is the additive identity.

- Inverse:

$\forall$ A$\in$ S $\exists$- A $\in$ S such that $-$A +A = 0 i.e.

$$\begin{bmatrix} -\xi_{11} & -\xi_{12} & \ldots\ldots & -\xi_{1n} \\ -\xi_{21} & -\xi_{22} & \ldots\ldots & -\xi_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ -\xi_{n1} & -\xi_{n2} & \ldots\ldots & -\xi_{nn} \end{bmatrix} + \begin{bmatrix} \xi_{11} & \xi_{12} & \ldots\ldots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \ldots\ldots & \xi_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \xi_{n1} & \xi_{n2} & \ldots\ldots & \xi_{nn} \end{bmatrix}. = \begin{bmatrix} 0 & 0 & \ldots\ldots & 0 \\ 0 & 0 & \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & \ldots\ldots & 0 \end{bmatrix}$$

Notes

Notes

Therefore "–A" is the additive inverse of A.

- *Commutative:*

Since

$$A + B = \begin{bmatrix} \xi_{11}+\eta_{11} & \xi_{12}+\eta_{12} & \cdots & \xi_{1n}+\eta_{1n} \\ \xi_{21}+\eta_{21} & \xi_{22}+\eta_{22} & \cdots & \xi_{2n}+\eta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1}+\eta_{n1} & \xi_{n2}+\eta_{n2} & \cdots & \xi_{nn}+\eta_{nn} \end{bmatrix}$$

And

$$B + A = \begin{bmatrix} \eta_{11}+\xi_{11} & \eta_{12}+\xi_{12} & \cdots & \eta_{1n}+\xi_{1n} \\ \eta_{21}+\xi_{21} & \eta_{22}+\xi_{22} & \cdots & \eta_{2n}+\xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \eta_{n1}+\xi_{n1} & \eta_{n2}+\xi_{n2} & \cdots & \eta_{nn}+\xi_{nn} \end{bmatrix}$$

Therefore A + B = B+A as C_2 is an algebra.
So S is abelian group under addition

b. *Ring structure*

- *Closure:*

Since

$$A\times B = \begin{bmatrix} \xi_{11}\eta_{11}+\cdots+\xi_{1n}\eta_{n1} & --- & \xi_{11}\eta_{1n}+\cdots+\xi_{1n}\eta_{nn} \\ \xi_{21}\eta_{11}+\cdots+\xi_{2n}\eta_{n2} & --- & \xi_{21}\eta_{1n}+\cdots+\xi_{2n}\eta_{nn} \\ --- & --- & --- \\ \xi_{n1}\eta_{11}+\cdots+\xi_{nn}\eta_{n1} & --- & \xi_{n1}\eta_{1n}+\cdots+\xi_{nn}\eta_{nn} \end{bmatrix}$$

Therefore it is evident that $A\times B \in S$
Therefore S is closed under multiplication

- *Associativity:*

$\forall$ A, B, C $\in$S

Let $A = [\xi_{ij}]_{n\times n}$, $B = [\eta_{ij}]_{n\times n}$ and $C = [\varsigma_{ij}]_{n\times n}$

The $i^{th}j^{th}$ entry of (A×B)×C = [i^{th}row of (A×B)]×[j^{th} column of C]

=[i^{th} row of A]×B×[j^{th}column of C]

Now the $i^{th}j^{th}$ entry of A×(B×C) = [i^{th} row of A]×[j^{th} column of (B×C)]

=[i^{th} row of A] × B × [j^{th}column of C]

Therefore the $i^{th}j^{th}$ entry of (A×B)×C =$i^{th}j^{th}$ entry of A×(B×C)

Hence (A×B)×C = A×(B×C)

Distribution law:

We can easily show that

(A + B)×C = A×C+B×C and A×(B+C)=A×B+A×C

c. *Linear space structure*

$$\alpha.(A+B)=\alpha.\begin{bmatrix} \xi_{11}+\eta_{11} & \xi_{12}+\eta_{12} & \cdots & \xi_{1n}+\eta_{1n} \\ \xi_{21}+\eta_{21} & \xi_{22}+\eta_{22} & \cdots & \xi_{2n}+\eta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1}+\eta_{n1} & \xi_{n2}+\eta_{n2} & \cdots & \xi_{nn}+\eta_{nn} \end{bmatrix} \quad (1)$$

$$=\begin{bmatrix} \alpha(\xi_{11}+\eta_{11}) & \alpha(\xi_{12}+\eta_{12}) & \cdots & \alpha(\xi_{1n}+\eta_{1n}) \\ \alpha(\xi_{21}+\eta_{21}) & \alpha(\xi_{22}+\eta_{22}) & \cdots & \alpha(\xi_{2n}+\eta_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ \alpha(\xi_{n1}+\eta_{n1}) & \alpha(\xi_{n2}+\eta_{n2}) & \cdots & \alpha(\xi_{nn}+\eta_{nn}) \end{bmatrix}=\begin{bmatrix} \alpha\xi_{11} & \alpha\xi_{12} & \cdots & \alpha\xi_{1n} \\ \alpha\xi_{21} & \alpha\xi_{22} & \cdots & \alpha\xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha\xi_{n1} & \alpha\xi_{n2} & \cdots & \alpha\xi_{nn} \end{bmatrix}+$$

$$\begin{bmatrix} \alpha\eta_{11} & \alpha\eta_{12} & \cdots & \alpha\eta_{1n} \\ \alpha\eta_{21} & \alpha\eta_{22} & \cdots & \alpha\eta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha\eta_{n1} & \alpha\eta_{n2} & \cdots & \eta_{nn} \end{bmatrix}=\alpha.\begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{bmatrix}+\alpha.\begin{bmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1n} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \eta_{n1} & \eta_{n2} & \cdots & \eta_{nn} \end{bmatrix}$$

Therefore $\alpha.(A+B)=\alpha.A+\alpha.B$

$$(\alpha\beta).A=(\alpha\beta).\begin{bmatrix} \xi_{11} & \xi_{12} & - & \xi_{1n} \\ \xi_{21} & \xi_{22} & - & \xi_{2n} \\ - & - & - & - \\ \xi_{n1} & \xi_{n2} & - & \xi_{nn} \end{bmatrix}=\begin{bmatrix} (\alpha\beta)\xi_{11} & (\alpha\beta)\xi_{12} & - & (\alpha\beta)\xi_{1n} \\ (\alpha\beta)\xi_{21} & (\alpha\beta)\xi_{22} & - & (\alpha\beta)\xi_{2n} \\ - & - & - & - \\ (\alpha\beta)\xi_{n1} & (\alpha\beta)\xi_{n2} & - & (\alpha\beta)\xi_{nn} \end{bmatrix}(\alpha\beta).A=\alpha.\begin{bmatrix} \beta\xi_{11} & \beta\xi_{12} & - & \beta\xi_{1n} \\ \beta\xi_{21} & \beta\xi_{22} & - & \beta\xi_{2n} \\ - & - & - & - \\ \beta\xi_{1n} & \beta\xi_{2n} & - & \beta\xi_{nn} \end{bmatrix} \quad (2)$$

Therefore $(\alpha\beta).A=\alpha.(\beta.A)$

$$(\alpha+\beta).A=(\alpha+\beta).\begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{bmatrix}=\alpha.\begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{bmatrix}+\beta.\begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{bmatrix} \quad (3)$$

Therefore $(\alpha+\beta).A=\alpha.A+\beta.A$

(4) It is evident that $1.A = A$ for all A in S and $1\in F$

d. Consistency between multiplication and scalar multiplication

$$\alpha.(A\times B)=(\alpha.A)\times B=A\times(\alpha.B)$$

$$\alpha.(A\times B)=\alpha.\left\{\begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{bmatrix}\times\begin{bmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1n} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \eta_{1n} & \eta_{2n} & \cdots & \eta_{nn} \end{bmatrix}\right\}$$

Notes

$$\alpha.(A\times B)=\alpha.\begin{bmatrix}\xi_{11}\eta_{11}+.....+\xi_{1n}\eta_{n1} & --- & \xi_{11}\eta_{1n}+......+\xi_{1n}\eta_{nn}\\ \xi_{21}\eta_{11}+.....+\xi_{2n}\eta_{n2} & --- & \xi_{21}\eta_{1n}+....+\xi_{2n}\eta_{nn}\\ --- & --- & ---\\ \xi_{n1}\eta_{11}+.....+\xi_{nn}\eta_{n1} & --- & \xi_{n1}\eta_{1n}+....+\xi_{nn}\eta_{nn}\end{bmatrix}$$

Notes

$$=\begin{bmatrix}\alpha(\xi_{11}\eta_{11})+.....+\alpha(\xi_{1n}\eta_{n1}) & --- & \alpha(\xi_{11}\eta_{1n})+......+\alpha(\xi_{1n}\eta_{nn})\\ \alpha(\xi_{21}\eta_{11})+.....+\alpha(\xi_{2n}\eta_{n2}) & --- & \alpha(\xi_{21}\eta_{1n})+....+\alpha(\xi_{2n}\eta_{nn})\\ --- & --- & ---\\ \alpha(\xi_{n1}\eta_{11})+.....+\alpha(\xi_{nn}\eta_{n1}) & --- & \alpha(\xi_{n1}\eta_{1n})+....+\alpha(\xi_{nn}\eta_{nn})\end{bmatrix}$$

Since C_2 is an algebra

Therefore $\alpha.(A\times B)=(\alpha.A)\times B=A\times(\alpha.B)$

Hence it proves that S is an algebra.

2.2.2 Theorem: Let $A = [\xi_{i\,j}]_{n\times n}$ be a bicomplex square matrix then det $A = (\det{}^1A)\, e_1 + (\det{}^2A)\, e_2$.

Proof:

Let

$$A=\begin{bmatrix}\xi_{11} & \xi_{12} & ..... & \xi_{1n}\\ \xi_{21} & \xi_{22} & .... & \xi_{2n}\\ ..... & ..... & ..... & .....\\ \xi_{n1} & \xi_{n2} & ..... & \xi_{nn}\end{bmatrix}$$

From 2.1.4,

$$|A|=\left|\begin{bmatrix}\xi_{11} & \xi_{12} & ..... & \xi_{1n}\\ \xi_{21} & \xi_{22} & ..... & \xi_{2n}\\ ..... & ..... & ..... & .....\\ \xi_{n1} & \xi_{n2} & ..... & \xi_{nn}\end{bmatrix}\right|.$$

$$\therefore\ detA=|A|=\sum_{\sigma\in S_n} Sig.(\sigma)\prod_{i=1}^{n}\xi_{i\,\sigma(i)}$$

$$=\left[\sum_{\sigma\in S_n} Sig.(\sigma)\prod_{i=1}^{n}{}^1\xi_{\,i\,\sigma(i)}\right]e_1+\left[\sum_{\sigma\in S_n} Sig.(\sigma)\prod_{i=1}^{n}{}^2\xi_{\,i\,\sigma(i)}\right]e_2$$

As $\xi.\eta = ({}^1\xi{}^1\eta)\, e_1 + ({}^2\xi{}^2\eta)\, e_2$ and $\xi+\eta = ({}^1\xi+{}^1\eta)\, e_1 + ({}^2\xi+{}^2\eta)\, e_2$

Since $\quad \det{}^1A = \left|\left[{}^1\xi_{i\,j}\right]_{n\times n}\right|$

$$= \left[\sum_{\sigma \in S_n} Sig.(\sigma) \prod_{i=1}^{n} {}^{1}\xi_{\, i\, \sigma(i)} \right]$$

And $\qquad \det{}^2 A = \left| \left[{}^{2}\xi_{ij} \right]_{n \times n} \right|$

$$= \left[\sum_{\sigma \in S_n} Sig.(\sigma) \prod_{i=1}^{n} {}^{2}\xi_{\, i\, \sigma(i)} \right]$$

Therefore det A = $(\det{}^1 A)\, e_1 + (\det{}^2 A)\, e_2$

2.2.3 Theorem: If the determinant of A is non-singular then $|{}^1A| \neq 0$ and $|{}^2A| \neq 0$.

Proof:

Suppose

$$A = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{bmatrix}$$

From 2.2.2,

det A = $(\det{}^1 A)\, e_1 + (\det{}^2 A)\, e_2$

since det A is non-singular

therefore det A = $[(\det{}^1 A)\, e_1 + (\det{}^2 A)\, e_2] \notin O_2$

(Since $({}^1\xi e_1 + {}^2\xi e_2) \notin O_2$ then ${}^1\xi e_1$ and ${}^2\xi e_2$ both are non-zero)

Hence $|{}^1A| \neq 0$ and $|{}^2A| \neq 0$

2.2.4 Theorem: Let A be a bicomplex matrix then $A^T = {}^1A^T\, e_1 + {}^2A^T\, e_2$.

Proof:

Let A = $[\xi_{ij}]_{m \times n}$ be a bicomplex matrix then

$$A^T = \begin{bmatrix} \xi_{11} & \xi_{21} & \cdots & \xi_{m1} \\ \xi_{12} & \xi_{22} & \cdots & \xi_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{1n} & \xi_{2n} & \cdots & \xi_{mn} \end{bmatrix}$$

$$A^T = \begin{bmatrix} {}^1\xi_{11} & {}^1\xi_{21} & \cdots & {}^1\xi_{m1} \\ {}^1\xi_{12} & {}^1\xi_{22} & \cdots & {}^1\xi_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ {}^1\xi_{1n} & {}^1\xi_{2n} & \cdots & {}^1\xi_{mn} \end{bmatrix} e_1 +$$

Notes

$$\begin{bmatrix} {}^2\xi_{11} & {}^2\xi_{21} & ..... & {}^2\xi_{m1} \\ {}^2\xi_{12} & {}^2\xi_{22} & ..... & {}^2\xi_{m2} \\ ..... & .... & .... & .... \\ {}^2\xi_{1n} & {}^2\xi_{2n} & .... & {}^2\xi_{mn} \end{bmatrix} e_2$$

Notes

Therefore $A^T = {}^1A^T e_1 + {}^2A^T e_2$

2.2.5 Theorem: Let A be a bicomplex square matrix then cofactor matrix of A = (cofactor matrix of ${}^1A$) e_1+(cofactor matrix of ${}^2A$) e_2.

Proof:

Let $A = [\xi_{i\,j}]_{n\,x\,n}$ be a bicomplex square matrix then

${}^1A = [{}^1\xi_{i\,j}]_{n\,x\,n}$ and ${}^2A = [{}^2\xi_{i\,j}]_{n\,x\,n}$

Now the $i^{th}\,j^{th}$ entry of cofactor matrix of A

=cofactor of the entry $\xi_{i\,j}$

$=(-1)^{i+j} \times$ the determinant obtained by leaving the row and the column(in the matrix A) passing through the entry $\xi_{i\,j}$

$= (-1)^{i+j} \times$ [{the determinant obtained by leaving the row and the column(in the matrix ${}^1A$) passing through the entry ${}^1\xi_{i\,j}$}e_1 +{the determinant obtained by leaving the row and the column(in the matrix ${}^2A$) passing through the entry ${}^2\xi_{i\,j}$}e_2]

= (cofactor of the entry ${}^1\xi_{i\,j}$ in the matrix ${}^1A$) e_1 +(cofactor of the entry ${}^2\xi_{i\,j}$ in the matrix ${}^2A$) e_2

Therefore the $i^{th}\,j^{th}$ entry of cofactor matrix of A = (The $i^{th}\,j^{th}$ entry of cofactor matrix of ${}^1A$) e_1 + (The $i^{th}\,j^{th}$ entry of cofactor matrix of ${}^2A$) e_2

Hence it proves that cofactor matrix of A = (cofactor matrix of ${}^1A$) e_1+(cofactor matrix of ${}^2A$) e_2

Theorem 2.2.4 and 2.2.5 submerge together to give a new corollary which is started below.

2.2.6 Corollary: If $A= [\xi_{i\,j}]_{m\,x\,n}$ is a bicomplex square matrix then adj A=[adj ${}^1A$]e_1 +[adj ${}^2A$] e_2.

c) Inversion of bicomplex matrix by two techniques

2.3.1 Inverse of a bicomplex square matrix with the help of adjoint matrix

Let $A = [\xi_{ij}]_{n\times n}$ be a square and non-singular matrix whose elements are in C_2 and From 2.1.1, 2.2.2 and 2.2.6

We have $A = {}^1A\ e_1 + {}^2A\ e_2$, $|A|=|{}^1A|e_1+|{}^2A|e_2$

And adj A = [adj ${}^1A$] e_1 + [adj ${}^2A$] e_2

Now

$A\times(Adj.A) = [{}^1A\ e_1 + {}^2A\ e_2]\times[Adj.({}^1A)e_1 + Adj.({}^2A)e_2]$

$= ({}^1A.Adj.{}^1A)\ e_1 + ({}^2A.Adj.{}^2A)\ e_2$ (since $e_1.e_2= 0$)

$= (|{}^1A|.I)\ e_1 + (|{}^2A|.I)\ e_2$,

Since ${}^1A$, ${}^2A$ and I are complex matrices.

Therefore $A\times(\text{Adj}.A)=(\left|{}^{1}A\right| e_1 + \left|{}^{2}A\right| e_2).I$, where the matrix I is a bicomplex matrix.(since there is no difference between the identity matrix in C_1 and the identity matrix in C_2 of same order.)

Therefore $A\times(\text{Adj}.A)=|A|.I$

$$\because |A| \notin O_2$$

$$\therefore A\times\frac{\text{AdjA}}{|A|}=I$$

$$\Rightarrow A^{-1}=\frac{\text{AdjA}}{|A|}$$

$$\because \left(\left|{}^{1}A\right|\neq 0 \text{ and } \left|{}^{2}A\right|\neq 0 \right)$$

Now construct

$$\frac{{}^{1}A.(\text{Adj }{}^{1}A)}{\left|{}^{1}A\right|} e_1 + \frac{{}^{2}A.(\text{Adj }{}^{2}A)}{\left|{}^{2}A\right|} e_2$$

$$= ({}^{1}A\ {}^{1}A^{-1})\ e_1 + ({}^{2}A\ {}^{2}A^{-1})\ e_2$$

$$= I\ e_1 + I\ e_2 = I$$

$$= A\times\frac{\text{AdjA}}{|A|}$$

Hence $A^{-1}=\frac{\text{AdjA}}{|A|}$ and

$$A\times\frac{\text{AdjA}}{|A|}=\frac{{}^{1}A.(\text{Adj}\,{}^{1}A)}{\left|{}^{1}A\right|} e_1 + \frac{{}^{2}A.(\text{Adj}\,{}^{2}A)}{\left|{}^{2}A\right|} e_2$$

2.3.2 Inverse of bicomplex square matrix with the help of idempotent technique

If $M = [\xi_{ij}]_{n\times n}$ is a square and nonsingular bicomplex matrix of order n

Therefore $M = {}^{1}M\ e_1 + {}^{2}M\ e_2$

Since $|M| \notin O_2$ therefore $\left|{}^{1}M\right|\neq 0$ and $\left|{}^{2}M\right|\neq 0$

i.e. ${}^{1}M$ and ${}^{2}M$ are invertible. Let the inverse of both ${}^{1}M$ and ${}^{2}M$ be $[z_{ij}]_{n\times n}$ and $[w_{ij}]_{n\times n}$ respectively. Now construct a new matrix with the help of $[z_{ij}]_{n\times n}$ and $[w_{ij}]_{n\times n}$ as

$$[z_{ij}]_{n\times n}\, e_1 + [w_{ij}]_{n\times n}\, e_2 = [\eta_{ij}]_{n\times n} \quad \text{(say)}$$

Now we claim that $[\eta_{ij}]_{n\times n}$ is the inverse of M.

Note that

$$[\xi_{ij}]_{n\times n}\times[\eta_{ij}]_{n\times n}=\left(\left[{}^{1}\xi_{ij}\right]_{n\times n} e_1 + \left[{}^{2}\xi_{ij}\right]_{n\times n} e_2\right)\times\left(\left[{}^{1}\eta_{ij}\right]_{n\times n} e_1 + \left[{}^{2}\eta_{ij}\right]_{n\times n} e_2\right)$$

$$=\left(\left[{}^{1}\xi_{ij}\right]_{n\times n}\left[{}^{1}\eta_{ij}\right]_{n\times n}\right) e_1 + \left(\left[{}^{2}\xi_{ij}\right]_{n\times n}\left[{}^{2}\eta_{ij}\right]_{n\times n}\right) e_2$$

Notes

Since $\left[{}^{1}\eta_{ij}\right]_{n\times n} = [z_{ij}]_{\text{n×n}}$ and $\left[{}^{2}\eta_{ij}\right]_{n\times n} = [w_{ij}]_{\text{n×n}}$

$$\therefore [\xi_{ij}]_{\text{n×n}} \times [\eta_{ij}]_{\text{n×n}} = \text{I e}_1 + \text{I e}_2 = \text{I}$$

Hence $[\eta_{ij}]_{\text{n×n}}$ is the inverse of M.

Notes

III. Some Special Bicomplex Matrices

a) Conjugates of a bicomplex matrix

As there are three types of conjugates of a bicomplex number, we have defined three types of conjugates of a bicomplex matrix.

3.1.1 Definition: i_1 Conjugate of a bicomlex matrix

Let A $= [\xi_{ij}]_{\text{m×n}}$ be the bicomplex matrix then the i_1 conjugate of matrix A written as $\bar{A}$ is the matrix obtained from A by taken i_1 conjugate of each entry of A. i.e.

$$\bar{A} = \begin{bmatrix} \bar{\xi}_{11} & \bar{\xi}_{12} & \cdots\cdots & \bar{\xi}_{1n} \\ \bar{\xi}_{21} & \bar{\xi}_{22} & \cdots\cdots & \bar{\xi}_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \bar{\xi}_{m1} & \bar{\xi}_{m2} & \cdots\cdots & \bar{\xi}_{mn} \end{bmatrix}$$

The idempotent representation of $\bar{A}$ can be obtained as follows

$$\bar{A} = \begin{bmatrix} \overline{{}^2\xi_{11}} & \overline{{}^2\xi_{12}} & \cdots\cdots & \overline{{}^2\xi_{1n}} \\ \overline{{}^2\xi_{21}} & \overline{{}^2\xi_{22}} & \cdots\cdots & \overline{{}^2\xi_{2n}} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \overline{{}^2\xi_{m1}} & \overline{{}^2\xi_{m2}} & \cdots\cdots & \overline{{}^2\xi_{mn}} \end{bmatrix} e_1 + \begin{bmatrix} \overline{{}^1\xi_{11}} & \overline{{}^1\xi_{12}} & \cdots\cdots & \overline{{}^1\xi_{1n}} \\ \overline{{}^1\xi_{21}} & \overline{{}^1\xi_{22}} & \cdots\cdots & \overline{{}^1\xi_{2n}} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \overline{{}^1\xi_{m1}} & \overline{{}^1\xi_{m2}} & \cdots\cdots & \overline{{}^1\xi_{mn}} \end{bmatrix} e_2$$

It is evident that

(a) $\overline{[(\bar{A})]} = A$

(b) $\overline{kA} = \bar{k}\,\bar{A}$ where $k \in C_2$

Similarly the definition of i_2 and $i_1 i_2$ conjugate of a bicomplex matrix is following.

3.1.2 Definition: i_2 Conjugate of a bicomlex matrix

Let A $= [\xi_{ij}]_{\text{m×n}}$ be the bicomplex matrix then the i_2 conjugate of matrix A written as $A^{\sim}$ is the matrix obtained from A by taken i_2 conjugate of each entry of A. i.e.

$$A^{\sim} = \begin{bmatrix} \xi_{11}^{\sim} & \xi_{12}^{\sim} & \cdots\cdots & \xi_{1n}^{\sim} \\ \xi_{21}^{\sim} & \xi_{22}^{\sim} & \cdots\cdots & \xi_{2n}^{\sim} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \xi_{m1}^{\sim} & \xi_{m2}^{\sim} & \cdots\cdots & \xi_{mn}^{\sim} \end{bmatrix}$$

The idempotent representation of $A^{\sim}$ can be obtained as follows:

$$A^{\sim} = \begin{bmatrix} {}^{2}\xi_{11} & {}^{2}\xi_{12} & \cdots & {}^{2}\xi_{1n} \\ {}^{2}\xi_{21} & {}^{2}\xi_{22} & \cdots & {}^{2}\xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ {}^{2}\xi_{m1} & {}^{2}\xi_{m2} & \cdots & {}^{2}\xi_{mn} \end{bmatrix} e_1$$

$$+ \begin{bmatrix} {}^{1}\xi_{11} & {}^{1}\xi_{12} & \cdots & {}^{1}\xi_{1n} \\ {}^{1}\xi_{21} & {}^{1}\xi_{22} & \cdots & {}^{1}\xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ {}^{1}\xi_{m1} & {}^{1}\xi_{m2} & \cdots & {}^{1}\xi_{mn} \end{bmatrix} e_2$$

It is evident that

(a) $(A^{\sim})^{\sim} = A$

(b) $(k\,A)^{\sim} = k^{\sim} A^{\sim}$ *where* $k \in C_2$

3.1.3 Definition: $i_1 i_2$ Conjugate of a bicomplex matrix

Let $A = [\xi_{ij}]_{m\times n}$ be the bicomplex matrix then the $i_1 i_2$ conjugate of matrix A written as $A^{\#}$ is the matrix obtained from A by taken $i_1 i_2$ conjugate of each entry of A. i.e.

$$A^{\#} = \begin{bmatrix} \xi_{11}^{\#} & \xi_{12}^{\#} & \cdots & \xi_{1n}^{\#} \\ \xi_{21}^{\#} & \xi_{22}^{\#} & \cdots & \xi_{2n}^{\#} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{m1}^{\#} & \xi_{m2}^{\#} & \cdots & \xi_{mn}^{\#} \end{bmatrix}$$

The idempotent representation of $A^{\#}$ can be obtained as follows:

$$A^{\#} = \begin{bmatrix} \overline{{}^{1}\xi_{11}} & \overline{{}^{1}\xi_{12}} & \cdots & \overline{{}^{1}\xi_{1n}} \\ \overline{{}^{1}\xi_{21}} & \overline{{}^{1}\xi_{22}} & \cdots & \overline{{}^{1}\xi_{2n}} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{{}^{1}\xi_{m1}} & \overline{{}^{1}\xi_{m2}} & \cdots & \overline{{}^{1}\xi_{mn}} \end{bmatrix} e_1 + \begin{bmatrix} \overline{{}^{2}\xi_{11}} & \overline{{}^{2}\xi_{12}} & \cdots & \overline{{}^{2}\xi_{1n}} \\ \overline{{}^{2}\xi_{21}} & \overline{{}^{2}\xi_{22}} & \cdots & \overline{{}^{2}\xi_{2n}} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{{}^{2}\xi_{m1}} & \overline{{}^{2}\xi_{m2}} & \cdots & \overline{{}^{2}\xi_{mn}} \end{bmatrix} e_2$$

It is evident that

(a) $(A^{\#})^{\#} = A$

(b) $(k\,A)^{\#} = k^{\#} A^{\#},\ k \in C_2$

b) Tranjugate of a bicomplex matrix

The transpose of conjugate of a bicomplex matrix is defined as the tranjugate of the matrix. There are three types of tranjugates of a bicomplex matrix.

3.2.1 Definition: i_1 tranjugate of a bicomplex matrix

The transpose of the i_1 conjugate of a bicomplex matrix is defined as the i_1 tranjugate of the matrix.

i.e. If $A = [\xi_{ij}]_{m\times n}$ then

Notes

$$i_1 \, tranjugate \ of \ A = [\bar{A}]^T = \begin{bmatrix} \bar{\xi}_{11} & \bar{\xi}_{21} & \cdots\cdots & \bar{\xi}_{m1} \\ \bar{\xi}_{12} & \bar{\xi}_{22} & \cdots\cdots & \bar{\xi}_{m2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \bar{\xi}_{1n} & \bar{\xi}_{2n} & \cdots\cdots & \bar{\xi}_{mn} \end{bmatrix}$$

Notes

Similarly

3.2.2 Definition: i_2 tranjugate of a bicomplex matrix

The transpose of the i_2 conjugate of a bicomplex matrix is defined as the i_2 tranjugate of the matrix.

i.e. If $A = [\xi_{ij}]_{m\times n}$ then

$$i_2 \, tranjugate \ of \ A = [A^{\sim}]^T = \begin{bmatrix} \xi_{11}^{\sim} & \xi_{21}^{\sim} & \cdots\cdots & \xi_{m1}^{\sim} \\ \xi_{12}^{\sim} & \xi_{22}^{\sim} & \cdots\cdots & \xi_{m2}^{\sim} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \xi_{1n}^{\sim} & \xi_{2n}^{\sim} & \cdots\cdots & \xi_{mn}^{\sim} \end{bmatrix}$$

3.2.3 Definition: $i_1 i_2$ tranjugate of a bicomplex matrix

The transpose of the $i_1 i_2$ conjugate of a bicomplex matrix is defined as the $i_1 i_2$ tranjugate of the matrix.

i.e. If $A = [\xi_{ij}]_{m\times n}$ then

$$i_1 i_2 \, tranjugate \ of \ A = [A^{\#}]^T = \begin{bmatrix} \xi_{11}^{\#} & \xi_{21}^{\#} & \cdots\cdots & \xi_{m1}^{\#} \\ \xi_{12}^{\#} & \xi_{22}^{\#} & \cdots\cdots & \xi_{m2}^{\#} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \xi_{1n}^{\#} & \xi_{2n}^{\#} & \cdots\cdots & \xi_{mn}^{\#} \end{bmatrix}$$

c) Symmetric and skew – symmetric matrix in C_2

3.3.1 Definition: Symmetric matrix

A bicomplex square matrix $A = [\xi_{ij}]_{n\times n}$ is said to be symmetric if $A = [A]^T$. Thus for a symmetric matrix A, we have $\xi_{ij} = \xi_{ji}$ for all i and j

3.3.2 Definition: Skew-symmetric matrix

A bicomplex square matrix $A = [\xi_{ij}]_{n\times n}$ is said to be skew-symmetric if $A = -[A]^T$. Thus for a skew-symmetric matrix A, we have $\xi_{ij} = -\xi_{ji}$ for all i and j

3.3.3 Theorem: The elements of principal diagonal of skew-symmetric matrix are zero.

Proof:

We know that a matrix $A = [\xi_{ij}]_{n\times n}$ is skew - symmetric if and only if $\xi_{ij} = -\xi_{ji}$ for all i and j. For diagonal element we have $\xi_{ii} = -\xi_{ii}$ therefore

If $\xi_{ii} = z_{ii} + i_2 w_{ii}$ then $(z_{ii} + i_2 w_{ii}) = -(z_{ii} + i_2 w_{ii})$

Therefore $z_{ii} = 0$, $w_{ii} = 0$

i.e. $\xi_{ii} = 0$ for all i

d) Hermitian matrix in C_2

Corresponding to the three types of conjugates in C_2, there are three types of Hermitian matrix in C_2.

3.4.1 Definition: i_1 Hermitian matrix

A bicomplex square matrix A is said to be i_1Hermitian matrix if $A = [\bar{A}]^T$

3.4.2 Theorem: The elements of principal diagonal of an i_1 Hermitian matrix are members of $C(i_2)$.

Proof:

Recall that $A = [\xi_{ij}]_{n\times n}$ is i_1Hermitian matrix if and only if $\xi_{ij} = \bar{\xi}_{ji}$ $\forall$i and j.

For diagonal element we have

$$\xi_{kk} = \bar{\xi}_{kk}$$

$$\text{If } \xi_{kk} = z_{kk} + i_2 w_{kk} \text{ then}$$

$$z_{kk} + i_2 w_{kk} = \bar{z}_{kk} + i_2 \bar{w}_{kk}$$

$$\Rightarrow z_{kk} \in C_0 \text{ and } w_{kk} \in C_0$$

$$\Rightarrow \xi_{kk} \in C(i_2)$$

3.4.3 Definition: i_2 Hermitian matrix

A bicomplex square matrix A is said to be i_2 Hermitian matrix if $A = [A^{\sim}]^T$

3.4.4 Theorem: The elements of principal diagonal of an i_2 Hermitian matrix are members of C (i_1).

Proof:

$A = [\xi_{ij}]_{n\times n}$ is i_2Hermitian matrix if and only if $\xi_{ij} = \xi_{ji}^{\sim}$ $\forall$i and j.

For diagonal element we have

$$\xi_{kk} = \xi_{kk}^{\sim}$$

$$\text{If } \xi_{kk} = z_{kk} + i_2 w_{kk}$$

$$\Rightarrow z_{kk} + i_2 w_{kk} = z_{kk} - i_2 w_{kk}$$

$$\Rightarrow \xi_{kk} \in C(i_1)$$

3.4.5 Definition: $i_1 i_2$ Hermitian matrix

A bicomplex square matrix A is said to be $i_1 i_2$ Hermitian matrix if $A = [A^{\#}]^T$

3.4.6 Theorem: The elements of principal diagonal of an $i_1 i_2$ Hermitian matrix are members of H.

Proof:

$A = [\xi_{ij}]_{n\times n}$ is $i_1 i_2$ Hermitian matrix if and only if $\xi_{ij} = \xi_{ji}^{\#}$ $\forall$i and j.

For diagonal element we have

Notes

$$\xi_{kk} = \xi_{kk}{}^{\#}$$

If $\xi_{kk} = z_{kk} + i_2 w_{kk}$

$$\Rightarrow z_{kk} + i_2 w_{kk} = \bar{z}_{kk} - i_2 \bar{w}_{kk}$$

$\Rightarrow z_{kk} \in C_0$ and w_{kk} is in the form of $i_1 y$ where y in C_0

$\Rightarrow \xi_{kk} \in H$

Notes

e) Skew-Hermitian matrix in C_2

Analogous to the theory of Hermitian matrices, we have defined three types of skew-Hermitian matrices in C_2.

3.5.1 Definition: i_1 skew-Hermitian matrix

A bicomplex square matrix A is said to be i_1 skew-Hermitian matrix if $A = -[\bar{A}]^T$

3.5.2 Theorem: The elements of principal diagonal of an i_1 skew-Hermitian matrix are of the type of $(i_1 \times s)$, where $s \in C(i_2)$.

Proof:

Let $A = [\xi_{ij}]_{n\times n}$ be an i_1 skew-Hermitian matrix then $\xi_{ij} = -\bar{\xi}_{ji}$;for all i and j.

For diagonal element we have

$$\xi_{kk} = -\bar{\xi}_{kk}$$

If $\xi_{kk} = z_{kk} + i_2 w_{kk}$ then

$$z_{kk} + i_2 w_{kk} = -(\bar{z}_{kk} + i_2 \bar{w}_{kk})$$

Therefore $z_{kk} = -\bar{z}_{kk}$ *and* $w_{kk} = -\bar{w}_{kk}$

Hence $\xi_{kk} = i_1\{\text{Im}(z_{kk})\} + i_2 i_1\{\text{Im}(w_{kk})\} = i_1.s$ where $s \in C(i_2)$

3.5.3 Definition: i_2 skew-Hermitian matrix

A bicomplex square matrix A is said to be i_2 skew-Hermitian matrix if $A = -[A^{\sim}]^T$

3.5.4 Theorem: The elements of principal diagonal of an i_2 skew-Hermitian matrix are of the type of $(i_2 \times s)$, where $s \in C(i_1)$.

Proof:

$A = [\xi_{ij}]_{n\times n}$ is an i_2 skew-Hermitian matrix if and only if $\xi_{ij} = -\xi_{ji}^{\sim}$; for all i and j.

For diagonal element we have

$$\xi_{kk} = -\xi_{kk}{}^{\sim}$$

If $\xi_{kk} = z_{kk} + i_2 w_{kk}$ then

$$z_{kk} + i_2 w_{kk} = -(z_{kk} - i_2 w_{kk})$$

Therefore $z_{kk} = -z_{kk}$ i.e. $z_{kk} = 0$

Hence $\xi_{kk} = i_2(w_{kk}) = i_2.s$ where $s = w_{kk} \in C(i_1)$

3.5.5 Definition: i_1i_2 skew-Hermitian matrix

A bicomplex square matrix A is said to be i_1i_2 skew-Hermitian matrix if $A = -[A^{\#}]^T$

3.5.6 Theorem: The elements of principal diagonal of an i_1i_2 skew-Hermitian matrix are of the type of $(i_1 \times s)$, where $s \in H$.

Proof:

$A = [\xi_{ij}]_{n\times n}$ is an $i_1i_2$skew-Hermitian matrix if and only if $\xi_{ij} = -\xi_{ji}^{\#}$; for all i and j.

For diagonal element we have

$$\xi_{kk} = -\xi_{kk}^{\#}$$

$$\text{If } \xi_{kk} = z_{kk} + i_2 w_{kk} \text{ then}$$

$$z_{kk} + i_2 w_{kk} = -(\bar{z}_{kk} - i_2\bar{w}_{kk})$$

Therefore $z_{kk} = -\bar{z}_{kk}$ and $w_{kk} = \bar{w}_{kk}$

Hence $\qquad \xi_{kk} = i_1\{\mathrm{Im}(z_{kk})\} + i_2\{\mathrm{Re}(w_{kk})\}$

i.e. $\qquad \xi_{kk} = i_1\{\mathrm{Im}(z_{kk})\} + i_1 i_1^{-1} i_2\{\mathrm{Re}(w_{kk})\}$

$$= i_1[\mathrm{Im}(z_{kk}) - i_1 i_2 \mathrm{Re}(w_{kk})]$$

$$= i_1.s;\ \text{where } s \in H$$

3.5.7 Theorem: A is i_1 Hermitian matrix if and only if i_1 A is i_1 skew - Hermitian matrix.

Proof:

Let A be an i_1 Hermitian matrix therefore

$$A = [\bar{A}]^T$$

$$\text{Now } [\overline{i_1 A}]^T = [\bar{i_1}\bar{A}]^T \qquad \ldots[\text{by } 3.1.1]$$

$$= -i_1[\bar{A}]^T$$

$$= -i_1 A$$

i.e. $i_1A = -[\overline{i_1 A}]^T$

$\Rightarrow i_1$ A is i_1 skew - Hermitian matrix.

Converse:

Let i_1 A be an i_1 skew - Hermitian matrix

i.e. $\qquad i_1 A = -[\overline{i_1 A}]^T$

$$= -[\bar{i_1}\bar{A}]^T$$

$$= i_1[\bar{A}]^T$$

i.e. $\qquad A = [\bar{A}]^T$

Hence A is i_1 Hermitian matrix.

Notes

3.5.8 Theorem: A is i_2 Hermitian matrix if and only if i_2 A is i_2 skew - Hermitian matrix.

Proof:

Let A be an i_2 Hermitian matrix therefore

$$A = [A^{\sim}]^T$$

Now $$[(i_2A)^{\sim}]^T = i_2^{\sim}[A^{\sim}]^T \quad \ldots\text{[by 3.1.2]}$$

$$= -i_2 A$$

i.e. $$i_2A = -[(i_2A)^{\sim}]^T$$

$\Rightarrow i_2$ A is i_2 skew - Hermitian matrix.

Converse:

Let i_2 A be an i_2 skew - Hermitian matrix

i.e. $$i_2 A = -[(i_2A)^{\sim}]^T$$

$$= -[i_2^{\sim} A^{\sim}]^T$$

$$= i_2[A^{\sim}]^T$$

i.e. $$A = [A^{\sim}]^T$$

Hence A is i_2 Hermitian matrix.

3.5.9 Theorem: A is i_1i_2 Hermitian matrix if and only if i_1i_2 A is i_1i_2 Hermitian matrix.

Proof:

Let A be an i_1i_2 Hermitian matrix therefore

$$A = [A^{\#}]^T$$

Now $$[(i_1i_2A)^{\#}]^T = i_1^{\#}i_2^{\#}[A^{\#}]^T \quad \ldots\text{[by 3.1.3]}$$

$$= (-i_1)(-i_2)A$$

$$= i_1i_2A$$

$\Rightarrow i_1i_2$ A is i_1i_2 Hermitian matrix.

Converse:

Let i_1i_2 A be i_1i_2 Hermitian matrix

i.e. $$i_1i_2A = [i_1i_2A]^{\#}$$

$$= i_1^{\#}i_2^{\#}[A^{\#}]^T$$

$$= (-i_1)(-i_2)[A^{\#}]^T$$

$$= i_1i_2[A^{\#}]^T$$

i.e. $$A = [A]^{\#}$$

Hence A is i_1i_2 Hermitian matrix.

Notes

It is evident that if A is $i_1 i_2$ skew-Hermitian matrix then $i_1 i_2 A$ will be also $i_1 i_2$ skew-Hermitian matrix and vice versa.

References Références Referencias

1. D. Rochon, M. Shapiro,"On Algebraic Properties of Bicomplex and Hyperbolic Numbers", Anal. Univ. Oradea, fasc.Math.,11, 2004, 71 – 110.
2. Futagawa, M.: On the theory of functions of a quaternary Variable, Tohoku Math. J., 29 (1928), 175-222.
3. Futagawa, M.: On the theory of functions of a quaternary Variable, II, Tohoku Math. J., 35 (1932), 69-120.
4. Lipschutz, S. and Lipson, M. ˝Schaum's "Outline of Theory and Problem of linear Algebra˝ Tata McGraw ˙Hill Education, 2005.
5. Price, G.B.: An introduction to multicomplex spaces and Functions, Marcel Dekker Inc., New York, 1991.
6. Riley, J.D.: Contributions to the theory of function of a bicomplex variable, Tohoku Math. J., 2nd series 5 (1953), 132-165.
7. Ringleb, F.: Beitrage Zur Funktionen theorie in Hyperkomplexon Systemen, I, Rend. Circ. Mat. Palermo, 57 (1933), 311-340.
8. Segre, C.: Le Rappresentazioni Reali Delle Forme Complesse e Gli enti Iperalgebrici, Math.Ann. 40 (1892), 413-467.
9. Srivastava, Rajiv K.: Bicomplex Numbers: Analysis and applications, Math.Student, 72 (1-4) 2003, 69-87.
10. Srivastava, Rajiv K "Certain topological aspects of Bicomplex space", Bull. Pure & Appl. Math. Dec., 2008

Notes