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Classification of Non-Oscillatory Solutions of Nonlinear Neutral Delay Impulsive Differential Equations

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$$\begin{cases} \left[y(t)-\sum_{i=1}^{m}p_i(t)y(t-\tau_i)\right]'' + \sum_{j=1}^{n} f_j\left(t, y\left(g_{jl}(t)\right), \cdots, y\left(g_{jl}\right)\right) = 0, \; t \geq t_0 \in R_+, t \neq t_k \\ \Delta\left[y(t_k)-\sum_{i=1}^{m}p_{ik}y(t_k-\tau_i)\right]' + \sum_{j=1}^{n} f_{jk}\left(t_k, y\left(g_{jl}(t_k)\right), \cdots, y\left(g_{jl}(t_k)\right)\right) = 0, \; t_k \geq t_0 \in R_+, t = t_k \end{cases}$$

is considered. We classifyits non-oscillatory solutions into four types of solution sets, namely $\Lambda^{(0,0,0)}, \Lambda^{(b,a,0)}, \Lambda^{(\infty,\infty,0)}$ and $\Lambda^{(\infty,\infty,d)}$ and establish necessary and sufficient conditions for the existence of these non-oscillatory solutions by means of Schauder-Tychonoff fixed point theorem and Lebesgue's Monotone Convergence Theorem. Some examples are given to illustrate the obtained results.

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Ref

2. B. G. Zhang, Zhu Shanliang, Oscillation of second-order nonlinear delay dynamic equations on time scales, *Comput. Math. Appl.*, 49 (2005), 599-609.

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I. Introduction

A survey of recent studies in neutral impulsive differential equations reveal that most of such works revolve around the quest for oscillatory conditions for impulsive differential equations, with or without delay, linear or nonlinear ([2], [3], [5], [6], [7], [8], [12], [13], [14]). The development of oscillatory and non-oscillatory criteria for nonlinear impulsive differential equations has so far attracted very little attention. In fact, the concept of non-oscillation for nonlinear neutral impulsive equations presently suffers almost complete neglect.

In this study, we attempt to classify the non-oscillatory solutions of a general class of second order nonlinear neutral delay impulsive differential equations into different solution sets and make conscious efforts to provide conditions for the existence of these solutions.

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In what follows, we recall some of the basic notions and definitions that will be of importance as we advance through the article.

Usually, the solution $y(t)$ for $t\in[t_0,T)$ of a given impulsive differential equation or its first derivative $y'(t)$ is a piece-wise continuous function with points of discontinuity $t_k\in[t_0,T)$, $t_k\neq t$. Therefore, in order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows:

Let $r\in N$, $D:=[T,\infty)\subset R$ and let $S:=\{t_k\}_{k\in E}$, where E is our subscript set which can be the set of natural numbers N or the set of integers Z, be fixed. Throughout this discussion, we will assume that the elements of the sequence $S:=\{t_k\}_{k\in E}$ are the moments of impulsive effects and satisfy the following properties:

C1.1: If $\{t_k\}$ is defined for all $k\in N$, then $0<t_1<t_2<\cdots$ and $\lim\limits_{k\to\infty} t_k=+\infty$.

C1.2: If $\{t_k\}$ is defined for all $k\in Z$, then $t_0\le 0<t_1, t_k<t_{k+1}$ for $k\in Z$, $k\neq 0$ and $\lim\limits_{k\to\pm\infty} t_k=\pm\infty$.

We denote by $PC(D,R)$ the set of all values $\psi:D\to R$ which is continuous for all $t\in D$, $t\notin S$. They are functions from the left and have discontinuity of the first kind at the points for $t\in S$. By $PC^r(D,R)$, we denote the set of functions $\psi:D\to R$ having derivative $\frac{d^j\psi}{dt^j}\in PC(D,R)$, $0\le j\le r$ ([1], [4]).

To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes use the symbols $PC(D,R;S)$ and $PC^r(D,R;S)$, $r\in N$.

The solution $y(t)$ of an impulsive differential equation is said to be

1. Finally positive (finally negative) if there exist $T\ge 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t\ge T$ ([9]);
2. Oscillatory, if it is neither finally positive nor finally negative; and
3. Non-oscillatory, if it is neither finally positive nor finally negative ([1], [10]).

II. Statement of the Problem

Here, we are considering the second order nonlinear neutral impulsive differential equation of the form

$$\begin{cases}\left[y(t)-\sum\limits_{i=1}^{m}p_i(t)y(t-\tau_i)\right]''+\sum\limits_{j=1}^{n}f_j\left(t,y\left(g_{jl}(t),\cdots,y\left(g_{jl}\right)\right)\right)=0,\ t\ge t_0\in R_+, t\notin S\\ \Delta\left[y(t_k)-\sum\limits_{i=1}^{m}p_{ik}y(t_k-\tau_i)\right]'+\sum\limits_{j=1}^{n}f_{jk}\left(t_k,y\left(g_{jl}(t_k),\cdots,y\left(g_{jl}(t_k)\right)\right)\right)=0,\ t_k\ge t_0\in R_+,\forall\, t_k\in S.\end{cases}\quad (2.1)$$

We introduce the following conditions:

C2.1: $\tau_i>0$, $p_{ik}\ge 0$, $p_i\in PC^1([t_0,\infty),R_+)$, $i=1,2,\cdots,m$ and there exists $\delta\in(0,1]$ such that

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1. D. D. Bainov and P. S. Simeonov, *Oscillation Theory of Impulsive Differential Equations*, International Publications Orlando, Florida, 1998.

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$$\sum_{i=1}^{m} p_i(t) + \sum_{j=1}^{n} p_j \le 1-\delta, \quad t, \ge t_0 \in R_+;$$

C2.2: $g_{js} \in C([t_0,\infty), R), \ \lim_{t\to\infty} g_{js}(t)=\infty, \ j=1,2,\cdots,n, \ s=1,2,\cdots,\ell;$

C2.3: $f_j \in PC([t_0,\infty)\times R^{\ell}, R), \ x_1 f_j(t,x_1,\cdots,x_\ell)>0;$

$x_1 f_{jk}(t_k,x_1,\cdots,x_l)>0$ for $x_1\ x_i>0, \ i=1,2,\cdots,\ell, \ j=1,2,\cdots,n$. Moreover,

$$\begin{cases} |f_j(t,y_1,\cdots,y_\ell)| \ge |f_j(t,x_1,\cdots,x_\ell)| \\ |f_{jk}(t_k,y_1\cdots,y_\ell)| \ge |f_{jk}(t_k,x_1,\cdots,x_\ell)| \end{cases}$$

whenever

$$|x_i| \le |y_i| \text{ and } y_i x_i > 0, \ i=1,2,\cdots,\ell, \ j=1,2,\cdots,n;$$

C2.4: Set

$$x(t)=y(t)-\sum_{i=1}^{m} p_i(t) y(t-\tau_i). \tag{2.2}$$

Our aim in this paper is to give the classification of non-oscillatory solutions of equation (2.1). But first, we define some concepts and establish the following lemmas which will be useful in the discussion of the main results.

Theorem 2.1: (Schauder-Tychonoff fixed point theorem) Let X be a locally convex linear space, S a compact convex subset of X, and let $T:S\to S$ be a continuous mapping with T(S) compact. Then T has a fixed point in S.

Theorem 2.2: (Lebesgue's Monotone Convergence Theorem) Let (A,Σ,μ) be a measure space and $f_1, f_2, f_3, \cdots$ a pointwise non-decreasing sequence of $[0,\infty)-$valued $\Sigma-$measurable functions. Let $\lim_{n\to\infty} f_n(t) := f(t)$ for all $t\in A$, then f is $\Sigma-$measurable and

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f\, d\mu.$$

Lemma 2.1 and 2.2 are extensions of Lemma 4.5.1 and 4.5.2 on pages 242 and 243 respectively of the monograph by Erbe et al [11]

Lemma 2.1: Let $y(t)$ be a finally positive (or negative) solution of equation (2.1). If $\lim_{t\to\infty} y(t)=0$, then $x(t)$ is finally negative (or positive) and $\lim_{t\to\infty} x(t)=0$. Otherwise, $x(t)$ is finally positive (or negative).

Proof: Let $y(t)$ be a finally positive solution of equation (2.1). From the same equation (2.1), $x''(t), \Delta x'(t_k)>0$ or $x'(t), \Delta x(t_k)<0$ finally. Also, $x(t)>0$ or $x(t)<0$ finally. If $\lim_{t\to\infty} y(t)=0$, from equation (2.2), it follows that $\lim_{t\to\infty} x(t)=0$. Since x(t) is monotonic, so $\lim_{t\to\infty} x'(t)=0$, $\lim_{t_k\to\infty} \Delta x(t_k)=0$ which implies that $x'(t)>0, \ \Delta x(t_k)>0$.

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11. L. H. Erbe, Q. Kong and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Dekker, New York, 1995.

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Therefore, $x(t)<0$ finally. If $\lim_{t\to\infty} y(t)\neq 0$, then $\limsup_{t\to\infty} y(t)>0$. We show that $x(t)>0$ finally. If not, then $x(t)<0$ finally. If $y(t)$ is unbounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n=\infty$, $y(t_n)=\max_{t_0\le t<t_n} y(t)$ and $\lim_{n\to\infty} y(t_n)=\infty$. From equation (2.2), we obtain

$$x(t_n)=y(t_n)-\sum_{i=1}^{m} p_i(t_n)y(t_n-\tau_i)\ge y(t_n)\left(1-\sum_{i=1}^{m} p_i(t_n)\right). \tag{2.3}$$

Thus, $\lim_{n\to\infty} x(t_n)=\infty$, which is a contradiction. If $y(t)$ is bounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n=\infty$ and $\lim_{n\to\infty} y(t_n)=\limsup_{t\to\infty} y(t)$. Since the sequences $\{p_i(t_n)\}$ and $\{y(t_n-\tau_i)\}$ are bounded, there exists convergent subsequences. Without loss of generality, we may assume that $\lim_{n\to\infty} y(t_n-\tau_i)$ and $\lim_{n\to\infty} p_i(t_n)$, $i=1,2\cdots,m$, exist. Hence

$$0\ge\lim_{n\to\infty} x(t_n)=\lim_{n\to\infty}\left(y(t_n)-\sum_{i=1}^{m} p_i(t_n)y(t_n-\tau_i)\right)\ge\limsup_{t\to\infty} y(t)\left(1-\sum_{i=1}^{m} p_i(t_n)\right)>0,$$

which, again, is a contradiction. Therefore, $x(t)>0$ finally. A similar proof can be repeated if $y(t)<0$ finally.

Lemma 2.2: Assume that $\lim_{t\to\infty}\sum_{i=1}^{m} p_i(t)=P\in(0,1]$, and $y(t)$ is a finally positive (or negative) solution of equation (2.1). If $\lim_{t\to\infty} x(t)=a\in R$, then $\lim_{t\to\infty} y(t)=\frac{a}{1-p}$. If $\lim_{t\to\infty} x(t)=\infty\ (\text{or } -\infty)$, then $\lim_{t\to\infty} y(t)=\infty\ (\text{or } -\infty)$.

Proof: Let $y(t)$ be a finally positive solution of equation (2.1), then $y(t)\ge x(t)$ finally. If $\lim_{t\to\infty} x(t)=\infty$, then $\lim_{t\to\infty} y(t)=\infty$. Now we consider the case that $\lim_{t\to\infty} x(t)=a\in R$. Thus, $x(t)$ is bounded which implies, by equation (2.3), that $y(t)$ is bounded. Therefore, there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n=\infty$ and $\lim_{n\to\infty} y(t_n)=\limsup_{t\to\infty} y(t)$. As before, without loss of generality, we may assume that $\lim_{n\to\infty} p_i(t_n)$ and $\lim_{n\to\infty} y(t_n-\tau_i)$, $i=1,2,\cdots,n$ exist. Hence

$$a=\lim_{n\to\infty} x(t_n)=\lim_{n\to\infty} y(t_n)-\sum_{i=1}^{m}\lim_{n\to\infty} p_i(t_n)\lim_{n\to\infty} y(t_n-\tau_i)\ge\limsup_{t\to\infty} y(t)(1-p),$$

that is,

$$\frac{a}{1-p}\ge\limsup_{t\to\infty} y(t). \tag{2.4}$$

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On the other hand, there exists $\{t'_n\}$ such that $\lim_{n\to\infty} y(t'_n)=\liminf_{t\to\infty} y(t)$. Without loss of generality, we assume that $\lim_{n\to\infty} p_i(t'_n)$ and $\lim_{n\to\infty} y(t'_n-\tau_i)$, $i=1,2,\cdots,m$ exist. Hence

$$a=\lim_{n\to\infty} x(t'_n)=\lim_{n\to\infty} y(t'_n)-\sum_{i=1}^{m}\lim_{n\to\infty} p_i(t'_n)\lim_{n\to\infty} y(t'_n-\tau_i)\le \liminf_{t\to\infty} y(t)(1-p)$$

or

$$\frac{a}{1-p}\le \liminf_{t\to\infty} y(t). \qquad (2.5)$$

Combining inequalities (2.4) and (2.5), we obtain $\lim_{t\to\infty} y(t)=\frac{a}{1-p}$. A similar argument can be repeated if $y(t)<0$.

We are now ready to prove the following results.

III. Main Results

Here, Theorem 3.1, 3.2, 3.3, 3.4, 3.5 are extensions of Theorem 4.5.1, 4.5.2, 4.5.3, 4.5.4, 4.5.5 found on pages 244, 245, 249, 251, 251, respectively, being their neutral delay versions as identified in the work by Erbe et al ([11]).

Theorem 3.1: Assume that $\lim_{t\to\infty}\sum_{i=1}^{m} p_i(t)=p\in[0,1)$. Let $y(t)$ be a non-oscillatory solution of equation (2.1). Let Λ denote the set of all non-oscillatory solutions of equation (2.1), and define

$$\Lambda^{(0,0,0)}=\left\{y\in\Lambda:\lim_{t\to\infty} y(t)=0,\ \lim_{t\to\infty} x(t)=0,\ \lim_{t,t_k\to\infty}\left(x'(t),\Delta x(t_k)\right)=0\right\},$$

$$\Lambda^{(b,a,0)}=\left\{y\in\Lambda:\ \lim_{t\to\infty} y(t)=b:=\frac{a}{1-p},\ \lim_{t\to\infty} x(t)=a,\ \lim_{t,t_k\to\infty}\left(x'(t),\Delta x(t_k)\right)=0\right\},$$

$$\Lambda^{(\infty,\infty,0)}=\left\{y\in\Lambda:\lim_{t\to\infty} y(t)=\infty,\ \lim_{t\to\infty} x(t)=\infty,\ \lim_{t,t_k\to\infty}\left(x'(t),\Delta x(t_k)\right)=0\right\},$$

$$\Lambda^{(\infty,\infty,d)}=\left\{y\in\Lambda:\lim_{t\to\infty} y(t)=\infty,\ \lim_{t\to\infty} x(t)=\infty,\ \lim_{t,t_k\to\infty}\left(x'(t),\Delta x(t_k)\right)=d\ne 0\right\}.$$

Then

$$\Lambda=\Lambda^{(0,0,0)}\cup\Lambda^{(b,a,0)}\cup\Lambda^{(\infty,\infty,0)}\cup\Lambda^{(\infty,\infty,d)}.$$

Proof: Without loss of generality, let $y(t)$ be a finally positive solution of equation (2.1). If $\lim_{t\to\infty} y(t)=0$, then by Lemma 2.1, $\lim_{t\to\infty} x(t)=0$ and $\lim_{t,t_k\to\infty}\left(x'(t),\Delta x(t_k)\right)=0$, that is, $y\in\Lambda^{(0,0,0)}$. If $\lim_{t\to\infty} y(t)\ne 0$, then by Lemma 2.1, $x(t)>0$ finally and it therefore implies that

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11. L. H. Erbe, Q. Kong and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Dekker, New York, 1995.

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$x'(t), \Delta x(t_k) > 0$ and $x''(t), \Delta x'(t_k) < 0$ finally. If $\lim_{t\to\infty} x(t) = a > 0$ exists, then $\lim_{t,t_k\to\infty}(x'(t), \Delta x(t_k)) = 0$. By Lemma 2.2, we have $\lim_{t\to\infty} y(t) = \frac{a}{1-p} = b$, that is, $y \in \Lambda^{(b,a,0)}$. If $\lim_{t\to\infty} x(t) = \infty$, then by Lemma 2.2, $\lim_{t\to\infty} y(t) = \infty$. Since $x''(t), \Delta x'(t_k) < 0$ and $x'(t), \Delta x(t_k) > 0$, we obtain $\lim_{t,t_k\to\infty}(x'(t), \Delta x(t_k)) = d$, where $d=0$ or $d > 0$. Then either $y \in \Lambda^{(\infty,\infty,0)}$ or $y \in \Lambda^{(\infty,\infty,d)}$.

This completes the proof of Theorem 3.1.

In what follows, we shall show some existence results for each kind of non-oscillatory solution of equation (2.1).

Theorem 3.2: Assume that there exist two constants $h_1 > h_2 > 0$ such that

$$|p_i(t_2) - p_i(t_1)| \le h_1 |t_2 - t_1|, |p_i(t_{2k}) - p_i(t_{1k})| \le h_1 |t_{2k} - t_{1k}|, \ i=1,2,\cdots,m,$$

$$\sum_{i=1}^{m} p_i(t)\exp(h_1\tau_i) + \exp(h_1 t)\sum_{i=1}^{m} p_{ik}\exp(-h_1(t_k - \tau_i)) > 1$$

$$\ge \sum_{i=1}^{m} p_i(t)\exp(h_2\tau_i) + \exp(h_2 t)\sum_{i=1}^{m} p_{ik}\exp(-h_2(t_k - \tau_i)) \tag{3.1}$$

and

$$\left(\sum_{i=1}^{m} p_i(t)\exp(h_1\tau_i) + \exp(h_1 t)\sum_{i=1}^{m} p_{ik}\exp(-h_1(t_k - \tau_i)) - 1\right)\exp(-h_1 t)$$

$$\ge \int_t^{x_n} (u-t)\sum_{j=1}^{m} f_j\left(u, \exp(-h_2 g_{j1}(u)), ..., \exp(-h_2 g_{jl}(u))\right) du +$$

$$+ \sum_{t\le t_k<\infty} (t_k - t)\sum_{j=1}^{n} f_{jk}\left(t_k, \exp(-h_2 g_{j1}(t_k))\right), \cdots, \exp(-h_2 g_{jl}(t_k))\Big) \tag{3.2}$$

finally. Then equation (2.1) has a solution $y \in \Lambda^{(0,0,0)}$.

Proof: Let us denote by B_p the space of all bounded piece-wise continuous functions in $PC([t_0,\infty))$ and define the sup norm in B_p as follows:

$$\|y\| := \sup_{t\ge t_0} |y(t)|.$$

Set

$$\Omega = \begin{cases} y \in B_p : \exp(-h_1 t) \le y(t) \le \exp(-h_2 t) \\ |y(t_2) - y(t_1)| \le L|t_2 - t_1|, |y(t_{2k}) - y(t_{1k})| \le L|t_{2k} - t_{1k}|, \end{cases}$$

for $t_1, t_2 \ge t_0$, $\forall k : t_{1k}, t_{2k} \ge t_0$ and for $L \ge h_1$. Then Ω is a nonempty, closed convex bounded set in B_p.

Notes

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For the sake of convenience, denote

$$\begin{cases} f\left(u,y\left(g\left(u\right)\right)\right)=\sum_{j=1}^{n} f_j\left(u,y\left(g_{j1}\left(u\right)\right),\cdots,y\left(g_{jl}\left(u\right)\right)\right) \\ f_k\left(t_k,y\left(g\left(t_k\right)\right)\right)=\sum_{j=1}^{n} f_{jk}\left(t_k,y\left(g_{j1}\left(t_k\right)\right),\cdots,y\left(g_{jl}\left(t_k\right)\right)\right), \end{cases} \tag{3.3}$$

Notes

$$\begin{cases} f\left(u,\exp\left(-h_2 g\left(u\right)\right)\right)=\sum_{j=1}^{n} f_j\left(u,\exp\left(-h_2 t_{j1}\left(u\right)\right),\cdots,\exp\left(-h_2 g_{jl}\left(u\right)\right)\right) \\ f_k\left(t_k,\exp\left(-h_2 g\left(t_k\right)\right)\right)=\sum_{j=1}^{n} f_{jk}\left(t_k,\exp\left(-h_2 g_{j1}\left(t_k\right)\right),\cdots,\exp\left(-h_2 g_{jl}\left(t_k\right)\right)\right). \end{cases} \tag{3.4}$$

Define a mapping J on Ω as follows:

$$(Jy)(t)=\begin{cases} \sum_{i=1}^{m} p_i\left(t\right)y\left(t-\tau_i\right)+\sum_{i=1}^{m} p_{ik}y\left(t_k-\tau_i\right)-\int_t^{\infty}\left(u-t\right)f\left(u,y\left(g\left(u\right)\right)\right)du- \\ \qquad -\sum_{t\le t_k<\infty}\left(t_k-t\right)+f_k\left(t_k,y\left(g\left(t_k\right)\right)\right),\ t,t_k\ge T \\ \exp\left(-K(y)t\right)+\exp\left(-K(y)t_k\right),\ t_0\le t,t_k<T, \end{cases} \tag{3.5}$$

where

$$K\left(y\right)=-\frac{\ln(Jy)(T)}{T},$$

T is sufficiently large such that $t-\tau_i\ge t_0$; $t_k-\tau_i\ge t_0$; $g_{js}\left(t_k\right)\ge t_0$; $i=1,2,\cdots,m$; $j=1,2,\cdots,n$; $s=1,2,\cdots,\ell$, for $t,\ t_k\ge T$.

Now, we see that condition (3.2) implies that

$$\int_T^{\infty} f\left(u,\exp\left(-h_2\, g\left(u\right)\right)\right)du+\sum_{T\le t_k<\infty} f_k\left(t_k,\exp\left(-h_2\, g\left(t_k\right)\right)\right)<\infty,$$

while from condition C2.1, it follows that for a given $\alpha\in\left(1-\delta,1\right)$,

$$\begin{cases} \left(\alpha-\sum_{i=1}^{m} p_i\left(t\right)\right)L\ge\left[\alpha-\left(1-\delta\right)\right]L>0 \\ \left(\alpha-\sum_{i=1}^{m} p_{ik}\right)\ge\left[\alpha-\left(t-\delta\right)\right]L. \end{cases} \tag{3.6}$$

Therefore, T can be chosen so large that for $t,\ t_k\ge T$,

$$\begin{cases} \int_T^{\infty} f\left(u,\exp\left(-h_2\, g\left(u\right)\right)\right)du\le\left(\alpha-\sum_{i=1}^{m} p_i\left(t\right)\right)L \\ \sum_{T\le t_k<\infty} f_k\left(t_k\right)\exp\left(-h_2\, g\left(t_k\right)\right)\le\left(\alpha-\sum_{i=1}^{m} p_{ik}\right)L, \end{cases} \tag{3.7}$$

and

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$$\begin{cases} \alpha+\sum_{i=1}^{m}\exp\left(-h_2\left(t-\tau_i\right)\right)\le\frac{1}{2} \\ \left(\alpha+\sum_{i=1}^{m}\exp\left(-h_2\left(t_k-\tau_i\right)\right)\right)\le\frac{1}{2}\frac{\left|t_2-t_1\right|}{\left|t_{2k}-t_{1k}\right|}. \end{cases}$$

Hence from inequalities (3.1) and (3.2), it follows that

$$(Jy)(t)\le\sum_{i=1}^{m}p_i(t)y(t-\tau_i)+\sum_{i=1}^{m}p_{ik}\,y(t_k-\tau_i)$$

$$\le\sum_{i=1}^{m}p_i(t)\exp\left(-h_2\left(t_k-\tau_i\right)\right)+\sum_{i=1}^{m}p_{ik}\exp\left(-h_2\left(t_k-\tau_i\right)\right)$$

$$\le\exp\left(-h_2t\right)\left[\sum_{i=1}^{m}p_i(t)\exp\left(h_2\tau_i\right)+\exp\left(h_2t\right)\sum_{i=1}^{m}p_{ik}\exp\left(-h_2\left(t_k-\tau_i\right)\right)\right]$$

$$\le\exp\left(-h_2t\right)\ \text{for}\ t,t_k\ge T,$$

and

$$(Jy)(t)\ge\sum_{i=1}^{m}p_i(t)\exp\left(-h_1\left(t-\tau_i\right)\right)+\sum_{i=1}^{m}p_{ik}\exp\left(-h_1\left(t_k-\tau_i\right)\right)-$$

$$-\int_t^{\infty}(u-t)f\left(u,\exp\left(-h_2g(u)\right)\right)du-\sum_{it\le t_k<\infty}(t_k-t)f_k\left(t_k,\exp\left(-h_2g(t_k)\right)\right)$$

$$=\exp\left(-h_1t\right)+\exp\left(-h_1t\right)\left(\sum_{i=1}^{m}p_i(t)\exp\left(h_1\tau_i\right)+\exp\left(h_1t\right)\sum_{i=1}^{m}p_{ik}\exp\left(-h_1\left(t_k-\tau_i\right)\right)\right)-$$

$$-\int_t^{\infty}(u-t)f\left(u,\exp\left(-h_2g(u)\right)\right)du-\sum_{t\le t_k<\infty}(t_k-t)f_k\left(t_k,\exp\left(-h_2g(t_k)\right)\right)$$

$$\ge\exp\left(-h_1t\right)\ \text{for}\ t,t_k\ge T.$$

That is,

$$\exp\left(-h_1t\right)\le(Jy)(t)\le\exp\left(-h_2t\right),\ t\ge T,$$

$$\exp\left(-h_1\left(t_k\right)\right)\le(Jy)(t_k)\le\exp\left(-h_2t_k\right),\ t_k\ge T.$$

By the definition of $K(y)$ and the statement

$$\exp\left(-h_1T\right)\le(Jy)(T)\le\exp\left(-h_2T\right),$$

It is clear that $h_2\le K(y)\le h_1$. Hence

$$\exp\left(-h_1t\right)\le(Jy)(t)\le\exp\left(-h_2t\right),\ \ t_0\le t,t_k<T.$$

Next, we show that

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$$\left|(Jy)(t_2)-(Jy)(t_1)\right|\le L\left|t_2-t_1\right|, \tag{3.8}$$

for $t_1,\, t_2\in[t_0,\infty)$ and $k{:}\ t_{1k},\, t_{2k}\in[t_0,\infty)$. Without loss of generality, we assume that $t_2\ge t_1\ge t_0$ and $\forall\, k: t_{2k}\ge t_{1k}\ge t_0$. Indeed, for $t_2\ge t_1\ge T$ and $\forall\, k{:}\, t_{2k}\ge t_{1k}\ge T$, using condition (3.7) and inequality (3.8), we have that

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$$\left|(Jy)(t_2)-(Jy)(t_1)\right|=\left|(Jy)(t_2)+(Jy)(t_{2k})-(Jy)(t_1)-(Jy)(t_{1k})\right|$$

$$\le\sum_{i=1}^{m}\left|p_i(t_1)y(t_1-\tau_i)+p_i(t_{1k})y(t_{1k}-\tau_i)-p_i(t_2)y(t_2-\tau_i)-p_i(t_{2k})y(t_{2k}-\tau_i)\right|+$$

$$+\int_{t_1}^{\infty}(u-t_1)f\big(u,y(g(u))\big)du+\sum_{t_1\le t_{1k}<\infty}(t_{1k}-t_1)f_k\big(t_{1k},y(g(t_{1k}))\big)-\int_{t_2}^{\infty}(u-t_2)f\big(u,y(g(u))\big)du-$$

$$-\sum_{t_2\le t_{2k}<\infty}(t_{2k}-t_2)f_k\big(t_{2k},y(g(t_{2k}))\big)$$

$$\le\sum_{i=1}^{m}\left|p_i(t_1)y(t_1-\tau_i)-p_i(t_2)y(t_2-\tau_i)\right|+\sum_{i=1}^{m}\left|p_i(t_{1k})y(t_{1k}-\tau_i)-p_i(t_{2k})y(t_{2k}-\tau_i)\right|+$$

$$+\left|\int_{t_1}^{\infty}(u-t_1)f\big(u,y(g(u))\big)du-\int_{t_2}^{\infty}(u-t_2)f\big(u,y(g(u))\big)du\right|+$$

$$+\left|\sum_{t_1\le t_{1k}<\infty}(t_{1k}-t_1)f_k\big(t_{1k},y(g(t_{1k}))\big)-\sum_{t_2\le t_{2k}<\infty}(t_{2k}-t_2)f_k\big(t_{2k},y(g(t_{2k}))\big)\right|$$

$$\le\sum_{i=1}^{m}p_i(t_2)\left|y(t_2-\tau_i)-y(t_1-\tau_i)\right|+\sum_{i=1}^{m}\left|p_i(t_2)-p_i(t_1)\right|y(t_1-\tau_i)+$$

$$+\left|\int_{t_1}^{t_2}(u-t_2)f\big(u,y(g(u))\big)du+\int_{t_2}^{\infty}(t_2-t_1)f\big(u,y(g(u))\big)du\right|+$$

$$+\sum_{i=1}^{m}p_i(t_{2k})\left|y(t_{2k}-\tau_i)-y(t_{1k}-\tau_i)\right|+\sum_{i=1}^{m}\left|p_i(t_{2k})-p_i(t_{1k})\right|y(t_{1k}-\tau_i)+$$

$$+\left|\sum_{t_1\le t_k\le t_2}(t_k-t_{1k})f\big(t_k,y(g(t_k))\big)+\sum_{t_2\le t_k<\infty}(t_{2k}-t_{1k})f_k\big(t_k,y(g(t_k))\big)\right|$$

$$\le\left[\sum_{i=1}^{m}\big(p_i(t_2)+\exp(-h_2(t_1-\tau_i))\big)L+\int_{t_1}^{\infty}f\big(u,\exp(-h_2g(u))\big)du\right]\left|t_2-t_1\right|+$$

$$+\left[\sum_{i=1}^{m}\big(p_i(t_{2k})+\exp(-h_2(t_{1k}-\tau_i))\big)L+\sum_{t_1\le t_k<\infty}f_k\big(t_k,\exp(-h_2g(t_k))\big)\right]\left|t_{2k}-t_{1k}\right|$$

$$\le\left\{\left[\sum_{i=1}^{m}p_i(t_2)+\sum_{i=1}^{m}\exp(-h_2(t_1-\tau_i))\right]+\left(\alpha-\sum_{i=1}^{m}p_i(t_2)\right)\right\}L\left|t_2-t_1\right|+$$

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$$+\left\{\left[\sum_{i=1}^{m}p_i(t_{2k})+\sum_{i=1}^{m}\exp(-h_2)(t_{1k}-\tau_i)\right]+\left(\alpha-\sum_{i=1}^{m}p_i(t_{2k})\right)\right\}L\left|t_{2k}-t_{1k}\right|$$

$$=\left[\sum_{i=1}^{m}\exp\left(-h_2(t_1-\tau_i)\right)+\alpha\right]L\left|t_2-t_1\right|+\left[\sum_{i=1}^{m}\exp\left(-h_2(t_{1k}-\tau_i)\right)+\alpha\right]L\left|t_{2k}-t_{1k}\right|$$

$$\leq\frac{L}{2}\left|t_2-t_1\right|+\frac{L}{2}\frac{\left|t_{2k}-t_{1k}\right|}{t}\cdot\frac{\left|t_2-t_1\right|}{\left|t_{2k}-t_{1k}\right|}$$

$$\leq L\left|t_2-t_1\right|.$$

For $t_0\leq t_1\leq t_2\leq T$ and $\forall k:t_0\leq t_{1k}\leq t_{2k}\leq T$, we have

$$\left|(Jy)(t_2)-(Jy)(t_1)\right|=\left|(Jy)(t_2)+(Jy)(t_{2k})-(Jy)(t_1)-(Jy)(t_{1k})\right|$$

$$=\left|\exp(-K(y)(t_2))+\exp(-K(y)(t_{2k}))-\exp(-K(y)(t_1))-\exp(-K(y)(t_{1k}))\right|$$

$$\leq\left|\exp(-K(y)(t_2))-\exp(-K(y)(t_1))\right|+\left|\exp(-K(y)(t_{2k}))-\exp(-K(y)(t_{1k}))\right|$$

$$\leq\frac{L}{2}\left|t_2-t_1\right|+\frac{L\left|t_{2k}-t_{1k}\right|}{2}\cdot\frac{\left|t_2-t_1\right|}{\left|t_{2k}-t_{1k}\right|}=L\left|t_2-t_1\right|.$$

For $t_0<t_1\leq T\leq t_2$ and $\forall\ k:t_0<t_{1k}\leq T\leq t_{2k}$, we obtain

$$\left|(Jy)(t_2)-(Jy)(t_1)\right|\leq\left|(Jy)(t_2)-(Jy)(t_1)\right|+\left|(Jy)(t_{2k})-(Jy)(t_{1k})\right|$$

$$\leq\left|(Jy)(t_2)-(Jy)(T)\right|+\left|(Jy)(T)-(Jy)(t_1)\right|+\left|(Jy)(t_{2k})-(Jy)(T)\right|+\left|(Jy)(T)-(Jy)(t_{1k})\right|$$

$$\leq\frac{L}{2}\left|t_2-T\right|+\frac{L}{2}\left|T-t_1\right|+\frac{L\left|t_2-t_1\right|}{2\left|t_{2k}-t_{1k}\right|}\left|t_{2k}-T\right|+\frac{L\left|t_2-t_1\right|}{2\left|t_{2k}-t_{1k}\right|}\left|T-t_{1k}\right|$$

$$=\frac{L}{2}\left|t_2-t_1\right|+\frac{L}{2}\left|t_2-t_1\right|=L\left|t_2-t_1\right|.$$

We have proved that inequality (3.8) holds for all $t_0\leq t_1\leq t_2$ and $\forall\ k:t_0\leq t_{1k}\leq t_{2k}$. Therefore, $J\Omega\subseteq\Omega$. Hence, J is piece-wise continuous. Since $J\Omega\subseteq\Omega$, $J\Omega$ is uniformly bounded.

Set $y\in\Omega$. It immediately implies that

$$\left|(Jy)(t)\right|\leq b_0,$$

where $b_0>0$ and

$$\left|(Jy)(t_2)-(Jy)(t_1)\right|\leq L\left|t_2-t_1\right|$$

for $t_2\geq t_1\geq t_0$ and $k:t_{2k}\geq t_0$. Without loss of generality, we set

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$$b_0 = \exp(-h_2 t),\ t, t_k \geq t_0 .$$

Hence, for any arbitrarily pre-assigned small positive number ε, there exists a sufficiently large $T' > t_0$ such that whenever $\exp(-h_2 t) < \frac{\varepsilon}{2}$,

$$\left|(Jy)(t_2) - (Jy)(t_1)\right| \leq \exp(-h_2 t_2) + \exp(-h_2 t_1) \leq \varepsilon \tag{3.9}$$

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for $t, t_k \geq T', t_2 \geq t_1 \geq T'$ and $k : t_{2k} \geq t_{1k} \geq T'$.

On the other hand, if we set $\lambda = \frac{\varepsilon}{L}$ and assume that $|t_2 - t_1| < \lambda$, then for all $t_0 \leq t_1 \leq t_2 \leq T'$ and $k : t_0 \leq t_{1k} \leq t_{2k} \leq T'$, it becomes clear that

$$\left|(Jy)(t_2) - (Jy)(t_1)\right| \leq \varepsilon \tag{3.10}$$

Thus, from inequalities (3.9) and (3.10), we can affirm that $J\Omega$ is quasi-equicontinuous. Therefore, $J\Omega$ is relatively compact. By virtue of Schauder-Tychonoff fixed point theorem, the mapping J has a fixed point $y^* \in J$ such that $y^* = Jy^*$. Then y^* is a positive solution of equation (2.1) and $y^* \in \Lambda^{(0,0,0)}$. This completes the proof of Theorem 3.2.

Theorem 3.3: Assume that $\lim_{t\to\infty} \sum_{i=1}^{m} p_i(t) + \lim_{t_k\to\infty} \sum_{i=1}^{m} p_{ik} = p \in [0,1)$. Then equation (2.1) has a non-oscillatory solution $y \in \Lambda^{(b,a,0)}$ $(b, a \neq 0)$ if and only if

$$\int_{t_0}^{\infty} u \left| \sum_{j=1}^{n} f_j(u, b_1, \ldots, b_1) \right| du + \sum_{t_0 \leq t_k < \infty} t_k \left| \sum_{j=1}^{n} f_{jk}(t_k, b_1, \ldots, b_1) \right| < \infty \tag{3.11}$$

for $b_1 \neq 0$.

Proof

i) *Necessity:* Without loss of generality, let $y(t) \in \Lambda^{(b,a,0)}$ be a finally positive solution of equation (2.1). From Theorem 3.1, we know that $b > 0$ and $a > 0$. Using notations in equations (3.3) and (3.4), we obtain from equations (2.1) and (2.2),

$$\begin{cases} x''(t) = -f(t, y(g(t))) \\ \Delta x'(t_k) = f_k(t_k, y(g(t_k))). \end{cases}$$

Integrating it from s to ∞ for $s \geq t_0$, we have

$$x'(s) = \int_s^{\infty} f(u, y(g(u)))du + \sum_{s \leq t_k < \infty} f_k(t_k, y(g(t_k))) . \tag{3.12}$$

Again, integrating equation (3.12) from T to t, where T is sufficiently large, we obtain

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$$x(t)=x(T)+\int_T^t (u-T)f\big(u,y(g(u))\big)du+\int_t^\infty (t-T)f\big(u,y(g(u))\big)du+$$

$$+\sum_{T\le t_k\le t}(t_k-T)f_k\big(t_k,y(g(t_k))\big)+\sum_{t\le t_k<\infty}(t-T)f_k\big(t_k,y(g(t_k))\big). \qquad (3.13)$$

Since $\lim_{u\to\infty} y\big(g_{jh}(u)\big)=b>0$ and $\lim_{t_k\to\infty} y\big(g_{jh}(t_k)\big)=b>0$, $j=1,2,\cdots,n$, $h=1,2,\cdots,\ell$, there exists a $T\ge t_0$ such that $y\big(g_{jh}(u)\big)\ge\frac{b}{2}$ for $u\ge T$ and $y\big(g_{jh}(t_k)\big)\ge\frac{b}{2}$ for $k:t_k\ge T$. Hence from equation (3.13) we have

$$\int_T^t (u-T)\left|\sum_{j=1}^n f_j\left(u,\frac{b}{2},\cdots,\frac{b}{2}\right)\right|du+\sum_{T\le t_k\le t}(t_k-T)\left|\sum_{j=1}^n f_{jk}\left(t_k,\frac{b}{2},\cdots,\frac{b}{2}\right)\right|<x(t)-x(T)$$

which implies that condition (3.11) holds.

ii) *Sufficiency:* Set $b_1>0$ and $A>0$ so that $A<(1-p)b_1$. From condition (3.11) there exists a sufficiently large T so that for $t,t_k\ge T$ we have $t-\tau_i\ge t_0$, $t_k-\tau_i\ge t_0$, $i=1,2,\cdots,m$, and $g_{jh}(t)\ge t_o$, $g_{jh}(t_k)\ge t_0$, $j=1,2,\cdots,n$, $h=1,2,\cdots,\ell$ and

$$\frac{A}{b_1}+\sum_{i=1}^m\big(p_i(t)+p_{ik}\big)+\frac{1}{b_1}\int_T^\infty u\sum_{j=1}^n f_j(u,b_1,...,b_1)du+\frac{1}{b}\sum_{T\le t_k<\infty}t_k\sum_{j=1}^n f_{jk}(t_k,b_1,...,b_1)\le 1. \qquad (3.14)$$

Let Ω be the set of all piece-wise continuous functions $y(t)\in[t_0,\infty)$ such that $0\le y(t)\le b_1$, $t,t_k\ge t_0$. Define a mapping J in Ω as follows:

$$(Jy)(t)=\begin{cases} A+\sum_{i=1}^m p_i(t)y(t-\tau_i)+\sum_{i=1}^m p_{ik}y(t_k-\tau_i)+\int_T^t uf\big(u,y(g(u))\big)du+ \\ \quad +\int_t^\infty tf\big(u,y(g(u))\big)du+\sum_{T\le t_k\le t}t_kf_k\big(t_k,y(g(t_k))\big)+\sum_{t\le t_k<\infty}tf_k\big(t_k,y(g(t_k))\big), \\ \quad t,t_k\ge T \\ (Jy)(T),\quad t_0\le t,t_k<T. \end{cases} \qquad (3.15)$$

Set

$$y_0(t)=0,\ t\ge t_0;$$

$$y_\ell(t)=(Ty_{\ell-1})(t),\ t\ge t_0,\ \ell=1,2,\cdots. \qquad (3.16)$$

It immediately follows that $y_0(t)<y_1(t)=A\le b_1$, $t\ge t_0$. By induction, we obtain

$$A\le y_\ell(t)\le y_{\ell+1}(t)\le b_1,\ t\ge t_0,\ \ell=1,2,\cdots.$$

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Thus, $\lim_{\ell\to\infty} y_\ell(t) \le y(t)$ exists and $A \le y(t) \le b_1,\ t\in[t_0,\infty)$. By Lebesgue's monotone convergence theorem, we obtain from equation (3.16) the result

$$y(t)=\begin{cases} A+\sum_{i=1}^{m} p_i(t)y(t-\tau_i)+\sum_{i=1}^{m} p_{ik}y(t_k-\tau_i)+\int_T^t uf(u,y(g(u)))du+\int_t^\infty tf(u,y(g(u)))du \\ \quad +\sum_{T\le t_k\le t} t_k f_k(t_k,y(g(t_k)))+\sum_{t\le t_k<\infty} tf_k(t_k,y(g(u))), \quad t,t_k\ge T \\ y(T), \quad t_0\le t,t_k<T. \end{cases}$$

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Hence, $y(t)$ is a positive solution of equation (2.1). Since $0<A\le y(t)<b_1$, from Theorem 3.1, $y\in\Lambda^{(b,a,0)}$. This completes the proof of Theorem 3.3.

Using reasoning analogous to that given in the proof of Theorem 3.3 above, we can verify the following results.

Theorem 3.4: Assume that $\lim_{t\to\infty}\sum_{i=1}^{m} p_i(t)+\lim_{t_k\to\infty}\sum_{i=1}^{m} p_{ik}=p\in[o,1)$. Then equation (2.1) has a non-oscillatory solution $y\in\Lambda^{(\infty,\infty,d)}$, $(d\ne 0)$ if and only if

$$\int_{t_0}^{\infty}\left|\sum_{j=1}^{n} f_j(u,d_1g_{j1}(u),\dots,d_1g_{j\ell}(u))\right|du+\sum_{t_0\le t_k<\infty}\left|\sum_{j=1}^{n} f_{jk}(t_k,d_1g_{ji}(t_k),\dots,d_1g_{j\ell}(t_k))\right|<\infty, \tag{3.17}$$

for some $d_1\ne 0$.

Theorem 3.5: Assume that $\lim_{t\to\infty}\sum_{i=1}^{m} p_i(t)+\lim_{t_k\to\infty}\sum_{i=1}^{m} p_{ik}=p\in[0,1)$. Further assume that

$$\int_{t_0}^{\infty}\left|\sum_{j=1}^{n} f_j(u,d_1g_{ji}(u),\dots,d_1g_{j\ell}(u))\right|du+\sum_{t_0\le t_k<\infty}\left|\sum_{j=1}^{n} f_{jk}(t_k,d_1g_{j1}(t_k),\dots,d_1g_{j\ell}(t_k))\right|<\infty \tag{3.18}$$

for some $d_1\ne 0$ and

$$\int_{t_0}^{\infty}u\left|\sum_{j=1}^{n} f_j(u,b_1,\dots,b_1)\right|du+\sum_{t_0\le t_k\infty} t_k\left|\sum_{j=1}^{n} f_{jk}(t_k,b_1,\dots,b_1)\right|=\infty \tag{3.19}$$

for some $d_1\ne 0$, where $b_1d_1>0$. Then equation (2.1) has a non-oscillatory solution $y\in\Lambda^{(\infty,\infty,0)}$.

We examine the following to help illustrate the obtained results.

Example 3.1: Consider

$$\begin{cases} \left[y(t)-\frac{1}{2}y(t-1)\right]''+\frac{2(t-1)^3-t^3}{(t-1)^6}y^3(t)=0 \\ \Delta\left[y(t_k)-\frac{1}{2}y(t_k-1)\right]'+\frac{1(t_k-1)^3-t_k^3}{(t_k-1)^6}y^3(t_k)=0, \end{cases} \tag{3.20}$$

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where

$$q(t)=\frac{2(t-1)^3-t^3}{(t-1)^6} \text{ and } q_k=\frac{2(t_k-1)^3-t_k^3}{(t_k-1)^6}.$$

It is obvious that inequality (3.11) holds. Therefore, equation (3.20) has a non-oscillatory solution $y\in\Lambda^{(b,a,0)}, b\neq 0, a\neq 0$. In fact, $y(t)=1-\frac{1}{t}$ is such a solution, where $a=\frac{1}{2}$ and $b=1$.

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Example 3.2: Consider

$$\begin{cases} \left[y(t)-\frac{1}{2}y(t-1)\right]''+q(t)y^{\frac{1}{3}}(t)=0 \\ \Delta\left[y(t_k)-\frac{1}{2}y(t_k-1)\right]'+q_k y^{\frac{1}{3}}(t_k)=0, \end{cases} \tag{3.21}$$

where

$$q(t)=\frac{1}{4}\left(t^{3/2}-\frac{1}{2}(t-1)^{-3/2}\right)t^{-1/6}, \quad q_k=\frac{1}{4}\left(t_k^{-3/2}-\frac{1}{2}(t_k-1)^{-3/2}\right)t_k^{-1/6}.$$

For large t and t_k, $q(t)\sim Mt^{-5/3}$ and $q_k\sim Mt_k^{-5/3}$. It is obvious that inequalities (3.18) and (3.19) are satisfied. From Theorem 3.5, equation (3.21) has a solution $y\in\Lambda^{(\infty,\infty,0)}$. In fact, $y(t)=\sqrt{t}$ is such a solution of equation (3.21).

Remark 3.1: The above arguments can be applied to the equation

$$\begin{cases} \left[y(t)-\sum_{i=1}^{m}p_i(t)y(t-\tau_i)\right]''=\sum_{j=1}^{n}f_j\left(t,y\left(g_{j1}(t)\right),\cdots,y\left(g_{j\ell}(t)\right)\right),\ t\geq t_0,\ t\notin S \\ \Delta\left[y(t_k)-\sum_{i=1}^{m}p_{ik}y(t_k-\tau_i)\right]'=\sum_{j=1}^{n}f_{jk}\left(t_k,y\left(g_{j1}(t_k)\right),\cdots,y\left(g_{j\ell}(t_k)\right)\right),\ t_k\geq t_0,\ \forall\, t_k\in S. \end{cases} \tag{3.22}$$

For instance, under the assumptions of Theorem 3.1, we have

$$\Lambda=\Lambda^{(0,0,0)}\cup\Lambda^{(b,a,0)}\cup\Lambda^{(\infty,\infty,\alpha)}\cup\Lambda^{(\infty,\infty,\infty)}.$$

Therefore, Theorems 3.3 and 3.4 hold for equation (3.22). Furthermore, equation (3.22) has a non-oscillatory solution $y(t)\in\Lambda^{(\infty,\infty,\infty)}$ if

$$\int_{t_0}^{\infty}\left|\sum_{j=1}^{n}f_j\left(t,d_1g_{j1}(t),\dots,d_1g_{j\ell}(t)\right)\right|dt+\sum_{t_0\leq t_k<\infty}\left|\sum_{j=1}^{n}f_{jk}\left(t_k,d_1g_{j1}(t_k),\dots,d_1g_{j\ell}(t_k)\right)\right|<\infty \tag{3.23}$$

for some $d_1\neq 0$.

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