Simple Proof of Two Identities of Ramanujan

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Abstract—Using two simple theta function identities we give simple proof of two identities of Ramanujan, one of which leads to the famous eight square theorem of Jacobi.

Key words and phrases: Theta functions, $q$-hypergeometric series

I. INTRODUCTION

In a fragment published with his lost notebook [5, pp. 353-355], Ramanujan provided a list of twenty identities. These are immediately followed by another fragment on pages 356 and 357 with another list of twenty one identities. Most of these, but not all, can be found in Ramanujan’s second notebook [4]. For more details see Andrews and Berndt [1, pp. 395-396]. The numbering are those given by Ramanujan in the fragments. The two identities mentioned in the title of the paper are

$$
\varphi^8(q) = 1 + 16 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - (-q)^n},
$$

(1.1)

writing $q$ for $q$ and then setting $q^2$ for $q$, we have

$$
\varphi^8(-q^2) = 1 + 16 \sum_{n=1}^{\infty} (-1)^n \frac{n^3 q^{2n}}{1 - q^{2n}},
$$

(1.2)

and

$$
\psi^2(q^4) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n}}{1 - q^{4n+2}}.
$$

(1.3)

The identity in (1.1) is Entry 18.2.3 [1, p. 397] which leads to Jacobi’s famous formula

$$
r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3,
$$

which is Jacobi’s eight square theorem. We shall, however, be proving the identity written in the form in (1.2). The second identity (1.3) is Entry 18.2.4 [1, p. 397]. In the present paper I give simple proof of these two identities (1.2) and (1.3), using the following simple identities [3, eq. (8.1), p. 117] and [6, p. 480], respectively:

$$
cot^2 y - \cot^2 x + 8 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n} (\cos 2nx - \cos 2ny)
$$

$$
= \theta_1'(0 | q)^2 \frac{\theta_1(x-y | q) \theta_1(x+y | q)}{\theta_1^2(x | q) \theta_1^2(y | q)},
$$

(1.4)

there is a slight misprint which I have corrected, and

$$
\frac{\theta_1'(z | q)}{\theta_1(z | q)} = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin 2nz.
$$

(1.5)

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II. SOME BASIC RESULTS

We shall use the following standard $q$-notation, ($q < 1$):

$$(a; q^k)_n = (1 - a)(1 - aq^k)\ldots(1 - aq^{k(n-1)}), \quad n \geq 1$$

$$(a; q^k)_0 = 1. \quad (2.1)$$

Jacobi $\theta^j$-function is defined as follows, see [6, p. 464]

$$\theta_1(z \mid q) = -i q^{1/8} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} e^{(2n+1)iz} \quad (2.2)$$

$$= 2q^{1/8} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z, \quad (2.3)$$

where $q = e^{2\pi i \tau}$ and $\text{Im}(\tau) > 0.$

The function $\theta_1(z \mid q)$ can also be expressed in terms of an infinite product

$$\theta_1(z \mid q) = 2q^{1/8} \sin z \left( q; q \right)_\infty \left( qe^{2iz}; q \right)_\infty \left( qe^{-2iz}; q \right)_\infty, \quad (2.4)$$

$$= i q^{1/8} e^{-iz} \left( q; q \right)_\infty \left( e^{2iz}; q \right)_\infty \left( qe^{-2iz}; q \right)_\infty. \quad (2.5)$$

$$\theta_1'(0 \mid q) = 2q^{1/8} \left( q; q \right)_\infty^3. \quad (2.6)$$

Ramanujan’s theta functions $\varphi(q)$ and $\psi(q)$ are defined as [1, p. 11]

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty = \frac{(-q; -q)_\infty}{(q; -q)_\infty}, \quad |q| < 1 \quad (2.7)$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \quad |q| < 1. \quad (2.8)$$

We shall require the following identities of Ramanujan:

$$\varphi(-q) = \frac{(q; q)_\infty}{(-q; q)_\infty}, \quad [2, \text{eq. (22.4)}, \text{p. 37}] \quad (2.9)$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \quad \text{Entry 25 (iii)} \ [2, \text{p. 40}] \quad (2.10)$$
Adding (v) and (vi) of Entry 25 [2, p. 40], we have
\[ \varphi^2(q) - \varphi^2(q^2) = 4q \psi^2(q^4) \] 

Putting \( z = \frac{\pi}{4} \) in (1.5), we have
\[ \frac{\theta_1'(\frac{\pi}{4} \mid q)}{\theta_1(q)} = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin \frac{n\pi}{2} \]
\[ = 1 + 4 \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n+1}}{1 - q^{2n+1}} \]
\[ = \varphi^2(q) \quad \text{by (2.11).} \] 

III. PROOF OF (1.2)

Differentiating (1.4) partially with respect to \( x \) and then putting \( y = x \), we have
\[ 2 \cot x \cos ec^2 x - 16 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1 - q^n} \sin 2nx = \frac{\theta_1'(0 \mid q)^3 \theta_1(2x \mid q)}{\theta_1^3(x \mid q)}. \]

Differentiating again the above expression partially with respect to \( x \) and then \( z = \frac{\pi}{4} \) putting we finally get
\[ 1 + 16 \sum_{n=1}^{\infty} (-1)^n \frac{n^3 q^{2n}}{1 - q^{2n}} = \theta_1'(0 \mid q)^3 \frac{\theta_1(\frac{\pi}{2} \mid q)}{4 \theta_1^4(\frac{\pi}{4} \mid q)} \theta_1'(\frac{\pi}{4} \mid q) \]
\[ = \frac{(q; q)_\infty^6 (-q; q)_\infty^2}{(-q^2; q^2)_\infty^4} \varphi^2(q), \quad \text{by (2.14)} \]
\[ = \frac{(q; q)_\infty^2 (q^2; q^2)_\infty^4}{(-q^2; q^2)_\infty^2 (-q^2; q^2)_\infty^4} \varphi^2(q) \]
\[ = \varphi^2(-q) \varphi^4(-q^2) \varphi^2(q), \quad \text{by (2.9)} \]
\[ = \varphi^8(-q^2), \quad \text{by (2.10)} \]

which proves (1.2).
IV. PROOF OF (1.3)

Writing (1.5) as

\[
\frac{\theta'_1(z)}{\theta_1(q)} = \cot z + 4 \sum_{n=1}^{\infty} q^n \left(1 + \frac{q^n}{1 - q^{2n}}\right) \sin 2nz
\]

\[
= \cot z + 4 \sum_{n=1}^{\infty} \left[\frac{q^n}{1 - q^{2n}} + \frac{q^{2n}}{1 - q^{2n}}\right] \sin 2nz \tag{4.1}
\]

and setting \(z = \frac{\pi}{4}\) in (4.1), we obtain

\[
\frac{\theta'_1\left(\frac{\pi}{4}, q^2\right)}{\theta_1(q)} = 1 + 4q \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n}}{1 - q^{4n+2}} + 4 \sum_{n=0}^{\infty} (-1)^n \frac{q^{4n+2}}{1 - q^{4n+2}}.
\]

Or

\[
4q \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n}}{1 - q^{4n+2}} = \frac{\theta'_1\left(\frac{\pi}{4}, q\right)}{\theta_1(q)} - 4 \sum_{n=0}^{\infty} (-1)^n \frac{q^{4n+2}}{1 - q^{4n+2}} - 1
\]

\[
= \varphi^2(q) - \varphi^2(q^2),
\]

here we have used (2.11) and (2.13).

Finally using (2.12) on the right hand side of (4.2), we have (1.3).

V. REFERENCES