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Discovering Thoughts, Inventing Future

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Some Further Developments in the Infinite Product Representation of Elementary Functions

By Viktor Reshniak

Middle Tennessee State University

Abstract - An innovatory approach has been recently proposed for the derivation of infinite product representation of elementary functions. The approach is based on the comparison of different alternative forms of Green's functions for boundary-value problems stated for the two-dimensional Laplace equation. A number of new infinite product representations of elementary functions was actually derived within the scope of that approach. The present study continues the trend: it aims at an analysis of the approach and exploring ways for its extending to some other problem statements that might also be efficiently treated.

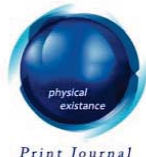
Keywords : *green's functions; infinite products; elementary functions.*

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Some Further Developments in the Infinite Product Representation of Elementary Functions

Viktor Reshniak

Abstract - An innovatory approach has been recently proposed for the derivation of infinite product representation of elementary functions. The approach is based on the comparison of different alternative forms of Green's functions for boundary-value problems stated for the two-dimensional Laplace equation. A number of new infinite product representations of elementary functions was actually derived within the scope of that approach. The present study continues the trend: it aims at an analysis of the approach and exploring ways for its extending to some other problem statements that might also be efficiently treated.

Keywords : green's functions; infinite products; elementary functions.

1. INTRODUCTION

In a series of recent works (see, for example, [6] and [7]), an innovatory approach was proposed to the derivation of infinite product representation of elementary functions. The approach is based on the multiplicity of forms of Green's functions for some boundary-value problems. To introduce the key idea of the approach, we turn to one of the simplest problem settings of that kind. That is the Dirichlet problem

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0, \quad (x, y) \in \Omega \quad (1)$$

$$u(x, 0) = u(x, b) = 0, \quad \lim_{x \rightarrow \pm\infty} u(x, y) < \infty \quad (2)$$

posed in the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$.

The classical representation

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{1 - 2e^{\omega(x-\xi)} \cos \omega(y+\eta) + e^{2\omega(x-\xi)}}{1 - 2e^{\omega(x-\xi)} \cos \omega(y-\eta) + e^{2\omega(x-\xi)}}, \quad \omega = \frac{\pi}{b} \quad (3)$$

of the Green's function for the setting in (1)-(2) can be obtained by one of the two standard methods available in the field. Indeed, either the conformal mapping method [1, 2] algorithm, or the method of eigenfunction expansion [4, 5] procedure, with subsequent summation of trigonometric series that represents the Green's function, appear successful.

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In [6], it was recalled that if the method of images routine (that is also considered workable, but efficient in a limited number of problem settings) is used instead, then we arrive at an alternative to (3) expression

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \prod_{n=-\infty}^{\infty} \sqrt{\frac{(x - \xi)^2 + (y + \eta - 2nb)^2}{(x - \xi)^2 + (y - \eta + 2nb)^2}} \quad (4)$$

Since the forms in (3) and (4) are equivalent, we obtain the multi-variable identity

$$\frac{1 - 2e^{\omega(x-\xi)} \cos \omega(y + \eta) + e^{2\omega(x-\xi)}}{1 - 2e^{\omega(x-\xi)} \cos \omega(y - \eta) + e^{2\omega(x-\xi)}} = \prod_{n=-\infty}^{\infty} \frac{(x - \xi)^2 + (y + \eta - 2nb)^2}{(x - \xi)^2 + (y - \eta + 2nb)^2} \quad (5)$$

which can be used as a starting point for the derivation of a number of infinite product representations of elementary functions.

Upon some trivial algebra, the identity in (5) can, for example, be transformed in the infinite product representation

$$\sin t = \frac{2t}{\pi} \prod_{k=1}^{\infty} \left[1 + \frac{4t^2 - \pi^2}{(1 - 4k^2)\pi^2} \right]$$

of the trigonometric sine function. The latter can be considered as an equivalent alternative form to the classical Euler infinite product expansion

$$\sin t = t \prod_{k=1}^{\infty} \left(1 - \frac{t^2}{k^2\pi^2} \right)$$

of the sine function.

A score of new as well as alternative to existing infinite product representations of elementary functions was presented in [7] by the approach just described. Note, that the author of [7] has claimed that research in the area is still open. Accepting the challenge, we will show in the next section, that the approach also appears efficient in a few other cases.

II. FURTHER EXTENSION OF THE APPROACH

a) *Semi-infinite Strip Region*

We start with the mixed boundary-value problem

$$u(0, y) = \frac{\partial u(a, y)}{\partial x} = 0 \quad (6)$$

$$u(x, 0) = 0, \quad \lim_{y \rightarrow \infty} u(x, y) < \infty \quad (7)$$

posed for the two-dimensional Laplace equation in the semi-infinite strip region $\Omega = \{0 < x < a, 0 < y < \infty\}$.

Following the technique proposed in [6] and [7], two equivalent representations of the Green's function for the problem in (6)-(7) can be found. The classical approach of eigenfunction expansion leads to a closed analytical form of its solution. On the other

Ref.

6. Yu. A. Melnikov, A new approach to the representation of some trigonometric and hyperbolic functions by infinite products, J. Math. Anal. Appl., 344, 1, 2008, 521-534.

hand, another classical approach – the method of images - provides an infinite product expression of the required Green's function.

Method of eigenfunction expansion. Let the Poisson equation

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = -f(x, y), \quad (x, y) \in \Omega \quad (8)$$

be subject to the boundary conditions in (6)-(7).

Once the solution of the stated problem is found as the integral

$$u(x, y) = \iint_{\Omega} G(x, y; \xi, \eta) f(\xi, \eta) d\Omega(\xi, \eta), \quad (x, y) \in \Omega \quad (9)$$

the kernel function $G(x, y; \xi, \eta)$ of the above represents the Green's function to the boundary value problem in (6)-(7).

According to the theory of Fourier series, when the eigenfunctions are known, the solution of the given boundary-value problem can be obtained by the superposition [5]

$$u(x, y) = \sum_{k=1}^{\infty} Y_k(y) X_k(x) \quad (10)$$

where X_k is a complete system of eigenfunctions, orthogonal in $[0, a]$

$$X_k = \sin \lambda_k x, \\ \lambda_k = \frac{\pi(2k-1)}{2a}, \quad k = 1, 2, \dots$$

Functions $Y_k(y)$ are to be determined from the Fourier series expansion of the right-hand side part in (8)

$$f(x, y) = \sum_{k=1}^{\infty} f_k(y) X_k(x), \quad f_k(y) = \frac{2}{a} \int_0^a f(\xi, y) X_k(\xi) d\xi \quad (11)$$

resulting in

$$\frac{d^2 Y_k(y)}{dy^2} - \lambda_k^2 Y_k^2(y) = -\frac{2}{a} \int_0^a f(\xi, y) \sin \lambda_k \xi d\xi \quad (12)$$

$$Y_k(0) = 0 \quad (13)$$

$$\lim_{y \rightarrow \infty} |Y_k(y)| < \infty \quad (14)$$

Clearly, solution of the problem in (12)-(14) has the form

$$Y_k(y) = \int_0^{\infty} f_k(\eta) g_k(y; \eta) d\eta \quad (15)$$

where $g_k(y; \eta)$ is the Green's function for the homogeneous equation corresponding to (12) with boundary conditions in (13) and (14) ([4],[7])

$$g_k(y; \eta) = \frac{1}{2\lambda_k} \begin{cases} e^{\lambda_k(y-\eta)} - e^{-\lambda_k(y+\eta)}, & y \leq \eta \\ e^{\lambda_k(\eta-y)} - e^{-\lambda_k(y+\eta)}, & y \geq \eta \end{cases}$$

Therefore the series in (10), along with (15) solves the problem posed in (6)-(8)

$$u(x, y) = \int_0^a \int_0^\infty \left[\frac{2}{a} \sum_{k=1}^\infty g_k(y; \eta) \sin \lambda_k \xi \sin \lambda_k x \right] f(\xi, \eta) d\xi d\eta$$

In light of (9) the kernel function of the above expression represents the Green's function to the problem in (6)-(7) posed for the Laplace equation in Ω

$$G(x, y; \xi, \eta) = \frac{2}{a} \sum_{k=1}^\infty g_k(y; \eta) \sin \lambda_k \xi \sin \lambda_k x \quad (16)$$

Upon the trigonometric identity

$$\sin \lambda_k \xi \sin \lambda_k x = \frac{1}{2} [\cos \lambda_k (\xi - x) - \cos \lambda_k (\xi + x)]$$

and (see, for example [3]) the summation formula

$$\sum_{k=1}^\infty \frac{p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \cos x + p^2}{1 - 2p \cos x + p^2}, \quad 0 < x < 2\pi, \quad p^2 \leq 1$$

the expression in (16) transforms into

$$\begin{aligned} G(x, y; \xi, \eta) = \frac{1}{4\pi} & \left(\ln \frac{1 + 2e^{\omega(y-\eta)} \cos \omega(\xi - x) + e^{2\omega(y-\eta)}}{1 - 2e^{\omega(y-\eta)} \cos \omega(\xi - x) + e^{2\omega(y-\eta)}} \right. \\ & - \ln \frac{1 + 2e^{\omega(y-\eta)} \cos \omega(\xi + x) + e^{2\omega(y-\eta)}}{1 - 2e^{\omega(y-\eta)} \cos \omega(\xi + x) + e^{2\omega(y-\eta)}} \\ & - \ln \frac{1 + 2e^{-\omega(y+\eta)} \cos \omega(\xi - x) + e^{-2\omega(y+\eta)}}{1 - 2e^{-\omega(y+\eta)} \cos \omega(\xi - x) + e^{-2\omega(y+\eta)}} \\ & \left. + \ln \frac{1 + 2e^{-\omega(y+\eta)} \cos \omega(\xi + x) + e^{-2\omega(y+\eta)}}{1 - 2e^{-\omega(y+\eta)} \cos \omega(\xi + x) + e^{-2\omega(y+\eta)}} \right) \end{aligned} \quad (17)$$

where

$$\omega = \frac{\pi}{2a}$$

After introducing the variables

$$\alpha = \frac{\pi}{2a}(y - \eta); \quad \beta = -\frac{\pi}{2a}(y + \eta); \quad \gamma = \frac{\pi}{2a}(x - \xi); \quad \theta = \frac{\pi}{2a}(\xi + x) \quad (18)$$

R_{ref.}

4. Dean G. Duffy, Green's Functions with Applications, Chapman and Hall/CRC, 2001.

the form in (17) reads

$$G(\alpha, \beta, \gamma, \theta) = \frac{1}{4\pi} \ln \frac{(1 + 2e^\alpha \cos \gamma + e^{2\alpha})(1 + 2e^\beta \cos \theta + e^{2\beta})}{(1 - 2e^\alpha \cos \gamma + e^{2\alpha})(1 - 2e^\beta \cos \theta + e^{2\beta})} \times \frac{(1 - 2e^\alpha \cos \theta + e^{2\alpha})(1 - 2e^\beta \cos \gamma + e^{2\beta})}{(1 + 2e^\alpha \cos \theta + e^{2\alpha})(1 + 2e^\beta \cos \gamma + e^{2\beta})} \quad (19)$$

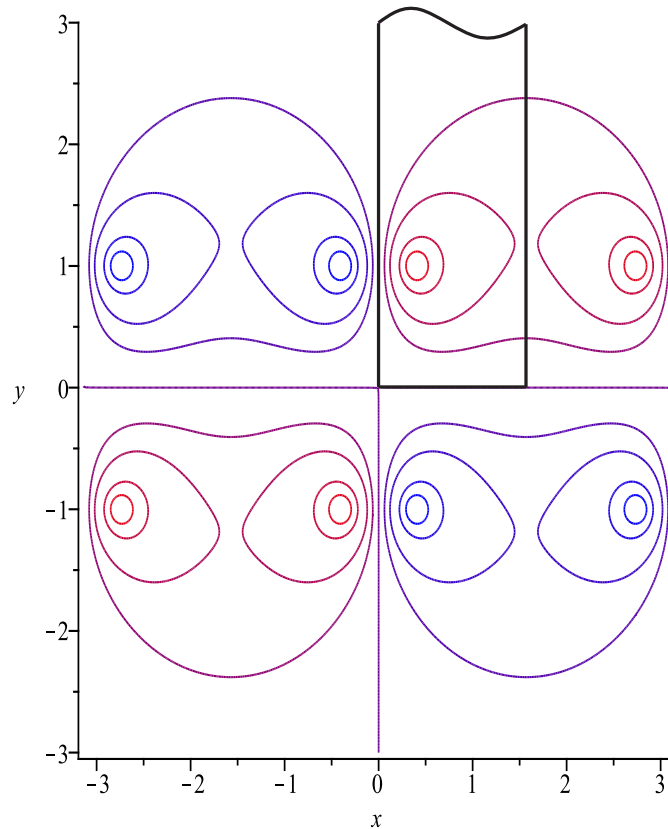


Figure 1 : Field generated in the semi-infinite strip $0 \leq x \leq \pi/2$, $y \geq 0$ by a source located at $\xi = \pi/8$, $\eta = 1$

By problem statement, the observation point (x, y) and the source point (ξ, η) are interior to the semi-infinite strip Ω . It means that parameters $\alpha, \beta, \gamma, \theta$ range as

$$-\infty < \alpha < \infty, \quad -\infty < \beta < 0, \quad -\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \pi \quad (20)$$

given that α and γ are not equal to zero at the same time.

But on the other hand the function in (19) is analytic everywhere except, of course, at the points of singularity, as it can be seen in Figure 1. Hence, the parameter ranges in (20) can be extended to the entire region of analyticity of (19).

Method of images. The Green's function for the two-dimensional Laplace equation has the form ([5, 7])

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln r(x, y, \xi, \eta) + \mu(x, y; \xi, \eta)$$

where $r(x, y, \xi, \eta)$ is the distance between the field point and the source point, and $\mu(x, y; \xi, \eta)$ is a harmonic in Ω function.

According to the method of images, the regular component $\mu(x, y; \xi, \eta)$ is sought as a superposition of responses at a field point P from singularities Q_j^* placed outside the region Ω

$$\mu(x, y) = \sum_{j=1}^m \pm \frac{1}{2\pi} \ln |P - Q_j^*|$$

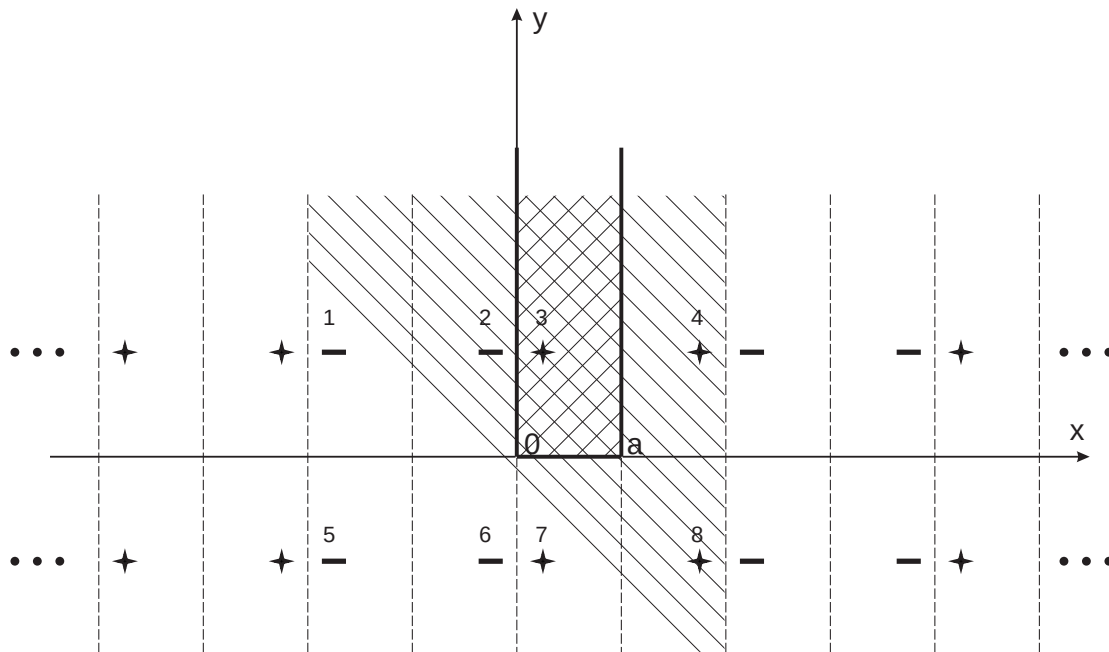


Figure 2: Sequence of sources and sinks that arises in the method of images applied to the problem in (6)-(7)

The sign of each term in the above as well as the positions of the corresponding singularities are chosen to satisfy boundary conditions of the given boundary-value problem.

Figure 2 illustrates this approach to the problem in (6)-(7). Unit sources are labeled with the plussign and sinks with the minus sign. Geometry of Ω leads to the infinite sequence of singularities, where the shaded strip of width $4a$ is repeated periodically in the direction of the x -axis. The singularities have the following coordinates.

$$(\xi_1^0 = -(2a - \xi); \eta_1^0 = \eta)$$

$$(\xi_5^0 = -(2a - \xi); \eta_5^0 = -\eta)$$

$$(\xi_2^0 = -\xi; \eta_2^0 = \eta)$$

$$(\xi_6^0 = -\xi; \eta_6^0 = -\eta)$$

$$(\xi_3^0 = \xi; \eta_3^0 = \eta)$$

$$(\xi_7^0 = \xi; \eta_7^0 = -\eta)$$

$$(\xi_4^0 = 2a - \xi; \eta_4^0 = \eta)$$

$$(\xi_8^0 = 2a - \xi; \eta_8^0 = -\eta)$$

Hence, the total effect on the field point P from the infinite sequence of suitably chosen sinks and sources Q_j^* is described by the following Green's function

$$G(x, y; \xi, \eta) = \sum_{k=-\infty}^{\infty} G^k(x, y; \xi, \eta)$$

$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \ln \sqrt{\frac{\left((x - \xi_1^k)^2 + (y - \eta_1^k)^2\right) \left((x - \xi_2^k)^2 + (y - \eta_2^k)^2\right)}{\left((x - \xi_3^k)^2 + (y - \eta_3^k)^2\right) \left((x - \xi_4^k)^2 + (y - \eta_4^k)^2\right)}} \times \sqrt{\frac{\left((x - \xi_7^k)^2 + (y - \eta_7^k)^2\right) \left((x - \xi_8^k)^2 + (y - \eta_8^k)^2\right)}{\left((x - \xi_5^k)^2 + (y - \eta_5^k)^2\right) \left((x - \xi_6^k)^2 + (y - \eta_6^k)^2\right)}} \quad (21)$$

with ξ_n^k representing the coordinates of singularities

$$\xi_n^k = \xi_n^0 + 4ak, \quad k = \pm 1, \pm 2, \dots$$

The form in (21) reads, in terms of the variables introduced in (18), as

$$G(\alpha, \beta, \gamma, \theta) = \frac{1}{4\pi} \ln \prod_{k=-\infty}^{\infty} \left(\frac{\left((\gamma + \pi(1 - 2k))^2 + \alpha^2\right) \left((\theta - 2\pi k)^2 + \alpha^2\right)}{\left((\gamma + \pi(1 - 2k))^2 + \beta^2\right) \left((\theta - 2\pi k)^2 + \beta^2\right)} \times \frac{\left((\theta - \pi(1 + 2k))^2 + \beta^2\right) \left((\gamma - 2\pi k)^2 + \beta^2\right)}{\left((\theta - \pi(1 + 2k))^2 + \alpha^2\right) \left((\gamma - 2\pi k)^2 + \alpha^2\right)} \right) \quad (22)$$

Infinite Products and Elementary Functions. As it follows directly from the equivalence of the representations of (19) and (22), the identity

$$\frac{(1 + 2e^\alpha \cos \gamma + e^{2\alpha}) (1 - 2e^\beta \cos \gamma + e^{2\beta})}{(1 - 2e^\alpha \cos \gamma + e^{2\alpha}) (1 + 2e^\beta \cos \gamma + e^{2\beta})} \times \frac{(1 + 2e^\beta \cos \theta + e^{2\beta}) (1 - 2e^\alpha \cos \theta + e^{2\alpha})}{(1 - 2e^\beta \cos \theta + e^{2\beta}) (1 + 2e^\alpha \cos \theta + e^{2\alpha})}$$

$$= \prod_{k=-\infty}^{\infty} \left(\frac{\left((\gamma + \pi(1 - 2k))^2 + \alpha^2\right) \left((\theta - 2\pi k)^2 + \alpha^2\right)}{\left((\gamma + \pi(1 - 2k))^2 + \beta^2\right) \left((\theta - 2\pi k)^2 + \beta^2\right)} \times \frac{\left((\theta - \pi(1 + 2k))^2 + \beta^2\right) \left((\gamma - 2\pi k)^2 + \beta^2\right)}{\left((\theta - \pi(1 + 2k))^2 + \alpha^2\right) \left((\gamma - 2\pi k)^2 + \alpha^2\right)} \right) \quad (23)$$

is valid at the region of analiticity of the function in (19).

The above identity involves four arbitrary parameters, and by a suitable choice of those some interesting representations of elementary functions can be obtained. To start with, we write its left-hand side in the form

$$\frac{(\cos \gamma + A_2)(\cos \gamma - A_1)(\cos \theta + A_1)(\cos \theta - A_2)}{(\cos \gamma - A_2)(\cos \gamma + A_1)(\cos \theta - A_1)(\cos \theta + A_2)}, \quad (24)$$

where

$$A_1 = \frac{1 + e^{2\beta}}{2e^\beta}, \quad A_2 = \frac{1 + e^{2\alpha}}{2e^\alpha}$$

This shows that it cannot be converted to an expression consisting of a single trigonometric function for any real value of the parameters $\alpha, \beta, \gamma, \theta$. It's also obvious from (24), that to avoid trivial results we should put some constraints on these parameters, specifically $\alpha \neq \pm\beta, \gamma \neq \pm\theta$.

It is worth noting that the trigonometric terms in the expression (24) depend only on the parameters γ and θ , and their successful choice allows that expression to be dependent only on the exponential terms.

As an illustration, let us consider the following parameter values: $\gamma = 0, \theta = \frac{\pi}{2}, \beta = 2u, \alpha = 2v$, which converts the expression in (23) into

$$\frac{\tanh^2 u}{\tanh^2 v} = \prod_{k=-\infty}^{\infty} \frac{(\pi^2(1+2k)^2 + 4v^2)(\pi^2k^2 + u^2)}{(\pi^2(1+2k)^2 + 4u^2)(\pi^2k^2 + v^2)} \quad (25)$$

Note that the above representation has already been derived in [6] and [7] upon considering a different problem.

If the parameter v in (25) is taken to infinity then we arrive at

$$\tanh^2 u = \prod_{k=-\infty}^{\infty} \frac{4(\pi^2k^2 + u^2)}{\pi^2(1+2k)^2 + 4u^2} \quad (26)$$

When the parameter u is taken to infinity, the following expansion arises

$$\coth^2 v = \prod_{k=-\infty}^{\infty} \frac{\pi^2(1+2k)^2 + 4v^2}{4(\pi^2k^2 + v^2)} \quad (27)$$

Relations in (26) and (27) have already been derived in [7] and their convergence has already been investigated.

Recalling the interrelation between hyperbolic and trigonometric functions

$$\tanh(x) = -i \tan(ix)$$

$$\coth(x) = i \cot(ix)$$

R_{ef.}

7. Yuri A. Melnikov, Green's functions and Infinite Products: Bridging the Divide, Birkhauser-Springer, New York - London, 2011.

the new infinite product representations

$$\tan^2 u = \prod_{k=-\infty}^{\infty} \frac{4(\pi^2 k^2 - u^2)}{4u^2 - \pi^2(1 + 2k)^2} \quad (28)$$

$$\cot^2 v = \prod_{k=-\infty}^{\infty} \frac{4v^2 - \pi^2(1 + 2k)^2}{4(\pi^2 k^2 - v^2)} \quad (29)$$

can be obtained.

To prove the convergence of the identity in (28), we rewrite it in the form

$$\begin{aligned} \tan^2 u &= \frac{4u^2}{4u^2 - \pi^2} \prod_{k=1}^{\infty} \frac{16(\pi^2 k^2 - u^2)^2}{(4u^2 - \pi^2(1 + 2k)^2)(4u^2 - \pi^2(1 - 2k)^2)} \\ &= \frac{4u^2}{4u^2 - \pi^2} \prod_{k=1}^{\infty} \left(1 + \pi^2 \frac{\pi^2(8k^2 - 1) + 8u^2}{(4u^2 - \pi^2(1 + 2k)^2)(4u^2 - \pi^2(1 - 2k)^2)} \right) \end{aligned} \quad (30)$$

Since the numerator in the second additive component in the product is a second degree polynomial in k , while the denominator is a polynomial of degree four the expansion in (30) converges uniformly for all $u \neq (1 \pm 2n)\pi/2$ at the rate of $1/k^2$.

A similar procedure shows the uniform convergence of the representation in (29)

$$\cot^2 v = \frac{\pi^2 - 4v^2}{4v^2} \prod_{k=1}^{\infty} \left(1 - \frac{\pi^2}{16} \left[\frac{\pi^2(8k^2 + 1) + 8v^2}{(\pi^2 k^2 - v^2)^2} \right] \right)$$

for all $v \neq \pm\pi n$ at the rate of $1/k^2$.

The representations in (28) and (29) deliver

$$\tan u = \pm \prod_{k=-\infty}^{\infty} \sqrt{\frac{4(\pi^2 k^2 - u^2)}{4u^2 - \pi^2(1 + 2k)^2}} \quad (31)$$

where the minus sign is valid for $u \in (-\frac{\pi}{2}; 0)$ while the plus sign holds for $u \in [0; \frac{\pi}{2})$.

$$\cot v = \pm \prod_{k=-\infty}^{\infty} \sqrt{\frac{4v^2 - \pi^2(1 + 2k)^2}{4(\pi^2 k^2 - v^2)}} \quad (32)$$

where the plus sign is valid for $u \in (0; \frac{\pi}{2})$ while the minus corresponds to $u \in [\frac{\pi}{2}; \pi)$.

Figure 3 provides a sense of the convergence of the representations in (31).

Two other expansions can be obtained using the standard identities

$$\sec u = \pm \sqrt{1 + \tan^2 u} = \pm \sqrt{1 + \prod_{k=-\infty}^{\infty} \frac{4(\pi^2 k^2 - u^2)}{4u^2 - \pi^2(1 + 2k)^2}} \quad (33)$$

where the plus sign is valid for $u \in [-\frac{\pi}{2}; \frac{\pi}{2})$, while the minus sign stays for $u \in [\frac{\pi}{2}; \frac{3\pi}{2}]$, and

$$\csc v = \pm \sqrt{1 + \cot^2 v} = \pm \sqrt{1 + \prod_{k=-\infty}^{\infty} \frac{4v^2 - \pi^2(1 + 2k)^2}{4(\pi^2 k^2 - v^2)}} \quad (34)$$

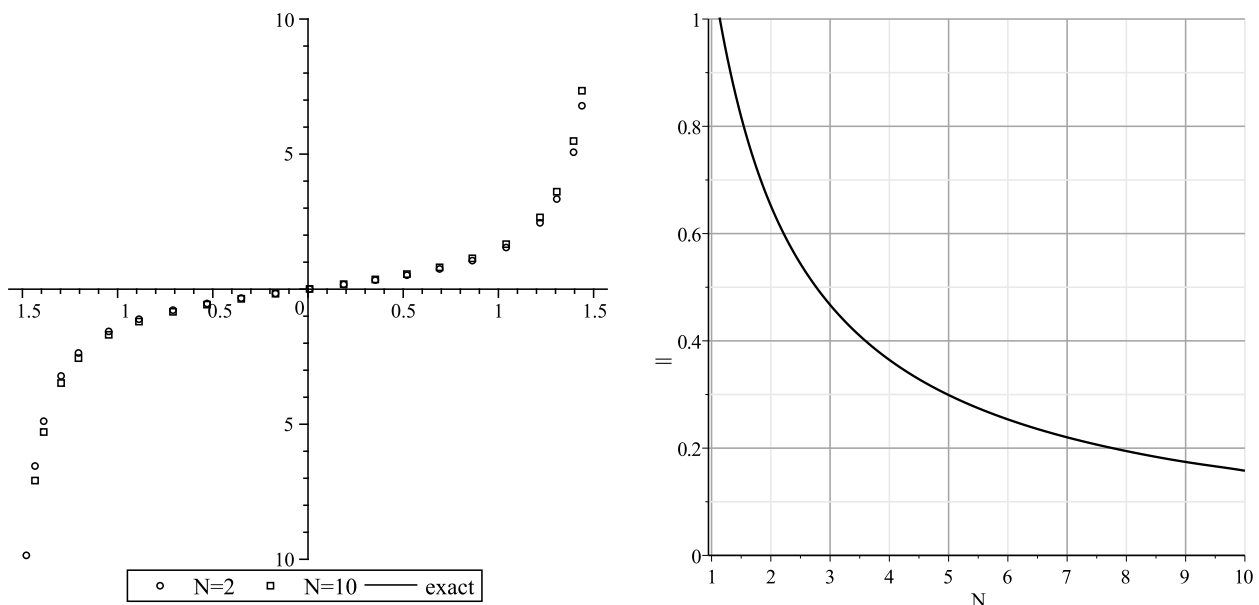
where the plus sign stays for $u \in [0; \pi)$ and the minus sign stays for $u \in [\pi; 2\pi]$.

The convergence rate of the representation in (33) is illustrated in Figure 4.

Similarly to the work done for the problem stated in (6)-(7), we will consider another mixed boundary-value problem

$$u(0, y) = \frac{\partial u(a, y)}{\partial x} = 0 \quad (35)$$

$$\frac{\partial u(x, 0)}{\partial x} = 0, \quad \lim_{y \rightarrow \infty} u(x, y) < \infty \quad (36)$$



(a) Convergence with the 2nd and 10th partial products.

(b) L^2 -norm of the error in $[-0.95\frac{\pi}{2}, 0.95\frac{\pi}{2}]$; N is a number of partial products

Figure 3 : Convergence of the representation in (31)

posed for the two-dimensional Laplace equation in the semi-infinite strip region $\Omega = \{0 < x < a, 0 < y < \infty\}$.

Method of eigenfunction expansion. Upon implementing the routine applied to the problem in (6)-(7) and using the same sequence of eigenfunctions, one obtains the Green's function for the problem in (35)-(36) in the form

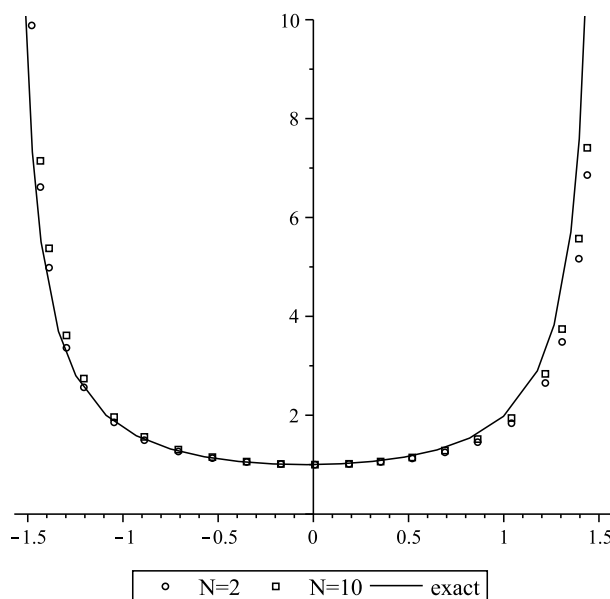
$$G(x, y; \xi, \eta) = \frac{2}{a} \sum_{k=1}^{\infty} g_k(y; \eta) \sin \lambda_k \xi \sin \lambda_k x$$

$$g_k(y; \eta) = \frac{1}{2\lambda_k} \begin{cases} e^{\lambda_k(y-\eta)} + e^{-\lambda_k(y+\eta)}, & y \leq \eta \\ e^{\lambda_k(\eta-y)} + e^{-\lambda_k(y+\eta)}, & y \geq \eta \end{cases}$$

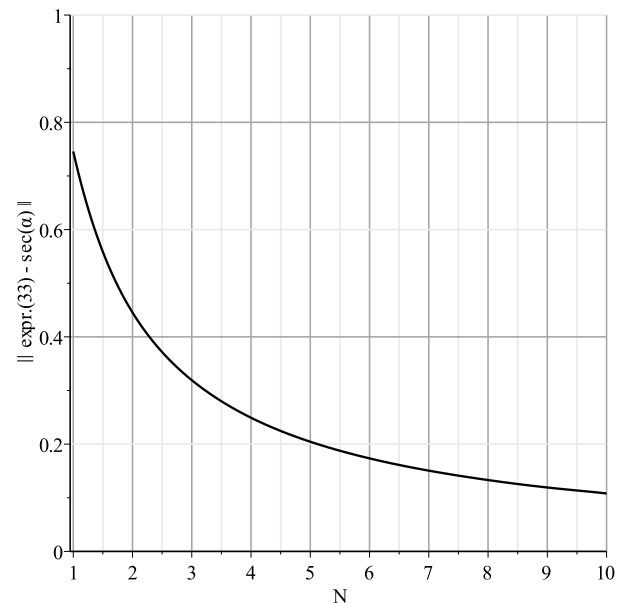
which reduces to

$$G(\alpha, \beta, \gamma, \theta) = \frac{1}{4\pi} \ln \frac{(1 + 2e^\alpha \cos \gamma + e^{2\alpha})(1 + 2e^\beta \cos \gamma + e^{2\beta})}{(1 - 2e^\alpha \cos \gamma + e^{2\alpha})(1 - 2e^\beta \cos \gamma + e^{2\beta})} \times \frac{(1 - 2e^\alpha \cos \theta + e^{2\alpha})(1 - 2e^\beta \cos \theta + e^{2\beta})}{(1 + 2e^\alpha \cos \theta + e^{2\alpha})(1 + 2e^\beta \cos \theta + e^{2\beta})} \quad (37)$$

where we use the variables introduced earlier in (18).



(a) Convergence with the 2nd and 10th partial products.



(b) L^2 -norm of the error in $[-0.95\frac{\pi}{2}, 0.95\frac{\pi}{2}]$; N is a number of partial products

Figure 4 : Convergence of the representation in (33)

Method of images. Distribution of singularities used in the method of images for the problem in (35)-(36) is shown in Figure 5. This gives rise to the infinite product version of the Green's function, written in terms of the variables introduced in (18), as

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \prod_{k=-\infty}^{\infty} \frac{((\gamma + \pi(1 - 2k))^2 + \alpha^2)((\theta - 2\pi k)^2 + \alpha^2)}{((\theta - \pi(1 + 2k))^2 + \beta^2)((\gamma - 2\pi k)^2 + \beta^2)} \frac{((\gamma + \pi(1 - 2k))^2 + \beta^2)((\theta - 2\pi k)^2 + \beta^2)}{((\theta - \pi(1 + 2k))^2 + \alpha^2)((\gamma - 2\pi k)^2 + \alpha^2)}$$

which being compared with (37) constitutes another identity valid for $\alpha \neq \pm\beta$, $\gamma \neq \pm\theta$.

Assuming $\gamma = 0$, $\theta = \frac{\pi}{2}$, $\beta = 2u$, $\alpha = 2v$, one arrives at

$$\coth^2 u \coth^2 v = \prod_{k=-\infty}^{\infty} \frac{(\pi^2(1 + 2k)^2 + 4v^2)(\pi^2(1 + 2k)^2 + 4u^2)}{(\pi^2 k^2 + u^2)(\pi^2 k^2 + v^2)}$$

Note that it has already been obtained earlier (see (25)) in this presentation.

b) Exterior of Circles

The key idea of the developments in the previous sections is to obtain an alternative to the classical expression for a Green's function as an appropriate arrangement of sinks and sources. For some regions such arrangements are periodic and described by infinite sequences of symmetrically placed images.

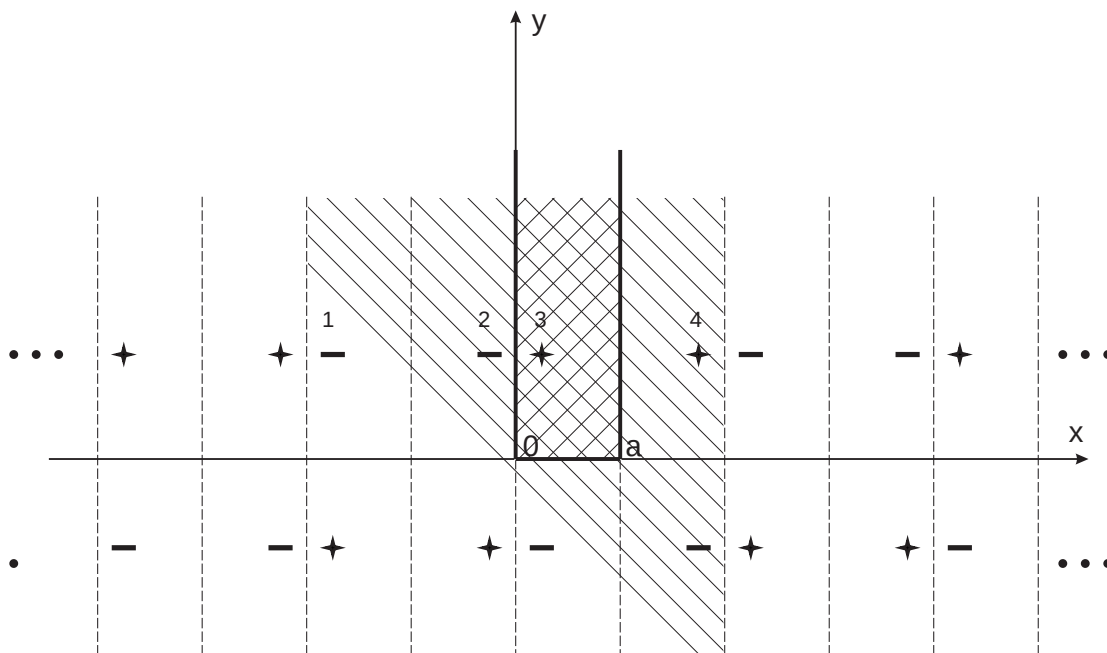


Figure 5: Sequence of sources and sinks that arises in the method of images for the problem in (35)-(36)

This approach was applied, for instance, in [8] to construct the potential due to a line charge in the infinite strip or in [2, 8] to construct the potential induced by a single line charge in the rectangular prism or in the circular ring. The same approach was used in [6, 7] to find a potential field induced by the Green's function to the boundary-value problem posed for the Laplace equation in infinite and semi-infinite strip region with different combinations of boundary conditions. Results obtained in the earlier works served as a basis for construction of infinite product representations of elementary functions. The present work is the logical continuation of [6, 7], and the expansions (31) - (32) and (33) - (34) complement the results derived there.

Working on further developments of the discussed approach, we have probably to turn to different geometries or different equations. One of such cases is described below.

Conformal mapping. Conformal mapping gives us a tool to derive Green's functions of the Dirichlet problem posed for the Laplace equation in a simply-connected region. If there is known a function $w(z, \zeta)$ which conformally maps the given region in the z -plane onto the interior of the unit disc in the w -plane with point ζ mapped on the center of the disc, the corresponding Green's function is presented ([1, 2, 7]) in terms of $w(z, \zeta)$ as

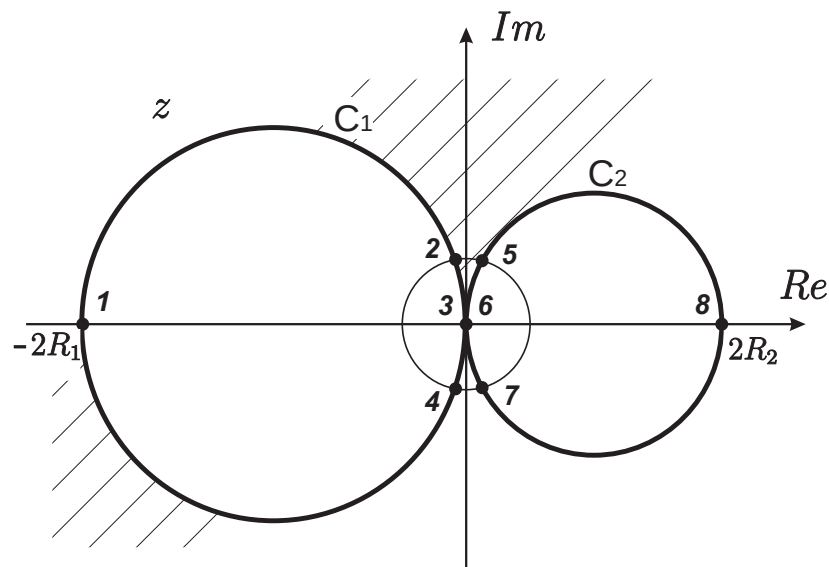
$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln |\omega(z, \zeta)| \quad (38)$$

where

$$z = x + iy$$

$$\zeta = \xi + i\eta$$

As an example, consider the exterior of two circles having external contact (Figure 6). Function $f(z) = 1/z$ transforms this region into the infinite strip $([1, 2])$ while the latter can be mapped onto the interior of a unit circle. Omitting cumbersome but trivial algebra, we just present the function that maps the exterior of two circles with external contact onto the interior of a unit circle as



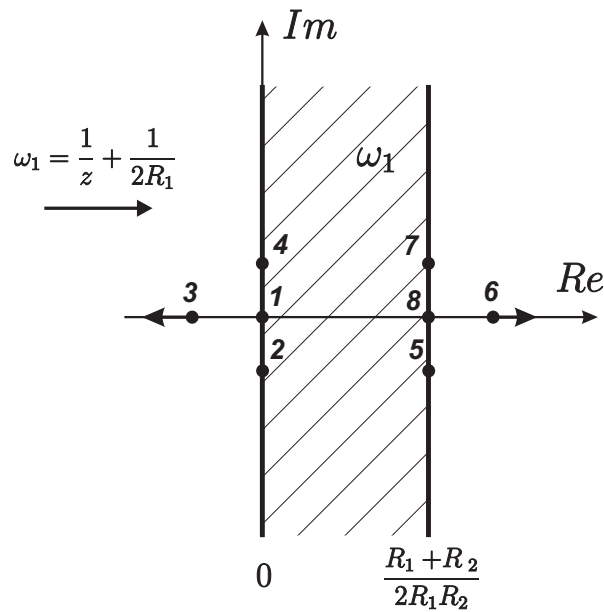


Figure 6 : Conformal mapping of the exterior of two circles with external contact onto the infinite strip

$$\omega(z, \zeta) = \frac{\exp\left(i\lambda\left(\frac{1}{z} + \frac{1}{2R_1}\right)\right) - \exp\left(i\lambda\left(\frac{1}{\zeta} + \frac{1}{2R_1}\right)\right)}{\exp\left(i\lambda\left(\frac{1}{z} + \frac{1}{2R_1}\right)\right) - \exp\left(i\lambda\left(\frac{1}{\zeta} + \frac{1}{2R_1}\right)\right)}$$

where

$$\lambda = 2\pi \frac{R_1 R_2}{R_1 + R_2}$$

The modulus of the above reads as

$$|\omega(z, \zeta)| = \sqrt{\frac{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta} \cos \lambda(\alpha - \beta)}{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta} \cos \lambda\left(\alpha + \beta + \frac{1}{R_1}\right)}}$$

where

$$\alpha = \frac{x}{x^2 + y^2}, \quad \beta = \frac{\xi}{\xi^2 + \eta^2}, \quad \gamma = \frac{y}{x^2 + y^2}, \quad \theta = \frac{\eta}{\xi^2 + \eta^2} \quad (39)$$

Hence, in light of (38), the Green's function to the Dirichlet problem for the region depicted in Figure 6 appears as

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta} \cos \left(\lambda \left[\alpha + \beta + \frac{1}{R_1} \right] \right)}{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta} \cos (\lambda(\alpha - \beta))} \quad (40)$$

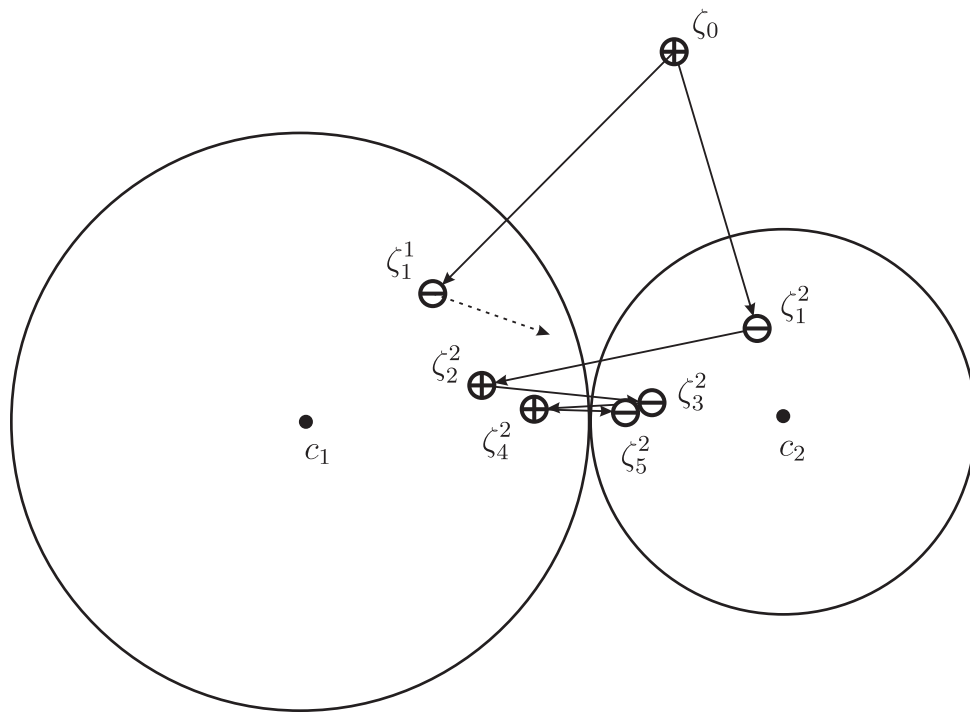


Figure 7: Sequence of sources and sinks that arises in the method of images

Method of images. In the case when boundaries of the region are formed by straight lines, images are placed symmetrically with respect to boundaries in a straightforward manner. But every straight line can be considered as a circumference with infinite radius. Thus the method of images can also be generalized to regions formed by circular arcs. In order to implement this idea, recall that an inversion with respect to a circle with radius R centered at a point c is given by the formula

$$z_1 = f(z_0) = \frac{R^2}{\overline{z_0 - c}} + c \quad (41)$$

where the points c, z_0, z_1 lie in the same plane.

Thus the Green's function to the Dirichlet problem posed for the Laplace equation in the exterior of a circle is given by the expression

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln \frac{|z - z_0|}{|z - z_1|} + \mu(x, y; \xi, \eta) \quad (42)$$

The compensatory function $\mu(x, y; \xi, \eta)$ can easily be derived from the fact that the argument of the logarithm in (42) has a constant value on the circumference of a circle and is equal to $\frac{|z_0 - c|}{R}$.

Thus the formula (42) converts to

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{|z - z_1||z_0 - c|}{|z - z_0|R}$$

In the case of two circles, each inversion of the form in (41) with respect to the first circle perturbs the potential on the circumference of the second circle and thus, should be compensated by another inversion with respect to the perturbed circle and so on. Consequently to satisfy the Dirichlet boundary conditions we should use the infinite sequence of inversions (Figure 7). This generates the potential field

$$F(z, \zeta_n^j) = \frac{1}{2\pi} \left(\ln \frac{|\zeta_1^1 - z| |\zeta_0^1 - c_1|}{|\zeta_0^1 - z| R_1} + \ln \frac{|\zeta_1^2 - z| |\zeta_0^2 - c_2|}{|\zeta_0^2 - z| R_2} \right. \\ \left. - \ln \frac{|\zeta_2^1 - z| |\zeta_1^1 - c_2|}{|\zeta_1^1 - z| R_2} - \ln \frac{|\zeta_2^2 - z| |\zeta_1^2 - c_1|}{|\zeta_1^2 - z| R_1} \right. \\ \left. + \ln \frac{|\zeta_3^1 - z| |\zeta_2^1 - c_1|}{|\zeta_2^1 - z| R_1} + \ln \frac{|\zeta_3^2 - z| |\zeta_2^2 - c_2|}{|\zeta_2^2 - z| R_2} - \dots \right)$$

Clearly, all the singularities appear twice in the above expression and hence the Green's function has the following infinite product form

$$G(x, y; \xi, \eta) = \frac{1}{2} F(z; \zeta_n^j) \\ = \frac{1}{4\pi} \ln \prod_{k=0}^{\infty} \frac{|\zeta_{2k+1}^1 - z|^2 |\zeta_{2k+1}^2 - z|^2 |\zeta_{2k}^1 - c_1| |\zeta_{2k}^2 - c_2|}{|\zeta_{2k}^1 - z| |\zeta_{2k}^2 - z| |\zeta_{2k+2}^1 - z| |\zeta_{2k+2}^2 - z| |\zeta_{2k+1}^1 - c_2| |\zeta_{2k+1}^2 - c_1|} \quad (43)$$

where $z = x + iy$ is a field point, c_1, c_2, R_1, R_2 are centres and radii of the circles, $\zeta_0^1 = \zeta_0^2 = z_0 = \xi + i\eta$ is a source point and ζ_k^j are points at which singularities are placed

$$\zeta_k^1 = \begin{cases} \frac{R_2^2}{\overline{\zeta_{k-1}^1} - c_2} + c_2, & \text{if } k \text{ is even} \\ \frac{R_1^2}{\overline{\zeta_{k-1}^1} - c_1} + c_1, & \text{if } k \text{ is odd} \end{cases} \\ \zeta_k^2 = \begin{cases} \frac{R_1^2}{\overline{\zeta_{k-1}^2} - c_1} + c_1, & \text{if } k \text{ is even} \\ \frac{R_2^2}{\overline{\zeta_{k-1}^2} - c_2} + c_2, & \text{if } k \text{ is odd} \end{cases} \quad (44)$$

In our case, $c_1 = -R_1, c_2 = R_2$ and the expressions in (44) transform to

$$\zeta_k^1 = \begin{cases} \frac{z_0}{\frac{k}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) z_0 + 1}, & \text{if } k \text{ is even} \\ \frac{-\bar{z}_0}{\frac{1}{2} \left(\frac{k+1}{R_1} + \frac{k-1}{R_2} \right) \bar{z}_0 + 1}, & \text{if } k \text{ is odd} \end{cases}$$

$$\zeta_k^2 = \begin{cases} \frac{-z_0}{\frac{k}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) z_0 - 1}, & \text{if } k \text{ is even} \\ \frac{\bar{z}_0}{\frac{1}{2} \left(\frac{k-1}{R_1} + \frac{k+1}{R_2} \right) \bar{z}_0 - 1}, & \text{if } k \text{ is odd} \end{cases}$$

When k approaches infinity both values ζ_k^1 and ζ_k^2 tend to zero, implying that the limit of the general term in (43) equals unity. That is

$$\lim_{k \rightarrow \infty} \left(\frac{|\zeta_{2k+1}^1 - z|^2 |\zeta_{2k+1}^2 - z|^2 |\zeta_{2k}^1 + R_1| |\zeta_{2k}^2 - R_2|}{|\zeta_{2k}^1 - z| |\zeta_{2k}^2 - z| |\zeta_{2k+2}^1 - z| |\zeta_{2k+2}^2 - z| |\zeta_{2k+1}^1 - R_2| |\zeta_{2k+1}^2 + R_1|} \right) = 1$$

and thus the infinite product in (43) is absolutely and uniformly convergent.

Infinite Products and Elementary Functions. Obviously, in (40) and (43) one finds just alternative representations of the same Green's function, and we arrive at the multivariable identity

$$\frac{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta} \cos \left(\lambda \left[\alpha + \beta + \frac{1}{R_1} \right] \right)}{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta} \cos (\lambda(\alpha - \beta))} = \prod_{k=0}^{\infty} \frac{|\zeta_{2k+1}^1 - z|^2 |\zeta_{2k+1}^2 - z|^2 |\zeta_{2k}^1 + R_1| |\zeta_{2k}^2 - R_2|}{|\zeta_{2k}^1 - z| |\zeta_{2k}^2 - z| |\zeta_{2k+2}^1 - z| |\zeta_{2k+2}^2 - z| |\zeta_{2k+1}^1 - R_2| |\zeta_{2k+1}^2 + R_1|} \quad (45)$$

Complex variables z and z_0 can be written in terms of real parameters $\alpha, \beta, \gamma, \theta$, which were introduced earlier in (39), as

$$z_0 = \xi + i\eta = \frac{\beta + i\theta}{\beta^2 + \theta^2}$$

$$z = x + iy = \frac{\alpha + i\gamma}{\beta^2 + \theta^2}$$

Thus, the expression in (45) has six arbitrary parameters: $\alpha, \beta, \gamma, \theta, R_1$ and R_2 . If $\lambda = 1$, then $R_2 = R_1/(2\pi R_1 - 1)$ and the number of independent parameters reduces to five.

Trigonometric functions. For any fixed value of the parameters γ and θ the expression in (45) involves only trigonometric functions. Particularly, when $\gamma = 0$ and $\theta = 0$ the left-hand side of (45) represents the following elementary function

$$\frac{1 - \cos \left(\alpha + \beta + \frac{1}{R_1} \right)}{1 - \cos(\alpha - \beta)} \quad (46)$$

If $R_1 = \frac{1}{n\pi} \left(R_2 = \frac{1}{\pi(2-n)}, n = 0, 1, \dots \right)$ the above expression converts to

$$\frac{1 - (-1)^n \cos(\alpha + \beta)}{1 - \cos(\alpha - \beta)} \quad (47)$$

Similarly, when $R_1 = \frac{2}{\pi(1+2n)} \left(R_2 = \frac{2}{\pi(3-2n)}, n = 0, 1, \dots \right)$, the function (46) converts to

$$\frac{1 + (-1)^n \sin(\alpha + \beta)}{1 - \cos(\alpha - \beta)}$$

So the functions (47) and (48) have the following infinite product representations

$$\frac{1 - (-1)^n \cos(\alpha + \beta)}{1 - \cos(\alpha - \beta)} = \prod_{k=0}^{\infty} \frac{[\alpha + \beta + \pi(2k + n)]^2 [\alpha + \beta - \pi(2(k + 1) - n)]^2}{|(\alpha - \beta)^2 - 4\pi^2 k^2| |(\alpha - \beta)^2 - 4\pi^2 (k + 1)^2|} \quad (49)$$

and

$$\frac{1 + (-1)^n \sin(\alpha + \beta)}{1 - \cos(\alpha - \beta)} = \prod_{k=0}^{\infty} \frac{[2(\alpha + \beta) + \pi(1 + 4k + 2n)]^2 [2(\alpha + \beta) - \pi(3 + 4k - 2n)]^2}{16 |(\alpha - \beta)^2 - 4\pi^2 k^2| |(\alpha - \beta)^2 - 4\pi^2 (k + 1)^2|} \quad (50)$$

Different interrelations between the parameters α and β are possible. For instance, when $\beta = \alpha \pm (2p + 1)\pi$ ($p = 0, 1, 2, \dots$), the expressions in (49) and (50) read

$$\frac{1 + (-1)^n \cos(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{[(2\alpha + \pi(2p + n))]^2 - \pi^2(2k + 1)^2}{\pi^4 |(2p + 1)^2 - 4k^2| |(2p + 1)^2 - 4(k + 1)^2|}, \quad (51)$$

$$\frac{1 + (-1)^n \cos(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{[(2\alpha - \pi(2(p + 1) - n))]^2 - \pi^2(2k + 1)^2}{\pi^4 |(2p + 1)^2 - 4k^2| |(2p + 1)^2 - 4(k + 1)^2|} \quad (52)$$

and

$$\frac{1 - (-1)^n \sin(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{[(4\alpha + \pi(2(2p + n) + 1))]^2 - 4\pi^2(2k + 1)^2}{16\pi^4 |(2p + 1)^2 - 4k^2| |(2p + 1)^2 - 4(k + 1)^2|},$$

$$\frac{1 - (-1)^n \sin(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{[(4\alpha - \pi(2(2p - n) + 3))]^2 - 4\pi^2(2k + 1)^2}{16\pi^4 |(2p + 1)^2 - 4k^2| |(2p + 1)^2 - 4(k + 1)^2|}$$

which, in turn, provide us with infinite product representations of the sine and cosine functions

$$\cos \alpha = (-1)^n \left(-1 + 2 \prod_{k=0}^{\infty} \frac{\left[\left(\alpha + \pi(2p+n) \right)^2 - \pi^2(2k+1)^2 \right]^2}{\pi^4 \left| (2p+1)^2 - 4k^2 \right| \left| (2p+1)^2 - 4(k+1)^2 \right|} \right) \quad (53)$$

$$\cos \alpha = (-1)^n \left(-1 + 2 \prod_{k=0}^{\infty} \frac{\left[\left(\alpha - \pi(2(p+1)-n) \right)^2 - \pi^2(2k+1)^2 \right]^2}{\pi^4 \left| (2p+1)^2 - 4k^2 \right| \left| (2p+1)^2 - 4(k+1)^2 \right|} \right) \quad (54)$$

and

$$\sin \alpha = (-1)^n \left(1 - 2 \prod_{k=0}^{\infty} \frac{\left[\left(2\alpha + \pi(2(2p+n)+1) \right)^2 - 4\pi^2(2k+1)^2 \right]^2}{16\pi^4 \left| (2p+1)^2 - 4k^2 \right| \left| (2p+1)^2 - 4(k+1)^2 \right|} \right) \quad (55)$$

$$\sin \alpha = (-1)^n \left(1 - 2 \prod_{k=0}^{\infty} \frac{\left[\left(2\alpha - \pi(2(2p-n)+3) \right)^2 - 4\pi^2(2k+1)^2 \right]^2}{16\pi^4 \left| (2p+1)^2 - 4k^2 \right| \left| (2p+1)^2 - 4(k+1)^2 \right|} \right) \quad (56)$$

where $n, p = 0, 1, 2, \dots$. For example, in the case of $n = 1$ and $p = 0$ the identities in (53)-(56) reduce to

$$\cos \alpha = 1 + 2 \prod_{k=0}^{\infty} \frac{\left((\alpha + \pi)^2 - \pi^2(2k+1)^2 \right)^2}{\pi^4 (4k^2 - 1) (4(k+1)^2 - 1)} \quad (57)$$

$$\cos \alpha = 1 + 2 \prod_{k=0}^{\infty} \frac{\left((\alpha - \pi)^2 - \pi^2(2k+1)^2 \right)^2}{\pi^4 (4k^2 - 1) (4(k+1)^2 - 1)} \quad (58)$$

and

$$\sin \alpha = -1 - 2 \prod_{k=0}^{\infty} \frac{\left((2\alpha + 3\pi)^2 - 4\pi^2(k+1)^2 \right)^2}{16\pi^4 (4k^2 - 1) (4(k+1)^2 - 1)} \quad (59)$$

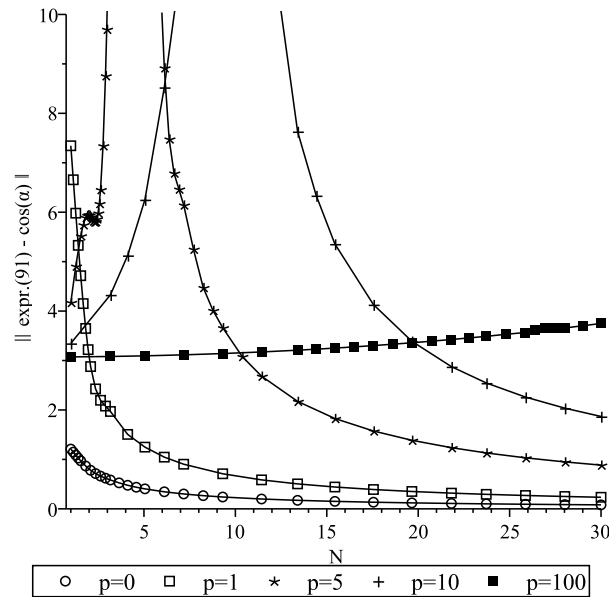
$$\sin \alpha = -1 - 2 \prod_{k=0}^{\infty} \frac{\left((2\alpha - \pi)^2 - 4\pi^2(k+1)^2 \right)^2}{16\pi^4 (4k^2 - 1) (4(k+1)^2 - 1)} \quad (60)$$

Figures 8 and 9 illustrate the sensitivity of the convergence of the representation in (53) to the choice of the parameters n and p . It is seen that the parameter values have the dramatical influence on the rate of convergence. The identities in (54)-(56) exhibit a similar behaviour.

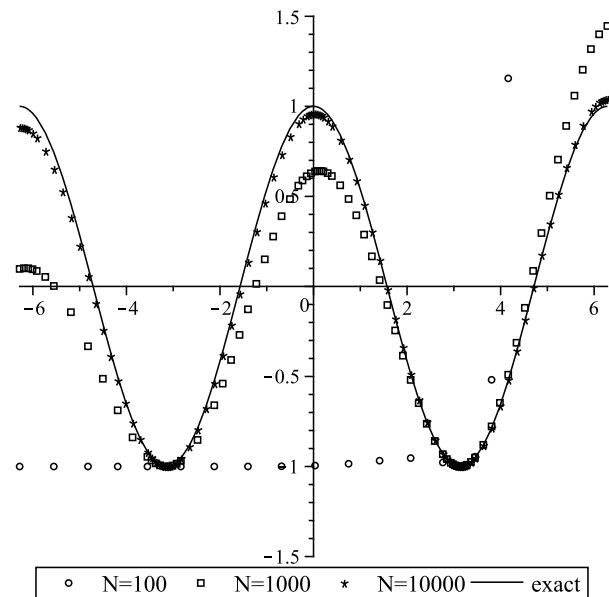
By applying the power reduction formula to the expressions in (51) and (52) several new infinite product expansions for the sine and cosine functions can be obtained.

If n in (51) and (52) is an even number, i.e. $n = 2n$, then the expansions of the cosine function can be written as

$$\cos \alpha = \pm \prod_{k=0}^{\infty} \frac{\left| 4 \left(\alpha + \pi(p+n) \right)^2 - \pi^2 (2k+1)^2 \right|}{\pi^2 \sqrt{\left| (2p+1)^2 - 4k^2 \right| \left| (2p+1)^2 - 4(k+1)^2 \right|}} \quad (61)$$

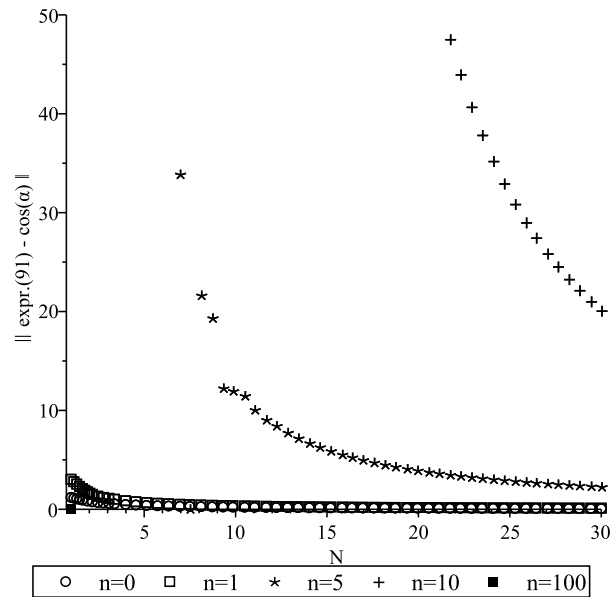


(a) L^2 -norm of the error in $[0, 2\pi]$; N is a number of partial products.

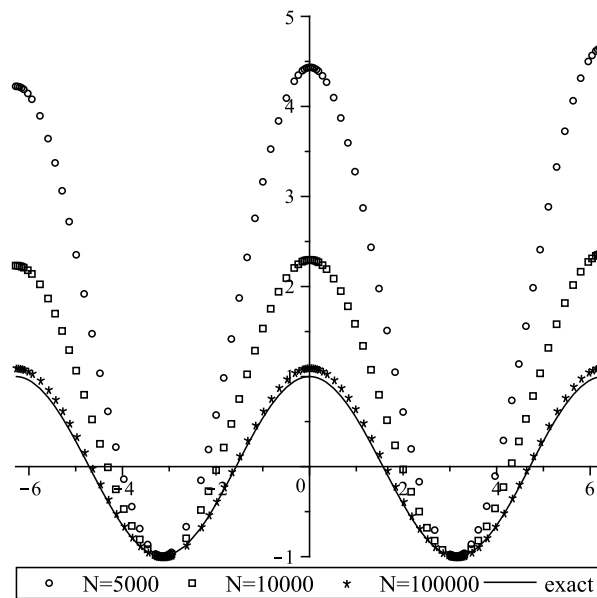


(b) Convergence of the infinite product expansion when $n = 0$, $p = 100$

Figure 8: Convergence of the representation in (53) for different values of the parameter p ; parameter $n = 0$ is fixed



(a) L^2 -norm of the error in $[0, 2\pi]$; N is a number of partial products



(b) Convergence of the infinite product expansion when $p=0$, $n=100$

Figure 9: Convergence of the representation in (53) for different values of the parameter n ; parameter $p=0$ is fixed

$$\cos \alpha = \pm \prod_{k=0}^{\infty} \frac{\left| 4(\alpha - \pi(p - n + 1))^2 - \pi^2(2k + 1)^2 \right|}{\pi^2 \sqrt{\left| (2p + 1)^2 - 4k^2 \right| \left| (2p + 1)^2 - 4(k + 1)^2 \right|}} \quad (62)$$

where the plus sign corresponds to $\alpha \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right)$ and the minus sign corresponds to $\alpha \in \left[\frac{\pi}{2}; \frac{3\pi}{2}\right]$.

Similarly, when n in (51) and (52) is an odd number, i.e. $n = 2n + 1$, then the expansion of the sine function can be written as

$$\sin \alpha = \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha + \pi(2(p+n)+1) \right)^2 - \pi^2(2k+1)^2 \right|}{\pi^2 \sqrt{\left| (2p+1)^2 - 4k^2 \right|} \left| (2p+1)^2 - 4(k+1)^2 \right|}} \quad (63)$$

$$\sin \alpha = \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha - \pi(2(p-n)+1) \right)^2 - \pi^2(2k+1)^2 \right|}{\pi^2 \sqrt{\left| (2p+1)^2 - 4k^2 \right|} \left| (2p+1)^2 - 4(k+1)^2 \right|}} \quad (64)$$

where the plus sign corresponds to $\alpha \in [0; \pi)$ and the minus sign corresponds to $\alpha \in [\pi; 2\pi]$.

The infinite product representations of the tangent function can also be obtained from the expansions in (61)-(64). Those are

$$\tan \alpha = \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha + \pi(2(p+n)+1) \right)^2 - \pi^2(2k+1)^2 \right|}{\left| 4 \left(\alpha + \pi(p+n) \right)^2 - \pi^2(2k+1)^2 \right|}}$$

$$\tan \alpha = \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha + \pi(2(p+n)+1) \right)^2 - \pi^2(2k+1)^2 \right|}{\left| 4 \left(\alpha - \pi(p-n+1) \right)^2 - \pi^2(2k+1)^2 \right|}}$$

and

$$\tan \alpha = \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha - \pi(2(p-n)+1) \right)^2 - \pi^2(2k+1)^2 \right|}{\left| 4 \left(\alpha + \pi(p+n) \right)^2 - \pi^2(2k+1)^2 \right|}}$$

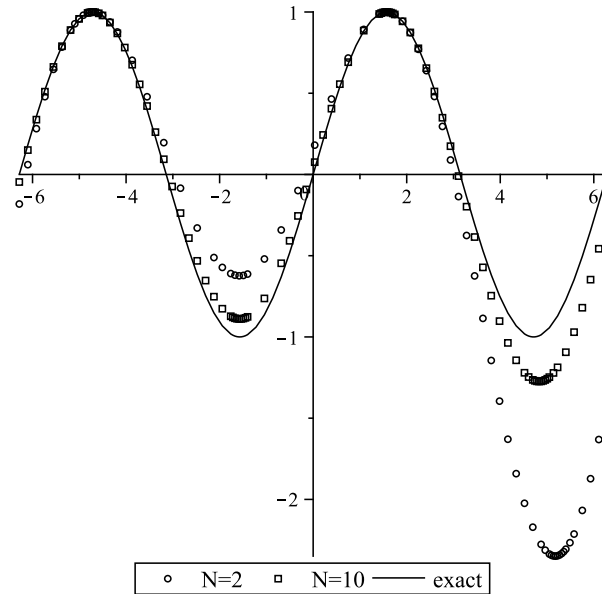
$$\tan \alpha = \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha - \pi(2(p-n)+1) \right)^2 - \pi^2(2k+1)^2 \right|}{\left| 4 \left(\alpha - \pi(p-n+1) \right)^2 - \pi^2(2k+1)^2 \right|}}$$

where the plus sign corresponds to $\alpha \in \left[-\frac{\pi}{2}; 0\right)$ and the minus sign corresponds to $\alpha \in \left[0; \frac{\pi}{2}\right]$.

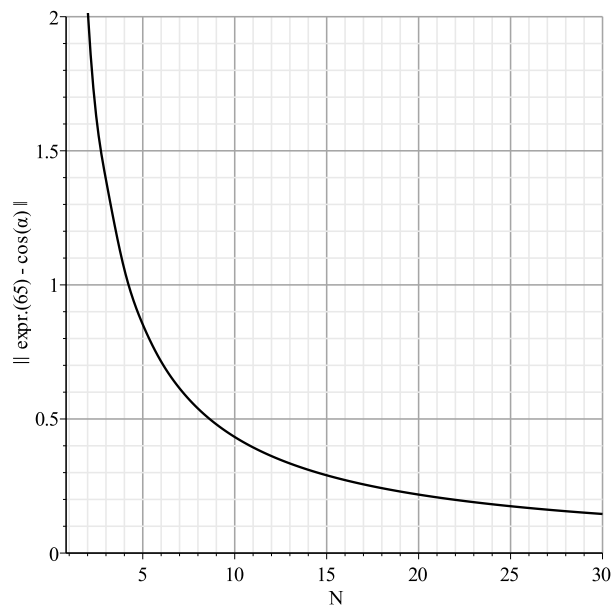
So far we assumed $\beta = \alpha \pm (2p+1)\pi$. Different representations can be derived for the relation between α and β as

$$\beta = \alpha \pm (2p+1)\frac{\pi}{2}, \quad p = 0, 1, 2, \dots$$

yielding in the following



(a) Convergence with the 2nd and 10th partial products.

(b) L_2 -norm of the error in $[0, 2\pi]$; N is a number of partial products*Figure 10* : Convergence of the representation in (65)

$$\sin \alpha = (-1)^{n+p} \left(-1 + \prod_{k=0}^{\infty} \frac{\left[\left(2\alpha + \pi(2(p+n) - 1) \right)^2 - 4\pi^2(2k+1)^2 \right]^2}{\pi^4 \left| (1+2p)^2 - 16k^2 \right| \left| (1+2p)^2 - 16(k+1)^2 \right|} \right) \quad (65)$$

$$\sin \alpha = (-1)^{n+p} \left(1 - \prod_{k=0}^{\infty} \frac{\left[\left(2\alpha - \pi(2(p-n) + 3) \right)^2 - 4\pi^2(2k+1)^2 \right]^2}{\pi^4 \left| (1+2p)^2 - 16k^2 \right| \left| (1+2p)^2 - 16(k+1)^2 \right|} \right) \quad (66)$$

and

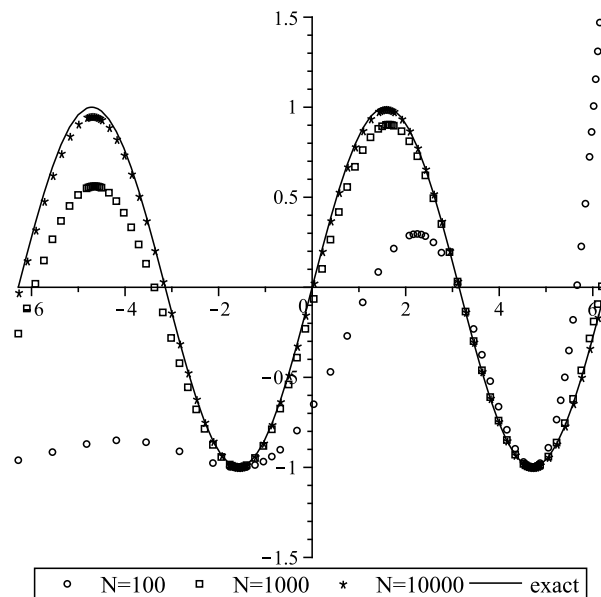
$$\cos \alpha = (-1)^{n+p} \left(-1 + \prod_{k=0}^{\infty} \frac{16 \left[\left(\alpha + \pi(p+n) \right)^2 - \pi^2(2k+1)^2 \right]^2}{\pi^4 \left| (1+2p)^2 - 16k^2 \right| \left| (1+2p)^2 - 16(k+1)^2 \right|} \right) \quad (67)$$

$$\cos \alpha = (-1)^{n+p} \left(1 - \prod_{k=0}^{\infty} \frac{16 \left[\left(\alpha - \pi(p-n+1) \right)^2 - \pi^2(2k+1)^2 \right]^2}{\pi^4 \left| (1+2p)^2 - 16k^2 \right| \left| (1+2p)^2 - 16(k+1)^2 \right|} \right) \quad (68)$$

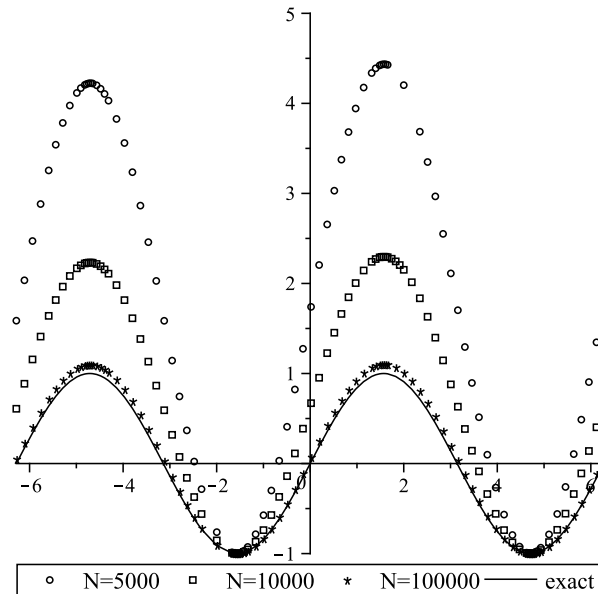
Notes

The convergence rate of the expansion in (65) with $n = 1$, $p = 0$ is illustrated in Figure 10. The sensitivity of the convergence in (65) to the choice of the parameters n and p is illustrated in Figure 11. The identities in (66)-(68) exhibit the similar behaviour.

By applying the power reduction formula to the identities in (67)-(68) one can derive alternative representations of the sine and cosine functions. When n and p are even numbers, i.e. $n = 2n$ and $p = 2p$, the expansions of the cosine and sine functions can be obtained in the form



(a) Convergence of the infinite product expansion when $p = 0$, $n = 100$; N is a number of partial products



(b) Convergence of the infinite product expansion when $p=0$, $n=100$; N is a number of partial products

Figure 11 : Convergence of the representation in (65) for different values of the parameters n and $p=0$

$$\cos \alpha = \pm \sqrt{2}/2 \prod_{k=0}^{\infty} \frac{4 \left| \left(2\alpha + 2\pi(p+n) \right)^2 - \pi^2(2k+1)^2 \right|}{\pi^2 \sqrt{\left| (1+4p)^2 - 16k^2 \right| \left| (1+4p)^2 - 16(k+1)^2 \right|}} \quad (69)$$

and

$$\sin \alpha = \pm \sqrt{2}/2 \prod_{k=0}^{\infty} \frac{4 \left| \left(2\alpha - \pi(2p-2n+1) \right)^2 - \pi^2(2k+1)^2 \right|}{\pi^2 \sqrt{\left| (1+4p)^2 - 16k^2 \right| \left| (1+4p)^2 - 16(k+1)^2 \right|}} \quad (70)$$

When n is odd and p is even, i.e. $n=2n+1$ and $p=2p$, one arrives at

$$\sin \alpha = \pm \sqrt{2}/2 \prod_{k=0}^{\infty} \frac{4 \left| \left(2\alpha + \pi(2p+2n+1) \right)^2 - \pi^2(2k+1)^2 \right|}{\pi^2 \sqrt{\left| (1+4p)^2 - 16k^2 \right| \left| (1+4p)^2 - 16(k+1)^2 \right|}} \quad (71)$$

and

$$\cos \alpha = \pm \sqrt{2}/2 \prod_{k=0}^{\infty} \frac{4 \left| \left(2\alpha - 2\pi(p-n) \right)^2 - \pi^2(2k+1)^2 \right|}{\pi^2 \sqrt{\left| (1+4p)^2 - 16k^2 \right| \left| (1+4p)^2 - 16(k+1)^2 \right|}} \quad (72)$$

When n is even and p is odd, i.e. $n=2n$ and $p=2p+1$, we arrive at

$$\sin \alpha = \pm \sqrt{2}/2 \prod_{k=0}^{\infty} \frac{4 \left| \left(2\alpha + \pi(2p + 2n + 1) \right)^2 - \pi^2(2k + 1)^2 \right|}{\pi^2 \sqrt{\left| (3 + 4p)^2 - 16k^2 \right| \left| (3 + 4p)^2 - 16(k + 1)^2 \right|}} \quad (73)$$

and

$$\cos \alpha = \pm \sqrt{2}/2 \prod_{k=0}^{\infty} \frac{4 \left| \left(2\alpha - 2\pi((p - n) + 1) \right)^2 - \pi^2(2k + 1)^2 \right|}{\pi^2 \sqrt{\left| (3 + 4p)^2 - 16k^2 \right| \left| (3 + 4p)^2 - 16(k + 1)^2 \right|}} \quad (74)$$

Finally, if both n and p are odd, i.e. $n = 2n+1$ and $p = 2p+1$, the expressions in (67)-(68) convert to

$$\cos \alpha = \pm \sqrt{2}/2 \prod_{k=0}^{\infty} \frac{4 \left| \left(2\alpha + 2\pi(p + n + 1) \right)^2 - \pi^2(2k + 1)^2 \right|}{\pi^2 \sqrt{\left| (3 + 4p)^2 - 16k^2 \right| \left| (3 + 4p)^2 - 16(k + 1)^2 \right|}} \quad (75)$$

and

$$\sin \alpha = \pm \sqrt{2}/2 \prod_{k=0}^{\infty} \frac{4 \left| \left(2\alpha - \pi(2p - 2n + 1) \right)^2 - \pi^2(2k + 1)^2 \right|}{\pi^2 \sqrt{\left| (3 + 4p)^2 - 16k^2 \right| \left| (3 + 4p)^2 - 16(k + 1)^2 \right|}} \quad (76)$$

In the above representations, the sign is selected similarly to those in (61)-(64)

Representations for the tangent function follow directly from (69)-(76). This provide us with the total number of sixteen expansions. Four of those are shown below.

$$\begin{aligned} \tan \alpha &= \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha - \pi(2p - 2n + 1) \right)^2 - \pi^2(2k + 1)^2 \right|}{\left| \left(\alpha + 2\pi(p + n) \right)^2 - \pi^2(2k + 1)^2 \right|} \\ \tan \alpha &= \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha - \pi(2p - 2n + 1) \right)^2 - \pi^2(2k + 1)^2 \right|}{\left| \left(\alpha - 2\pi(p - n) \right)^2 - \pi^2(2k + 1)^2 \right|} \\ \tan \alpha &= \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha + \pi(2p + 2n + 1) \right)^2 - \pi^2(2k + 1)^2 \right|}{\left| \left(\alpha + 2\pi(p + n) \right)^2 - \pi^2(2k + 1)^2 \right|} \\ \tan \alpha &= \pm \prod_{k=0}^{\infty} \frac{\left| \left(2\alpha + \pi(2p + 2n + 1) \right)^2 - \pi^2(2k + 1)^2 \right|}{\left| \left(\alpha - 2\pi(p - n) \right)^2 - \pi^2(2k + 1)^2 \right|} \end{aligned}$$

while the others can be derived in a similar manner.

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Comparison of Numerical Schemes for Shallow Water Equation

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Abstract - Many problems of river management and civil protection consist of the evaluation of the maximum water levels and discharges that may be attained at particular locations during the development of an exceptional meteorological event. Numerical methods have become a useful tool to predict discharges and water levels in hydraulic systems. The shallow water or St. Venant equations, being a hyperbolic quasi- linear partial differential system represents a good candidate for the application of many of the techniques developed originally for Fluid Dynamics.

In this paper we will present different numerical schemes such as Lax-Friedrich scheme, Lax-Wendroff scheme, Leap-Frog scheme for the shallow water equation and implement the numerical schemes by computer programming. Next we will compare these different schemes with respect to their efficiency and the quality of the solution indeed. Here we apply linear advection equation in order to test the accuracy of these schemes for shallow water equation.

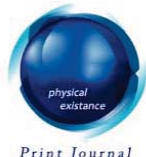
Keywords : *equation of continuity, linear advection equation, lax-friedrich scheme, lax-wendroff scheme, leap-frog scheme, shallow water equations. stencil.*

GJSFR-F Classification : *MSC 2010: 32W50*



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Md.Saiduzzaman^α & Sobuj Kumar Ray^σ

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1. INTRODUCTION

The shallow water equations (SWEs) describe the evolution of a hydrostatic homogeneous (constant density), incompressible fluid in response to gravitational and rotational accelerations and they are derived from the principles of conservation of mass and conservation of momentum. The SWEs (also called Saint-Venant equations) are one of the simplest form of the equations of motion that can be used to describe the horizontal structure of an atmosphere and ocean that model the propagation of disturbances in fluids. They are widely used to model the free surface water flows such as periodic (tidal) flows, transient wave phenomena [1] (tsunamis, flood waves, and dam-break waves) etc. The shallow water equations are only relevant when the horizontal scale of the flow is much smaller than the depth of the fluid.

In fluid dynamics the flow of the fluid is known as the Navier – Stokes equation. The shallow water equations are good approximation to the fluid motion equation when fluid density is homogeneous and depth is small in comparison to characteristic horizontal distance. So the most attention will be given to the flow of shallow water equation [2].

Shallow water equation is a system of first order partial differential equation. It is linear in derivative but non-linear in unknowns and so is called quasi-linear partial differential equation [3], [4].

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II. MATHEMATICAL MODELS FOR FLUID MOTIONS

a) Symbols and Notations

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2\}$ be a region occupied by a fluid flow and let $[t_0, T]$ be time interval with $0 \leq t_0 \leq T$. An arbitrary point in Ω is denoted by $\mathbf{x} = (x_1, \dots, x_d)^T$. For the description of a general unsteady compressible fluid flow, we introduce the quantities:

The density $\rho = \rho(\mathbf{x}, t)$, the velocity vector $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (v_1(\mathbf{x}, t), \dots, v_d(\mathbf{x}, t))^T$, the pressure $p = p(\mathbf{x}, t)$, the energy density $E = E(\mathbf{x}, t)$. We denote the external forces by $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) = (f_1(\mathbf{x}, t), \dots, f_d(\mathbf{x}, t))^T$, the mass flux by $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$. For the description of the viscous flow, let λ and μ denote the coefficient of viscosity and coefficient of kinematic viscosity respectively.

All the above quantities describing fluid flows are functions of space and time. The density of such a quantity is generally written as a function of $\Phi = \Phi(\mathbf{x}, t)$, where for given any time instant $t \in [t_0, T]$, $\mathbf{x} \in G(t)$ denote the points of set $G(t) \subset \mathbb{R}^d$ occupied by the fluid at time t . So the domain of definition of the function ϕ is the set

$$G = \{(\mathbf{x}, t); \mathbf{x} \in G(t), t \in [t_0, T]\} \in \mathbb{R}^{d+1} \quad (1)$$

In particular $G(t_0)$ represents the domain occupied by the fluid at the initial time t_0 . For the motion of a particular fluid particle, the trajectory can be described by an application $\mathbf{x} = \mathbf{X}(x_0, t_0; t)$ where $\mathbf{x}_0 \in G(t_0)$ represents the initial position of the particle at time t_0 . Any bounded domain

$$V(t) = \{\mathbf{x} = \mathbf{X}(x_0, t_0; t) \mid \mathbf{x}_0 \in V(t_0) \subset G(t_0)\} \subset G(t) \quad (2)$$

occupied by the fluid at any instant t is called a control volume. The total amount of a quantity with density $\phi(\mathbf{x}, t)$ contained in the volume $V(t)$ at time t is given by the integral

$$\phi(t) = \int_{V(t)} \phi(\mathbf{x}, t) d\mathbf{x} \quad (3)$$

In what follows, we are interested in the rate of change of $\phi(t)$ for the analysis of derivative

$$\frac{d}{dt} \phi(t) = \frac{d}{dt} \int_{V(t)} \phi(\mathbf{x}, t) d\mathbf{x} \quad (4)$$

The so-called Reynolds transport theorem plays a crucial rule:

b) The Equation of Continuity

The mass $m(V(t))$ of the fluid contained in an arbitrary control volume $V(t)$ at some given time t is given by:

$$m(V(t)) = \int_{V(t)} \rho(\mathbf{x}, t) d\mathbf{x}, \quad (5)$$

where $\rho(\mathbf{x}, t)$ is the mass density. As the domain $V(t)$ contains the same particles at each time instant t (i.e. no mass is generated or destroyed), the law of conservation of mass can be written as:

The mass of a amount of fluid within the control volume $V(t)$ is independent of time, i.e.

$$\frac{dmV(t)}{dt} = 0, \text{ for all } t \in [t_0, T]$$

Assuming $\rho(\mathbf{x}, t)$ is smooth, one obtain from the Reynolds transport theorem the integral of the mass conservation law

$$\int_{V(t)} \left[\frac{\partial \rho}{\partial t}(\mathbf{x}, t) + \nabla \cdot (\rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t)) \right] d\mathbf{x} = 0 \quad (6)$$

For all $t \in [t_0, T]$

As the control volume can be chosen arbitrary, whenever the integrand is continuous we obtain the differential form of the law of conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (7)$$

which is known as the continuity equation.

c) The Linear Advection Equation

In fluid dynamics the flow of the fluid is known as the Navier- Stokes equation. Shallow water equations are good approximation to the fluid motion equation when the flow is one dimensional and incompressible unsteady fluid and derived from the water wave by assuming that the water depth is sufficiently small compared to the water wave length. Now we will discuss the mathematical description of the shallow water equation for flood modeling.

We now consider the linear advection equation, which is given by

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad (8)$$

The Cauchy problem is defined by this equation on the domain $-\infty < x < \infty, t \geq 0$ together with initial conditions $U(x, 0) = u_0(x)$

Here we assume that the velocity c is constant. For simplicity we assume that the velocity c is positive.

$$c > 0$$

If u_0 is differentiable, then the solution is simply

$$u(t, x) = u_0(x + ct) \quad (9)$$

for $t \geq 0$ (which is based on [5]). As time involved, the initial data simply propagates unchanged to the right ($c > 0$), The solution $u(t, x)$ is constant along each ray $x - ct = x_0$, which are known as characteristics of the equation.

The characteristics are curves in the $x-t$ plane satisfying the ordinary differential equations $\dot{x}(t) = c, x(0) = x_0$.

If we differentiate $u(x, t)$ along one of this curves to find the rate of change of u along the characteristic, we find that

$$\frac{d}{dt} u(t, x(t)) = \frac{\partial}{\partial t} u(t, x(t)) + \frac{\partial}{\partial x} u(t, x(t)) \dot{x} = \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0,$$

Confirming that u is constant along these characteristics.

More generally, we might consider a variable coefficient advection equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial(c(x)u)}{\partial x} = 0 \quad (10)$$

Where $c(x)$ is a function, then we have

$$\frac{\partial u}{\partial t} + c(x) \frac{\partial u}{\partial x} = -c'(x)$$

$$\text{or, } \left(\frac{\partial}{\partial t} + c(x) \frac{\partial}{\partial x} \right) u(x, t) = -c'(x) u(x, t)$$

It follows that the evaluation of u along any curve $x(t)$ satisfying

$$x'(t) = c(x(t))$$

$$x(0) = x_0$$

Satisfies a simple ordinary differential equation:

$$\frac{d}{dt} u(t, x(t)) = -c'(x(t)) u(x(t), t)$$

The curves determined by (11) are again called characteristics.

In this case the solution is not constant along these curves, but can be easily determined by solving two sets of ordinary differential equations.

It can be shown that if $u_0(x)$ is a smooth function, say $u_0 \in C^k(-\infty, \infty)$ then the solution $u(x, t)$ is equally smooth in space and time, $u \in C^k((-\infty, \infty) \times (0, \infty))$

In the above discussion we also mention the model of Linear advection equation as well as its solution. [6]

d) Shallow Water Equation

Shallow water waves which arise if the water height is much smaller than the wave length of the water. The flow of water is distributed process because the flow rate, velocity and depth in space and time. To derive the one dimensional equation, we consider,

1. The velocity parallel to x - direction and almost independent of y : $v(t, x) \underline{e}_x$
2. The fluid is incompressible, so the density ρ is constant.

The shallow water equations are based on the laws of conservation of mass and the laws of conservation of momentum [6]. Now we will show the mathematical description of the shallow water equation which shows the direction of the propagation of the wave length as well as the flow of the fluid.

e) Mathematical Description of the Model

Conservation of mass, simply state that the mass of the volume at a given time t equal to the mass of source inside it.

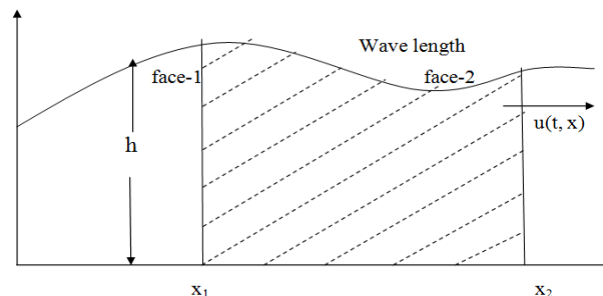


Figure 1: Shallow Water

R_{ref.}

6. Randall J. Le Veque, "Numerical Methods for Conservation Laws".

Now the change of mass inside the volume $((\Delta x h(t, x)))$ is

$$\frac{\partial}{\partial t}(\rho \Delta x h(t, x)) \quad (12)$$

Mass flow in/out the volume:

Mass flux vector $\rho v \underline{e}_x$ then integrate $\rho v \underline{e}_x$ at face-1 we get influx: $= q_1$

$$\text{So } q_1 = \rho v h(t, x_1)$$

Integrate $\rho v \underline{e}_x$ at face-2 we get influx: $= q_2$

$$\text{So } q_2 = -\rho v h(t, x_2)$$

So the change of mass flux $= \rho v h(t, x_1) - \rho v h(t, x_2)$ (2.2)

Hence we obtain from (2.1) and (2.2),

$$\frac{\partial}{\partial t}(\rho \Delta x h(t, x)) = \rho v h(t, x_1) - \rho v h(t, x_2)$$

$$\rho \frac{\partial}{\partial t} (h(t, x)) = \rho -v h(t, x_1) - v h(t, x_2) / \Delta x$$

Taking $\Delta x \rightarrow 0$ we have

$$\frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} \quad (13)$$

Conservation of momentum state that the rate of change of the total momentum of a volume of fluid formed by the same particles at any time and occupying the control volume $V(t)$ at the time instant t is equal to the resultant of the forces acting on the fluid in $V(t)$.

Now the change of momentum inside the volume $((\Delta x h(t, x)))$ is

$$\frac{\partial}{\partial t} (\rho v \Delta x h(t, x)) \quad (14)$$

Net flux of momentum:

Momentum flux vector $(\rho v)v \underline{e}_x$, then the net flux momentum at $[x_1, x_2]$ is

$$\rho v^2 h(t, x_1) - \rho v^2 h(t, x_2) \quad (15)$$

The pressure P is determined from a hydrostatic law, stating that the pressure at depth y is $\rho g h$, where g is gravitational force. Then pressure force for dy is,

$$P(t, x, y) = \rho g (h(t, x) - y) + p_0$$

Where p_0 is atmospheric pressure. Consider $P_1 = P(t, x_1, y)$ and $P_2 = P(t, x_2, y)$. Now force acting at $[x_1, x_2]$ is

$$P_1 dy - p_2 dy = \rho g (h(t, x_1) - h(t, x_2)) dy$$

Integrating from $y=0$ to $y = h(t, x)$ we have total forces

$$\rho g (h(t, x_1) - h(t, x_2)) h(t, x) \quad (16)$$

using (14), (15) and (16) we have,

$$\rho \frac{\partial}{\partial t} (\rho \Delta x h(t, x)) = \rho v^2 h(t, x_1) - \rho v^2 h(t, x_2) + \rho g (h(t, x_1) - h(t, x_2)) h(t, x)$$

$$\frac{\partial}{\partial t} (hv) = \rho -vh(t, x_1) -vh(t, x_2)/\Delta x'' + \rho g h(t, x) - h(t, x_1) - h(t, x_2)/\Delta x''$$

Taking $\Delta x \rightarrow 0$ and dividing by ρ then we have,

$$\frac{\partial}{\partial t} (hv) + \frac{\partial}{\partial t} (hv^2) + gh\left(\frac{\partial h}{\partial x}\right) = 0$$

$$h \frac{\partial v}{\partial t} + v \frac{\partial h}{\partial t} + hv \frac{\partial v}{\partial x} + v \frac{\partial (hv)}{\partial x} + gh\left(\frac{\partial h}{\partial x}\right) = 0$$

$$v \left(\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} \right) + h \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + gh \left(\frac{\partial h}{\partial x} \right) = 0$$

$$h \left(\frac{\partial v}{\partial x} + g \frac{\partial h}{\partial x} + v \frac{\partial v}{\partial x} \right) = 0 \quad [\text{using (13)}]$$

Hence Shallow water equation become,

$$\frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + g \frac{\partial h}{\partial x} + v \frac{\partial v}{\partial x} = 0 \quad (17)$$

If we denote $u = \begin{pmatrix} h \\ v \end{pmatrix}$ and $A(u) = \begin{pmatrix} v & h \\ g & v \end{pmatrix}$

Then we have

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0$$

This is a quasi-linear system of partial differential equations and this is the mathematical description of shallow water equation.

Now we construct two dimensional shallow water model and also add with source or rainfall term (based on [7]). Next we write vector form of shallow water equation. We are considering water flow into the system. Flood is caused by overflow of water. The heavy rainfall is one of the cause of it. So we have

$$\begin{aligned} \frac{\partial h}{\partial t} + h \frac{\partial u}{\partial x} + u \frac{\partial h}{\partial x} + h \frac{\partial v}{\partial y} + v \frac{\partial h}{\partial y} &= f_s \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial h}{\partial x} + g \frac{\partial z}{\partial x} &= a_{fx} \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial h}{\partial y} + g \frac{\partial z}{\partial y} &= a_{fy} \end{aligned} \quad (18)$$

And in vector form

$$\frac{dh}{dt} + \nabla \cdot (Vh) = f_s$$

Ref.

7. C. B. Vreugdenhil, "Numerical Method for Shallow Water Flow" Kluwer Academic Publisher, Dordrecht 1994.

$$\frac{dh}{dt} + V \cdot \nabla V + g \nabla h + g \nabla z - a_f = 0$$

Where

$V(x, t)$ [m/s]: velocity vector, $h(x, t)$ [m]: water level above ground, f_s : source term (Rainfall), z [m]: topographical height, g [m/s²]: acceleration due to gravity, c_f [m]: Manning coefficient, a_f [m]: friction slope.

Where $a_f = V|V| / c_f$

III. NUMERICAL METHODS

a) Finite Difference Methods

We will develop approximations to solutions of general nonlinear scalar conservation laws on a bounded domain,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \text{ for } a < x < b, 0 < t \quad (19)$$

Initially we will assume that the characteristic speeds satisfy $\lambda = f'(u) > 0$ for all states that occur in the solution of (19). Thus we also provide initial data

$$u(x, 0) = u_0(x)$$

and data at the left hand boundary

$$u(t, a) = u_a(t) \quad (20)$$

The numerical techniques have been applied to find their approximate solution of (19), i.e density $u(x, t)$ and flux $f(u(x, t))$ at a certain number of points in the time space domain, in such way as to satisfy the basic equations as far as possible. This procedure is called the discretization (based on [8]). Each point is called a grid point and the set of grid points is called a grid. We will discretize space

$$a = x_{-1/2} < x_{1/2} < x_{3/2} < \dots < x_{l-1/2} = b$$

and time

$$0 = t^0 < t^1 < t^2 < \dots < t^N = T$$

We will define the componential grid cells to be the intervals $(x_{i+1/2}, x_{i-1/2})$ with cell weights

$$\Delta x_i \equiv (x_{i+1/2} - x_{i-1/2})$$

We will also define the time steps to be

$$\Delta t^{n+1/2} \equiv t^{n+1} - t^n$$

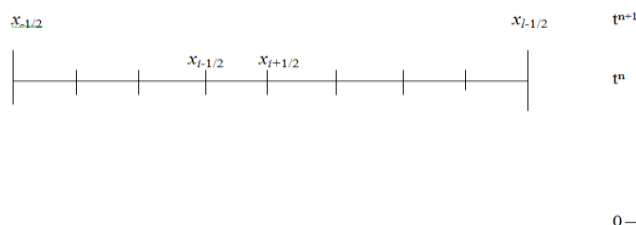


Figure 2 : Spatial and temporal discretization

The main numerical technique applied to the hyperbolic system of partial differential equations is the finite difference method (FDM). Its foundation is the following functions of continuous arguments which describe the state of the flow are replaced by functions defined on a finite number of the grid points within the considered domain. The derivatives are then replaced by divided differences. Thus the differential equations are replaced by algebraic finite difference relationships. The different ways in which derivatives and integrals are expressed by discrete functions are called finite difference schemes or methods [9].

b) The CFL Condition

One of the first papers on finite difference methods for PDEs was written in 1928 by Courant, Friedrich and Levy [6]. They use finite difference methods as an analytic tool for proving existence of solutions of certain PDEs. The idea is to define a sequence of approximate solutions (via finite difference equation), prove that they converge as the grid is refined, and then show that the limit function must satisfy the PDE, giving existence of a solution.

In the course of proving convergence of this sequence, they recognized that a necessary stability condition for any numerical method of is that the domain of dependence of the finite difference method should be the domain of dependence of the PDEs, at least in the limit as $k, h \rightarrow 0$. This condition is known as the CFL, condition after Courant, Friedrich and Lewy.

The domain of dependence $D(\tilde{x}, \tilde{t})$ for the PDE has already been described in the previous section. The set $D(\tilde{x}, \tilde{t})$ consists of the points $\tilde{x} - c\tilde{t}$, since only initial data at these points can affect the solution at $(\tilde{x}, \tilde{t})$.

c) Lax-Friedrich Scheme

We now describe the Lax-Friedrich scheme for example we take linear advection equation [4]. For discretization we use

1. Forward difference for time derivative

$$\text{i.e. } \frac{\partial u}{\partial t} \approx (u_i^n - u_{i-1}^n) / \Delta t^{n+1/2} \quad (21)$$

2. Central difference for spatial derivative

$$\text{i.e. } \frac{\partial u}{\partial x} \approx (u_{i+1}^n - u_{i-1}^n) / 2\Delta x^{n+1/2} \quad (22)$$

Substituting these values in (8) we get

$$(u_i^n - u_{i-1}^n) / \Delta t^{n+1/2} + c(u_{i+1}^n - u_{i-1}^n) / 2\Delta x^{n+1/2} = 0 \quad (23)$$

$$u_i^{n+1} = u_i^n - c\Delta t^{n+1/2} / 2\Delta x_i (u_{i+1}^n - u_{i-1}^n) \quad (24)$$

Unfortunately, despite the quite natural derivation of this method, it suffers from severe stability problem and is useless in practice. Now we replace u_i^n by $1/2 (u_{i+1}^n + u_{i-1}^n)$ and is stable provided $\Delta t^{n+1/2} / \Delta x$ is sufficiently small. Hence we have

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - c\Delta t^{n+1/2} / 2\Delta x_i (u_{i+1}^n - u_{i-1}^n) \quad (25)$$

Stencil:

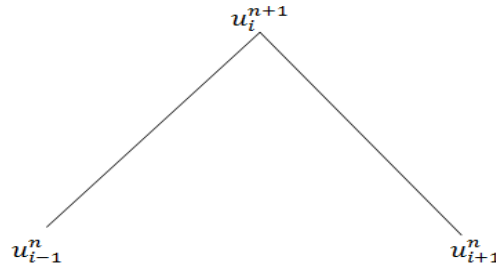


Figure 3 : Stencil of Lax-Friedrich Scheme

Stability condition: The method is stable provided that Δt and Δx are related by the following CFL (The Courant - Friedrichs - Lewy) condition

$$\gamma_i^{n+1} \equiv -c \Delta t^{n+1/2} / \Delta x_i \leq 1$$

If we take equidistance grid and denote $\Delta x = h$ and $\Delta t = k$ then we have,

$$|\frac{ck}{h}| \leq 1 \quad (26)$$

This is the stability restriction for this method.

d) *Lax – wendroff scheme for linear system*

In a search for stable and more accurate shock capturing numerical schemes, P. Lax and B. Wendroff proposed the idea of combining the spatial and temporal discretization in order to globally achieve second order. Lax-Wendroff's scheme is an explicit second order method [10].

We consider the time dependent Cauchy problem in one space dimension [6],

$$u_t + Au_x = 0, -\infty < x < \infty, t \geq 0 \quad (27)$$

$$u(x, 0) = u_0(x). \quad (28)$$

We discretize that x - t plane by choosing a mesh width $h \equiv \Delta x$ and a time step $k \equiv \Delta t$.

A wide variety of methods can be devised for the linear system (27) by using different finite difference approximations. Most of these are based directly on finite difference approximations to the PDE. An exception is the Lax-Wendroff scheme [10], which is based on the Taylor series expansion

$$u(x, t+k) = u(x, t) + k u_t(x, t) + \frac{1}{2} k^2 u_{tt}(x, t) + \quad (29)$$

The Lax-Wendroff scheme is just the first three terms of (29).

Hence the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{k}{2h} A(U_{j+1}^n - U_{j-1}^n) + \frac{k^2}{2h^2} A^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n) \quad (30)$$

Stencil:

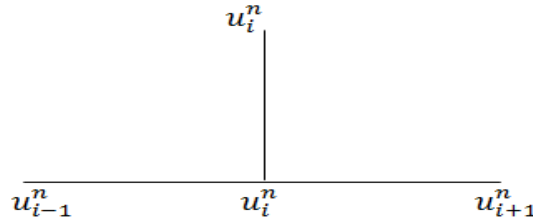


Figure 4 : Stencil of Lax-Wendroff Scheme

Stability condition: The method is stable provided that Δt and Δx are related by the following relation:

It turns out that the resulting scheme is second-order in both space and time and stable for

$$-\frac{\partial f}{\partial u} \Delta t \leq \Delta x$$

e) *Lax – wendroff scheme for non-linear system*

Our intention is to apply the Lax-Wendroff scheme for shallow water equation which is a quasi-linear equation i.e. non-linear [10]. It was further simplified by some authors in order to avoid evaluation of the celerity (Jacobian) at an intermediate position. In the case of homogeneous systems of equations, the same guidelines lead to the following scheme [11]

$$U_j^{n+1} = U_j^n - \Delta t / 2 \Delta x (F_{i+1}^n - F_{i-1}^n) + \frac{1}{2} (\Delta t / \Delta x)^2 [J_{i+1/2}^n (F_{i+1}^n - F_i^n) - J_{i+1/2}^n (F_i^n - F_{i-1}^n)] \quad (31)$$

where now $J = \frac{\partial F}{\partial U}$ is the Jacobian matrix of the system. If the system is linear, the matrix is constant $F = JU$ with $J = \text{constant}$ and in the non-linear case JU must be evaluated at an intermediate position $J_{i+1/2} = J(U_{i+1/2})$. The numerical flux can be written as

$$F_{i+1/2}^n = \frac{1}{2} (F_{i+1}^n - F_i^n) + \frac{1}{2} \frac{\Delta t}{\Delta x} (J_{i+1/2}^n)^2 (F_{i+1}^n - F_{i-1}^n) \quad (32)$$

The scheme is non-dissipative for $J = \text{constant}$, and displays oscillations near strong gradients (shocks). It can also lead to numerical difficulties near critical or sonic points. Several authors have recommended the addition of extra dissipative terms (pseudo viscosity) in these cases [12]. Lax-Wendroff's scheme is one of the most frequently encountered in the literature related to classical shock-capturing schemes. Difficulties have been reported when trying to include source terms in the discretization and to keep second order of accuracy at the same time.

Hence finally for hyperbolic system the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{1}{2} A(U_{j+1}^n - U_{j-1}^n) \left(\frac{\Delta t}{\Delta x} \right) + \frac{1}{2} A^2 (U_{j+1}^n - 2U_j^n + U_{j-1}^n) \left(\frac{\Delta t}{\Delta x} \right) \quad (33)$$

R_{ef.}

10. John A. Trangenstein "Numerical Solution for Partial Differential Equations".

f) *Leap-Frog scheme for linear system*

Leap-Frog scheme is a special form of Lax-Wendroff scheme. It is obtained from Lax-Wendroff scheme. It is exceptional to the above schemes [10]. It is also based on the Taylor series expansion as mentioned above. The Leap-Frog scheme is just the first two terms of (31).

Hence the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{k}{2h} A(U_{j+1}^n - U_{j-1}^n) \quad (34)$$

Stencil:

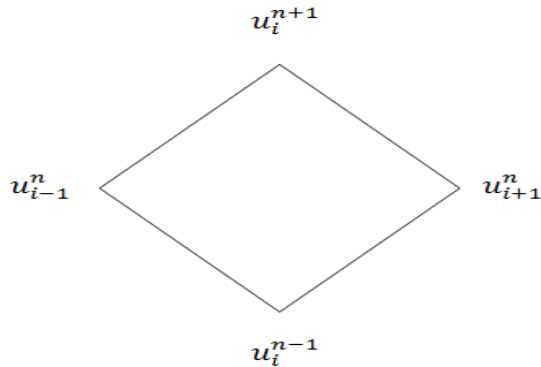


Figure 5 : Stencil of Leap-Frog scheme

Stability condition: The method is stable provided that Δt and Δx are related by the following relation:

It turns out that the resulting scheme is second-order in both space and time and stable for

$$-\frac{\partial f}{\partial u} \Delta t \leq \Delta$$

g) *Leap-Frog scheme for non-linear system*

For hyperbolic system the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{1}{2} A(U_{j+1}^n - U_{j-1}^n) \left(\frac{\Delta t}{\Delta x} \right) \quad (35)$$

h) *Finite Difference schemes for shallow water equation*

In the following, we specify the finite difference methods for solving the shallow water equation [15].

Here $h(t^n, x_i)$ is abbreviated by h_i^n and $v(t^n, x_i)$ is abbreviated by v_i^n . We denote flux $f(t^n, x_i)$ by f_i^n for first equation and flux $f1(t^n, x_i)$ by $f1_i^n$ for second equation.

In Lax-Friedrich scheme, the shallow water equation can be written as

$$h_i^n = \frac{1}{2} (h_{i+1}^{n-1} + h_{i-1}^{n-1}) - \frac{1}{2} (f_{i+1}^{n-1} - f_{i-1}^{n-1}) \frac{\Delta t}{\Delta x}$$

$$v_i^n = \frac{1}{2} (v_{i+1}^{n-1} + v_{i-1}^{n-1}) - \frac{1}{2} (f1_{i+1}^{n-1} - f1_{i-1}^{n-1}) \frac{\Delta t}{\Delta x}$$

Where $f = h * v$

$$f1 = g * h + \frac{1}{2} v^2$$

In Lax-Wendroff scheme, the shallow water equation can be written as

$$h_i^n = h_i^{n-1} - \frac{1}{2}(f_{i+1}^{n-1} - f_{i-1}^{n-1}) \frac{\Delta t}{\Delta x} + \frac{1}{2} \left[\left(\frac{\Delta t}{\Delta x} \right)^2 \left\{ \frac{1}{2}(v_{i+1}^{n-1} + v_i^{n-1})(f_{i+1}^{n-1} - f_i^{n-1}) - \frac{1}{2}(v_{i-1}^{n-1} + v_i^{n-1})(f_i^{n-1} - f_{i-1}^{n-1}) \right\} \right]$$

$$v_i^n = v_i^{n-1} - \frac{1}{2}(f1_{i+1}^{n-1} - f1_{i-1}^{n-1}) \frac{\Delta t}{\Delta x} + \frac{1}{2} \left[\left(\frac{\Delta t}{\Delta x} \right)^2 \left\{ \frac{1}{2}(v_{i+1}^{n-1} + v_i^{n-1})(f1_{i+1}^{n-1} - f1_i^{n-1}) - \frac{1}{2}(v_{i-1}^{n-1} + v_i^{n-1})(f1_i^{n-1} - f1_{i-1}^{n-1}) \right\} \right]$$

Where

$$f = h * v$$

$$f1 = g * h + \frac{1}{2} v^2$$

In Leap-Frog scheme, the shallow water equation can be written as

$$h_i^n = h_i^{n-1} - \frac{1}{2}(f_{i+1}^{n-1} - f_{i-1}^{n-1}) \frac{\Delta t}{\Delta x}$$

$$v_i^n = v_i^{n-1} - \frac{1}{2}(f1_{i+1}^{n-1} - f1_{i-1}^{n-1}) \frac{\Delta t}{\Delta x}$$

Where

$$f = h * v$$

$$f1 = g * h + \frac{1}{2} v^2$$

Here we consider $t_n = n \cdot \Delta t$, $x_i = i \cdot \Delta x$, where $n = 0, \dots, \Delta t + 1$, $i = 0, \dots, \Delta x + 1$.

IV. RESULT

Shallow water equation solved by analytical method is too complex that's why in this paper we apply two numerical schemes known as Lax-Friedrich scheme and Lax-Wendroff scheme. To Test the accuracy of the implementation of the numerical schemes we consider the linear advection equation whose analytical solution is known to us.

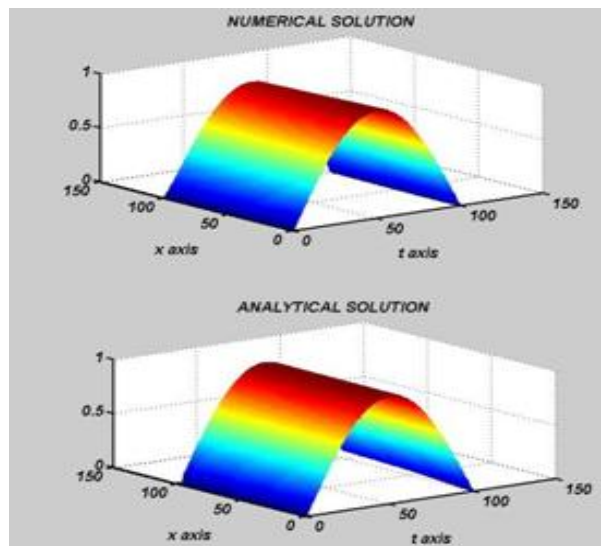


Figure 6 : Numerical and Analytical solution of linear advection equation using Lax-Friedrich scheme

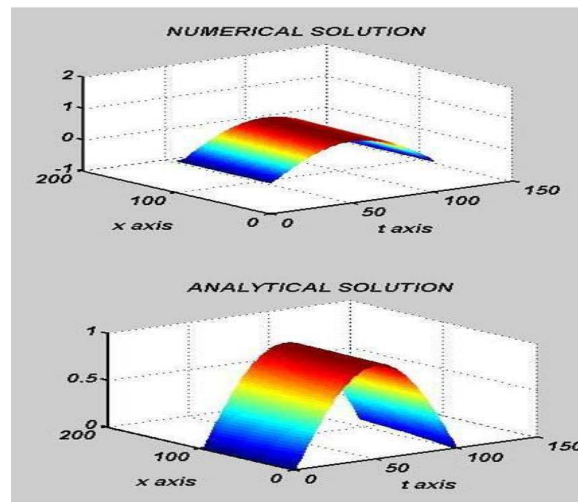


Figure 7 : Numerical and Analytical solution of linear advection equation using Lax-Wendroff scheme

Fig. 6 and fig.7 shows the analytical and numerical solution of linear advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme respectively. From these figure we observe that there is no difference between the analytical and numerical solution. Now we define error term as follows:

$$\text{Error} = \left(\frac{\text{Analytical solution} - \text{numerical solution}}{\text{Analytical solution}} \right) \times 100.$$

From the program we obtain, the error of the numerical solution of linear advection equation is 0.05, 0.048 and .99 using Lax-Friedrich scheme and Lax-Wendroff scheme and Leap-Frog scheme respectively. So we can say the error between them is very negligible and if we take $\Delta x \rightarrow 0$ then the error $\rightarrow 0$.

Figure 6. and 7 shows numerical and analytical solution of advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme, respectively.

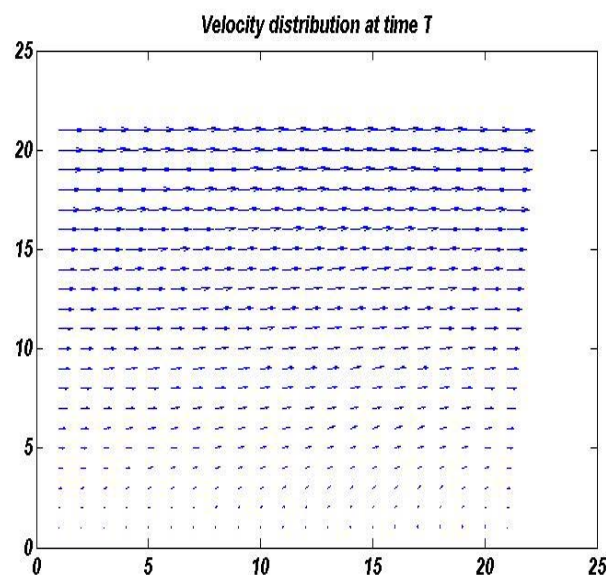


Figure 8 : Velocity distribution for numerical solution of linear advection equation using Lax-Friedrich scheme

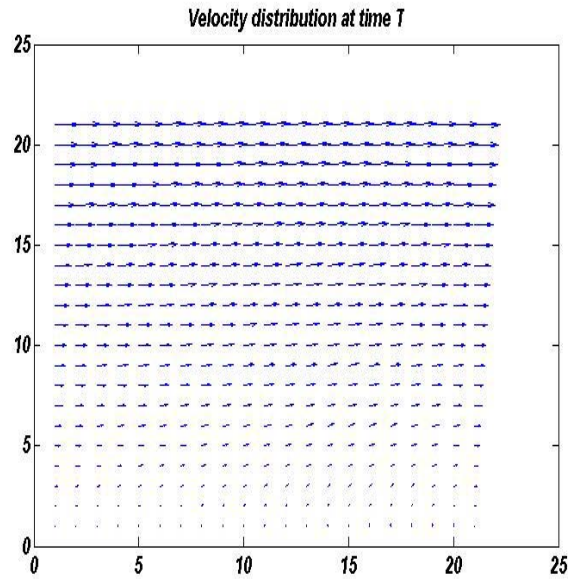


Figure 9 : Velocity distribution for numerical solution of linear advection equation using Lax-Wendroff scheme

Fig. 8. and Fig. 9. shows water velocity distribution for solution of advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme, respectively.

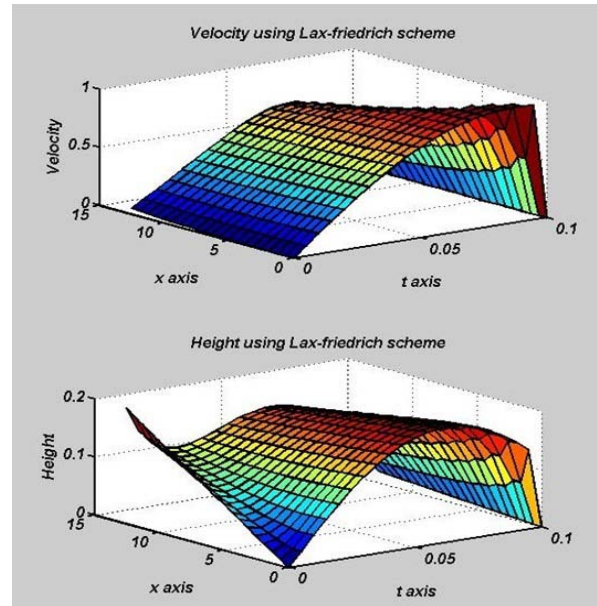


Figure 10 : Water velocity and height of shallow water equation using Lax-Friedrich scheme

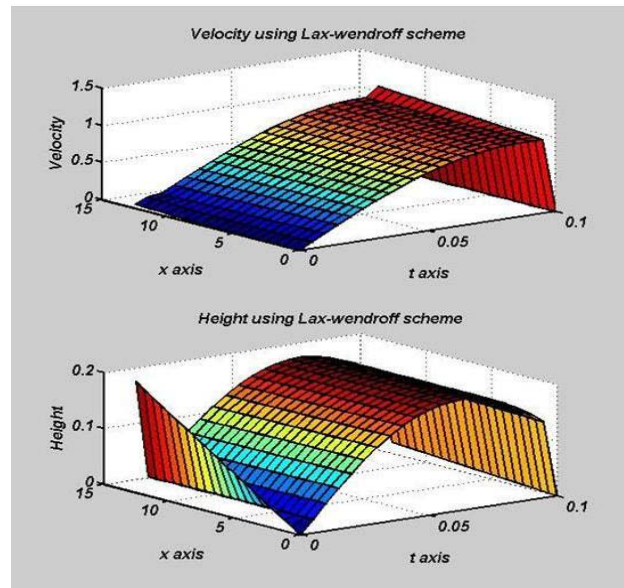


Figure 11 : Water velocity and height of shallow water equation using Lax-Wendroff scheme

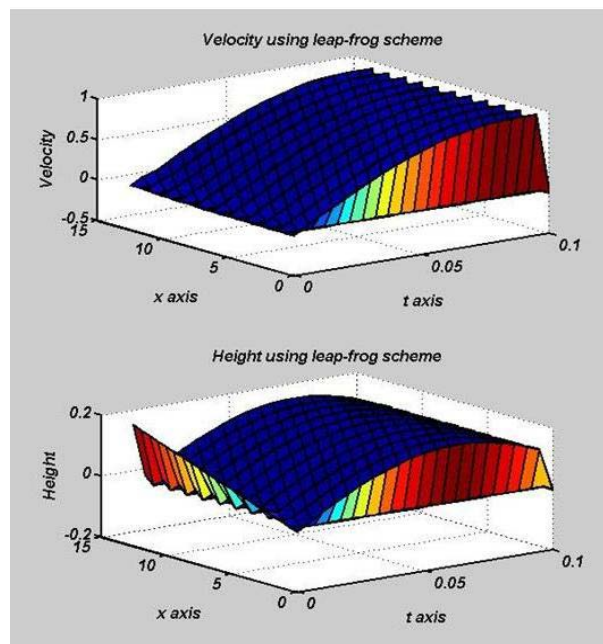


Figure 12 : Water velocity and height of shallow water equation using Leap-Frog scheme

Fig. 10 and 11 and 12 shows water velocity and height of shallow water equation using Lax-Friedrich scheme, Lax-Wendroff scheme and Leap-Frog scheme, respectively.

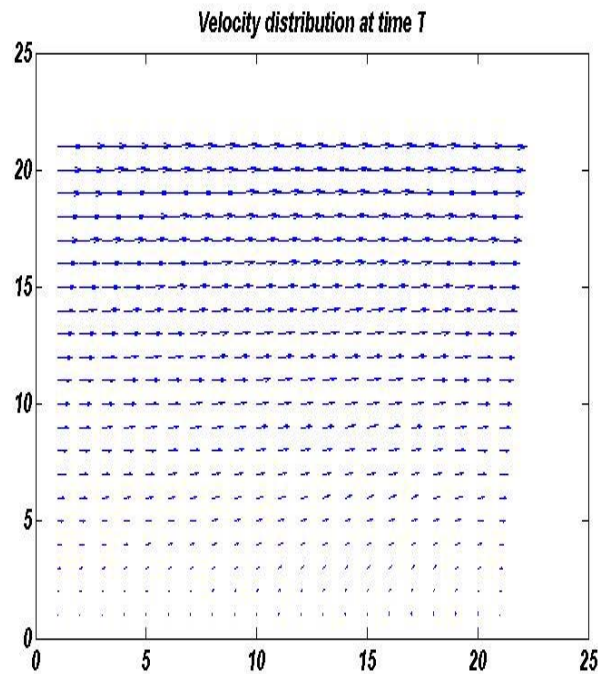


Figure 13 : Velocity distribution for numerical solution of shallow water equation using Lax-Friedrich scheme

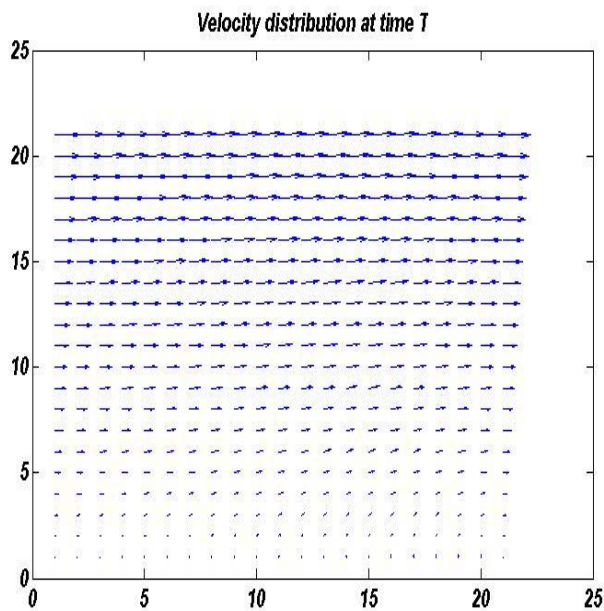


Figure 14 : Velocity distribution for numerical solution of shallow water equation using Lax-Wendroff scheme

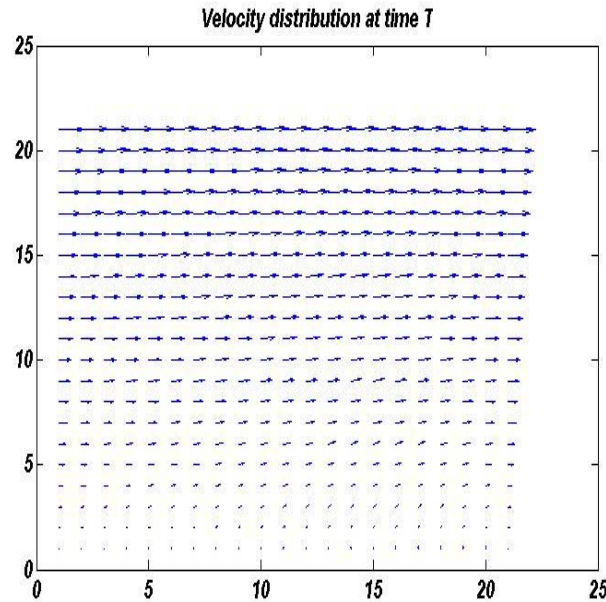


Figure 15 : Velocity distribution for numerical solution of shallow water equation using Leap-Frog scheme

Fig. 13, 14 and 15 shows velocity distribution for numerical solution of shallow water equation using Lax-Friedrich scheme and Lax-Wendroff scheme and Leap-Frog scheme respectively in forms of mesh, which we have simulated from our numerical model.

Fig. 8, and 9 implies that the velocity distribution for numerical solution of linear advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme is identical.

We also observe from the scheme of Leap-Frog scheme that it is obtain from Lax-Wendroff scheme.

V. CONCLUSION

From the above experiment it has revealed clearly that Lax-Wendroff scheme is more efficient than Lax-Friedrich scheme and Leap-Frog scheme because the error term in Lax-Wendroff scheme is less than the Lax-Friedrich scheme and Leap-Frog scheme.

In this paper we have consider only the 1D shallow water equation. First we have shown the numerical and analytical solution of linear advection equation using Lax-Friedrich scheme, Lax-Friedrich scheme and their qualitative graphs where we have obtained the error terms of these schemes which is essential to find the acceptability of these schemes. Finally we have shown the Lax-Friedrich scheme, Lax-Wendroff scheme and Leap-Frog scheme and their respective graphs for shallow water equation. It may be more effective experiment for flood modeling using higher order shallow water equation and we left this as our future work.

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Helical-One, Two, Three-Revolutional Cyclical Surfaces

By T. Olejníková

Technical University of Košice

Abstract - This paper describes method for modelling of helical- n -revolutional cyclical surfaces. The axis of the cyclical surface Φ_1 is the helix s_1 created by revolving the point about n each other revolving axes o_n ($n = 1, 2, 3$), that move together with Frenet-Serret moving trihedron along the cylindrical helix s . Particular evolutions are determined by its angular velocity and orientation. The moving circle along the helix s or s_1 , where its center lies on the helix and circle lies in the normal plane of the helix creates the cyclical surface.

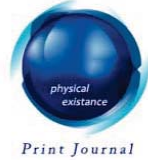
Keywords : cyclical surface, helix, frenet-serret moving trihedron, transformation matrices.

GJSFR-F Classification : MSC 2010: 05B20



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1. INTRODUCTION

Let three-dimensional Euclidean space E^3 is determined by Cartesian coordinate system $(0, x, y, z)$. In this space is given cylindrical helix s with axis identical with coordinate axis z determined by vector function (Fig.1)

$$\mathbf{r}(v) = (x_s, y_s, z_s, 1) = (a \cos mv, sg a \sin mv, bv, 1), \quad v \in \langle 0, 2\pi \rangle, \quad (1)$$

where parameter a is radius of the helix, b is the reduced pitch, sg determined orientation of the helix, ($sg = +1$ for right-handed and $sg = -1$ for left-handed revolution), m is number of pitches. Let $(0', n, b, t)$ be Frenet-Serret moving trihedron of the cylindrical helix s represented by regular square matrix

$$\mathbf{M}(v) = \begin{pmatrix} n_x(v) & n_y(v) & n_z(v) & 0 \\ b_x(v) & b_y(v) & b_z(v) & 0 \\ t_x(v) & t_y(v) & t_z(v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2)$$

where the matrix elements are the coordinates of unit vectors of the principle normal n , binormal b and tangent t of the helix s in the point $0' \in s$ in the coordinate system $(0, x, y, z)$

$$\mathbf{t}(v) = (t_x(v), t_y(v), t_z(v)) = \frac{\mathbf{r}'(v)}{|\mathbf{r}'(v)|}, \quad (3)$$

$$\mathbf{b}(v) = (b_x(v), b_y(v), b_z(v)) = \frac{\mathbf{r}'(v) \times \mathbf{r}''(v)}{|\mathbf{r}'(v) \times \mathbf{r}''(v)|}, \quad (4)$$

$$\mathbf{n}(v) = (n_x(v), n_y(v), n_z(v)) = \mathbf{b}(v) \times \mathbf{t}(v). \quad (5)$$

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Transformations of revolutions about coordinate axes x, y, z are represented by matrices $\mathbf{T}_x(\varphi, \psi)$, $\mathbf{T}_y(\varphi, \psi)$, $\mathbf{T}_z(\varphi, \psi)$, where φ is angle and ψ is orientation of the revolution, transformation of translation is represented by matrix $\mathbf{T}(\pm d_x, \pm d_y, \pm d_z)$, where $(\pm d_x, \pm d_y, \pm d_z)$ is translation vector determined by its coordinates (6), (7):

$$\mathbf{T}_x(\varphi, \psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & \psi \sin \varphi & 0 \\ 0 & -\psi \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{T}_y(\varphi, \psi) = \begin{pmatrix} \cos \varphi & 0 & \psi \sin \varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\psi \sin \varphi & 0 & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (6)$$

$$\mathbf{T}_z(\varphi, \psi) = \begin{pmatrix} \cos \varphi & \psi \sin \varphi & 0 & 0 \\ -\psi \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{T}(\pm d_x, \pm d_y, \pm d_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm d_x & \pm d_y & \pm d_z & 1 \end{pmatrix}. \quad (7)$$

The moving circle $c = (O', r)$ along the helix s , where its center O' lies in the normal plane determined by principal normal n and binormal b of the helix in the point $O' \in s$ creates the cyclical surface Φ . The vector function of this surface is

$$\mathbf{P}(u, v) = \mathbf{r}(v) + \mathbf{c}(u) \cdot \mathbf{M}(v), \quad u \in \langle 0, 2\pi \rangle, \quad v \in \langle 0, 2\pi \rangle, \quad (8)$$

where $\mathbf{r}(v)$ is vector function of the helix s expressed in equation (1), $\mathbf{M}(v)$ is transformation matrix of the coordinate system (O', n, b, t) into coordinate system (O, x, y, z) (2) and $\mathbf{c}(u) = (r \cos u, r \sin u, 0, 1), u \in \langle 0, 2\pi \rangle$ is vector function of the circle c determined by its center O' and radius r (Fig.2). In Fig.3 there are displayed two screws of the right-handed cyclical surface Φ together with the cylindrical surface on which helix s is wound.

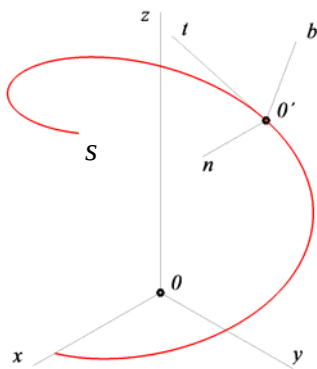


Fig. 1 : Helix s with Trihedron

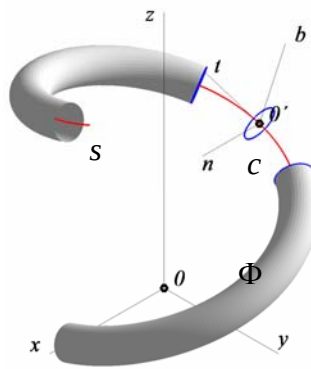


Fig. 2 : Cyclical Surface Φ

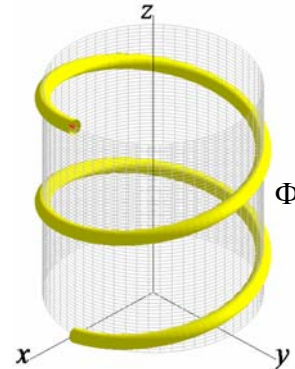


Fig. 3 : Surface Φ and Cylinder

II. CYCLICAL HELICAL SURFACE CREATED BY ONE REVOLUTION

The helix s_1 created by revolution of the point $P(x_0, y_0, z_0, 1)$ about the axis o_1 connected to the moving trihedron of the helix s , is represented by vector function

$$\mathbf{r}_1(v) = \mathbf{r}(v) + (x_0, y_0, z_0, 1) \cdot \mathbf{T}_1(m_1 v, sg_1) \cdot \mathbf{M}(v) \quad (9)$$

and cyclical surface Φ_1 created in a similar way as surface Φ by vector function

$$\mathbf{P}_1(u, v) = \mathbf{r}_1(v) + \mathbf{c}_1(u) \cdot \mathbf{M}_1(v), \quad u \in \langle 0, 2\pi \rangle, \quad v \in \langle 0, 2\pi \rangle, \quad (10)$$

where $\mathbf{r}_1(v)$ is vector function of the helix s_1 expressed in equation (9), $\mathbf{M}_1(v)$ is transformation matrix of the coordinate system $(0'', n', b', t')$ into coordinate system $(0, x, y, z)$ (11), $\mathbf{c}_1(u) = (r_1 \cos u, r_1 \sin u, 0, 1), u \in \langle 0, 2\pi \rangle$ is vector function of the circle c_1 determined by center $0'' \in s_1$ and radius r_1

$$\mathbf{M}_1(v) = \begin{pmatrix} n'_x(v) & n'_y(v) & n'_z(v) & 0 \\ b'_x(v) & b'_y(v) & b'_z(v) & 0 \\ t'_x(v) & t'_y(v) & t'_z(v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (11)$$

Elements of this matrix are coordinates of unit vectors of the principle normal n' , binormal b' and tangent t' of the helix s_1 in the point $0'' \in s_1$ in the coordinate system $(0', n, b, t)$

$$\mathbf{t}'(v) = (t'_x(v), t'_y(v), t'_z(v)) = \frac{\mathbf{r}'_1(v)}{|\mathbf{r}'_1(v)|} \quad (12)$$

$$\mathbf{b}'(v) = (b'_x(v), b'_y(v), b'_z(v)) = \frac{\mathbf{r}'_1(v) \times \mathbf{r}_1(v)}{|\mathbf{r}'_1(v) \times \mathbf{r}_1(v)|}, \quad (13)$$

$$\mathbf{n}'(v) = (n'_x(v), n'_y(v), n'_z(v)) = \mathbf{b}'(v) \times \mathbf{t}'(v). \quad (14)$$

a) Revolution about tangent t of the helix s

The helix created by the revolution of the point P about the axis $o_1 = t$ is expressed by vector function (9), in which matrix $\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_z(m_1 v, sg_1)$. In Fig.4 is displayed cyclical surface Φ , whose axis is helix s with parameters $m = 2, sg = +1$ and surface Φ_1 , whose axis is helix s_1 created by revolution of the point $P = (d, 0, 0, 1)$ about tangent t of the helix s with parameters $m_1 = 8m, sg_1 = +1$.

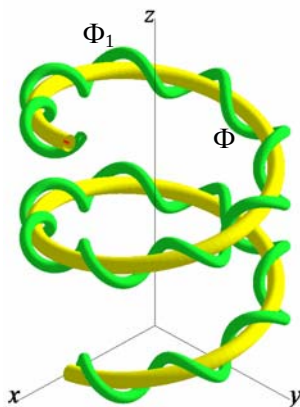


Fig. 4 : Cyclical Surfaces Φ, Φ_1

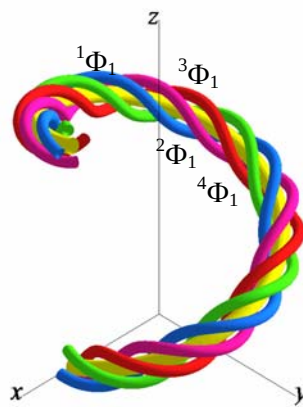


Fig. 5 : 4 Surfaces $^1\Phi_1$

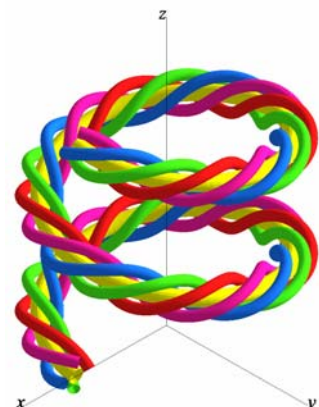


Fig. 6 : Left-handed Surfaces $\Phi, ^1\Phi_1$

In Fig.5 are displayed $k = 4$ surfaces ${}^i\Phi_1$, whose axes are helix ${}^i s_1$, $i = 1, \dots, k$ created by revolution of the points ${}^i P = (d \cos i\alpha, d \sin i\alpha, 0, 1)$, $\alpha = 2\pi/k$ about tangent t of the helix s with parameters $m_1 = 4m, sg_1 = +1$, in Fig.6 are displayed the same surfaces with altered orientation of the revolution $m = 2, sg = -1$, $m_1 = 4m, sg_1 = -1$.

b) *Revolution about principal normal n of the helix s*

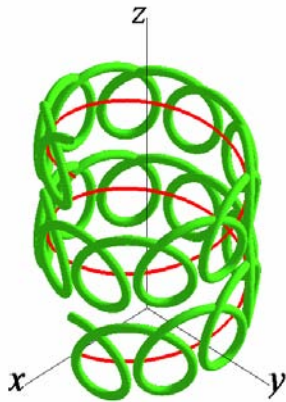


Fig. 7 : Normal Surface Φ_1

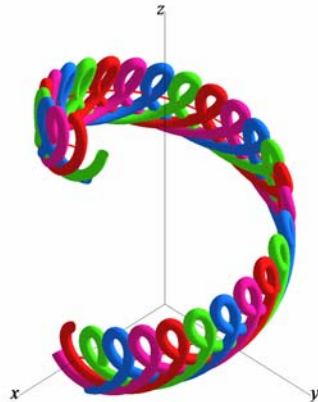


Fig. 8 : 4 Surfaces ${}^i\Phi_1$

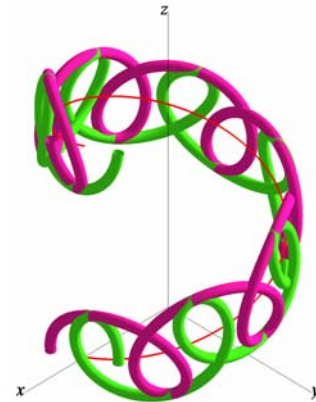
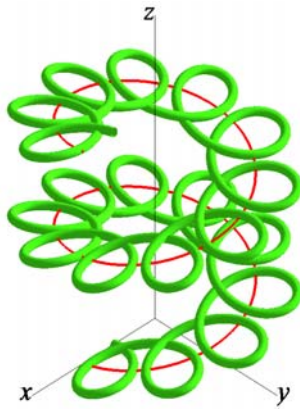
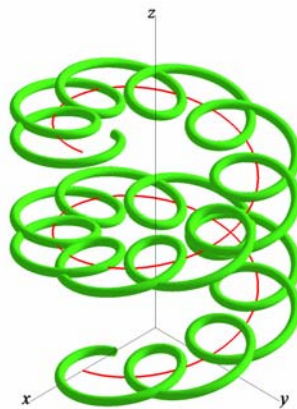
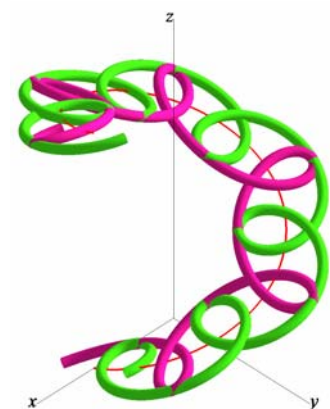


Fig. 9 : Normal Surfaces ${}^1\Phi_1, {}^3\Phi_1$

The helix s_1 created by the revolution of the point P about the axis $o_1 = n$ is expressed by vector function (9), in which matrix $T_1(m_1v, sg_1) = T_x(m_1v, sg_1)$. In Fig.7 is displayed helix s with parameters $m = 2, sg = +1$ and normal surface Φ_1 , whose axis is helix s_1 created by revolution of the point $P = (0, d, 0, 1)$ about normal n of the helix s with parameters $m_1 = 10m, sg_1 = +1$, in Fig.8 are displayed $k = 4$ normal surfaces ${}^i\Phi_1$, whose axes are helix ${}^i s_1$, $i = 1, \dots, k$ created by revolution of the points ${}^i P = (0, d \cos i\alpha, d \sin i\alpha, 1)$, $\alpha = 2\pi/k$ about normal n of the helix s with parameters $m_1 = 7m, sg_1 = -1$, in Fig.9 are displayed surfaces ${}^1\Phi_1, {}^3\Phi_1$ with altered orientation of the revolution $sg_1 = \pm 1$.

c) *Revolution about binormal b of the helix s*

The helix created by the revolution of the point P about axis $o_1 = b$ is expressed by vector function (9), in which matrix $T_1(m_1v, sg_1) = T_y(m_1v, sg_1)$. In Fig.10 is displayed helix s with parameters $m = 2, sg = +1$ and binormal surface Φ_1 , whose axis is helix s_1 created by revolution of the point $P = (d, 0, 0, 1)$ about binormal b of the helix s with parameters $m_1 = 10m, sg_1 = +1$, in Fig.11 is displayed binormal surface with parameters $m_1 = 8m, sg_1 = -1$, in Fig.12 are surfaces ${}^1\Phi_1, {}^3\Phi_1$ created by revolution of the points ${}^i P = (d \cos i\alpha, d \sin i\alpha, 0, 1)$, $\alpha = 2\pi/k$ about binormal b of the helix with altered orientation of the revolution $sg_1 = \pm 1$.


 Fig. 10: Binormal Surface Φ_1

 Fig. 11: 4 Surfaces $^1\Phi_1$

 Fig. 12: Binormal Surfaces $^1\Phi_1, ^3\Phi_1$

III. CYCLICAL HELICAL SURFACE CREATED BY TWO REVOLUTIONS

The helix s_1 created by revolution of the point $P(x_0, y_0, z_0, 1)$ about axis o_2 , which revolves about the axis o_1 identical with one edge of the moving trihedron of the helix s is represented by vector function

$$\mathbf{r}_1(v) = \mathbf{r}(v) + (x_0, y_0, z_0, 1) \cdot \mathbf{T}_2(m_2v, sg_2) \cdot \mathbf{T}_1(m_1v, sg_1) \cdot \mathbf{M}(v), \quad (15)$$

where matrix $\mathbf{T}_2(m_2v, sg_2)$ represents revolution of the point P about the axis o_2 and matrix $\mathbf{T}_1(m_1v, sg_1)$ represents revolution of the axis o_2 about the axis o_1 .

a) Revolution about two parallel axes

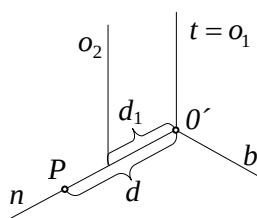


Fig. 13

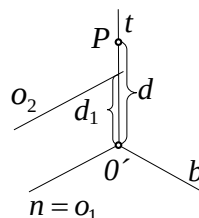


Fig. 14

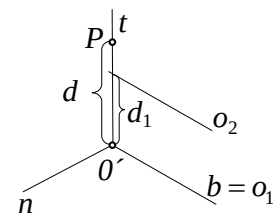


Fig. 15

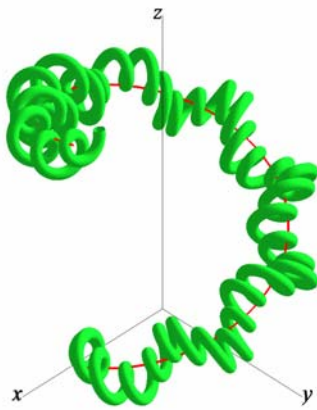
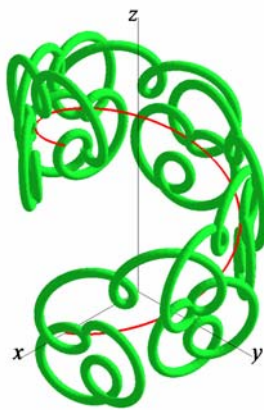
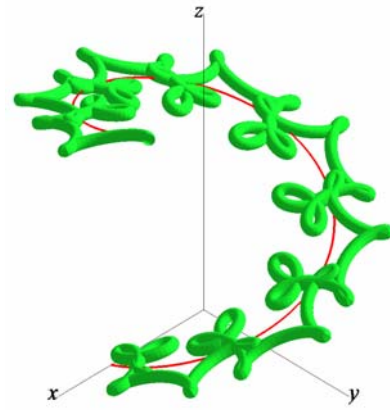
If the helix s_1 is created by revolution of the point P about two parallel axes $o_2 \parallel o_1$ and $o_1 = t$, where $d_1 = |o_1o_2|$ is the distance between them, then (Fig.13)

$$\mathbf{T}_2(m_2v, sg_2) = \mathbf{T}(-d_1, 0, 0) \cdot \mathbf{T}_z(m_2v, sg_2) \cdot \mathbf{T}(+d_1, 0, 0), \quad \mathbf{T}_1(m_1v, sg_1) = \mathbf{T}_z(m_1v, sg_1). \quad (16)$$

In Fig.16 is displayed this surface Φ_1 with parameters $m_1 = 8m, sg_1 = +1, m_2 = 4m_1, sg_2 = +1$.

If the helix s_1 is created by revolution of the point P about parallel axes $o_2 \parallel o_1$ and $o_1 = n$, where $d_1 = |o_1o_2|$, then (Fig.14)

$$\mathbf{T}_2(m_2v, sg_2) = \mathbf{T}(0, 0, -d_1) \cdot \mathbf{T}_x(m_2v, sg_2) \cdot \mathbf{T}(0, 0, +d_1), \quad \mathbf{T}_1(m_1v, sg_1) = \mathbf{T}_x(m_1v, sg_1). \quad (17)$$


 Fig. 16 : $o_2 \parallel o_1, o_1 = t$

 Fig. 17 : $o_2 \parallel o_1, o_1 = n$

 Fig. 18 : $o_2 \parallel o_1, o_1 = b$

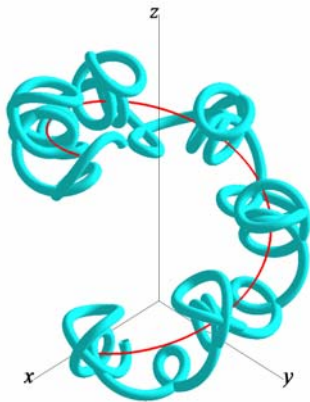
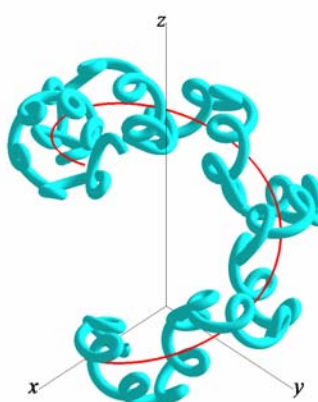
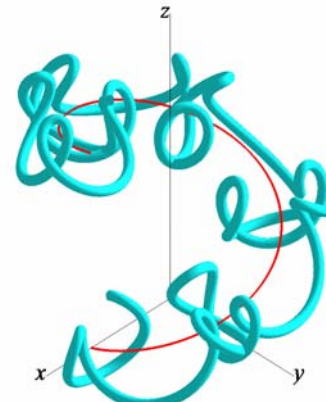
In Fig.17 is displayed this surface Φ_1 with parameters $m_1 = 6m, sg_1 = +1, m_2 = 4m_1, sg_2 = +1$.

If the helix s_1 is created by revolution of the point $P = (0, 0, d_1)$ about parallel axes $o_2 \parallel o_1$ and $o_1 = b$, where $d_1 = |o_1 o_2|$, then (Fig.15)

$$T_2(m_2 v, sg_2) = T(0, 0, -d_1) \cdot T_y(m_2 v, sg_2) \cdot T(0, 0, +d_1), \quad T_1(m_1 v, sg_1) = T_y(m_1 v, sg_1). \quad (18)$$

In Fig.18 is displayed this surface Φ_1 with parameters $m_1 = 8m, sg_1 = -1, m_2 = 5m_1, sg_2 = +1$.

b) Revolution about two intersecting axes


 Fig. 19 : $(o_2 = t) \perp (o_1 = n)$

 Fig. 20 : $(o_2 = n) \perp (o_1 = t)$

 Fig. 21 : $(o_2 = n) \perp (o_1 = b)$

In Fig.19 is displayed surface created by revolution of the point $P = (2, 2, 0, 1)$ about mutually perpendicular axes $(o_2 = t) \perp (o_1 = n)$ determined by the parameters $m_1 = 6m, sg_1 = -1, m_2 = 4m_1, sg_2 = -1$, where matrices $T_2(m_2 v, sg_2) = T_z(m_2 v, sg_2), T_1(m_1 v, sg_1) = T_x(m_1 v, sg_1)$, in Fig.20

is displayed surface created by revolution of the point $P = (2, 2, 1, 2, 0, 1)$ about mutually perpendicular axes $(o_2 = n) \perp (o_1 = t)$ determined by parameters $m_1 = 6m, sg_1 = +1, m_2 = 6m_1, sg_2 = +1$, and matrices $T_2(m_2 v, sg_2) = T_x(m_2 v, sg_2), T_1(m_1 v, sg_1) = T_z(m_1 v, sg_1)$, here we see action of changing the order of the revolutions to form of the surfaces.

In Fig.21 is displayed surface created by revolution of the point $P = (2, 5, 2, 5, 0, 1)$ about mutually perpendicular axes $(o_2 = n) \perp (o_1 = b)$ determined by parameters

$m_1 = 5m, sg_1 = -1, m_2 = 3m_1, sg_2 = +1, T_2(m_2v, sg_2) = T_x(m_2v, sg_2), T_1(m_1v, sg_1) = T_y(m_1v, sg_1)$. In Figs.22,23 is displayed surface created by revolution of the point $P = (d, 0, 0, 1)$ about intersecting axes $o_2 \times (o_1 = t)$ determined by the parameters $m_1 = 6m, sg_1 = +1, m_2 = 6m_1, sg_2 = -1, T_1(m_1v, sg_1) = T_z(m_1v, sg_1), T_2(m_2v, sg_2) = T_y(\alpha, +1), T_x(m_2v, sg_2) = T_y(\alpha, -1)$.

c) *Revolution about two skew axes*

In Figs.24,25 is displayed surface created by revolution of the point $P = (d, 0, 0, 1)$ about mutually skew axes $(o_2 \parallel n) / (o_1 = t)$, determined by parameters $m_1 = 4m, sg_1 = +1, m_2 = 8m_1, sg_2 = +1$, where transformation matrices of two revolutions are

$$T_1(m_1v, sg_1) = T_z(m_1v, sg_1), T_2(m_2v, sg_2) = T(0, 0, -d_1) \cdot T_x(m_2v, sg_2) \cdot T(0, 0, +d_1).$$

In Figs.26,27 is displayed surface created by revolution of the point $P = (d, 0, 0, 1)$ about mutually skew axes $(o_2 \times t, o_2 \times n) / (o_1 = t)$ determined by parameters $m_1 = 6m, sg_1 = +1, m_2 = 4m_1, sg_2 = -1$, and transformation matrices $T_2(m_2v, sg_2) = T_y(\alpha, +1) \cdot T_x(m_2v, sg_2) \cdot T_y(\alpha, -1), T_1(m_1v, sg_1) = T_y(m_1v, sg_1)$.

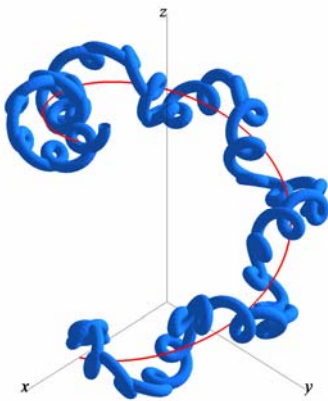


Fig. 22: $o_2 \times (o_1 = t)$

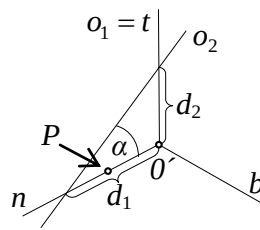


Fig. 23

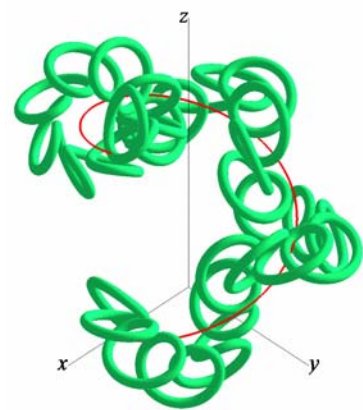


Fig. 24: $(o_2 \parallel n) / (o_1 = t)$

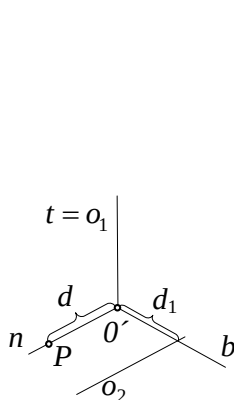


Fig. 25

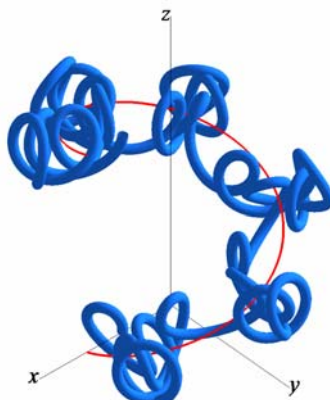


Fig. 26: $(o_2 \times t, o_2 \times n) / (o_1 = t)$

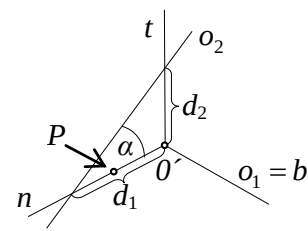


Fig. 27

IV. CYCLICAL HELICAL SURFACE CREATED BY THREE REVOLUTIONS

The helix s_1 created by the revolution of the point $P = (x_0, y_0, z_0, 1)$ about the axis o_3 , which revolves about the axis o_2 and this revolves about the axis o_1 identical with any edge of the moving trihedron of the helix s is represented by vector function

$$\mathbf{r}(v) = \mathbf{r}(v) + (x_0, y_0, z_0, 1) \cdot \mathbf{T}_3(m_3v, sg_3) \cdot \mathbf{T}_2(m_2v, sg_2) \cdot \mathbf{T}_1(m_1v, sg_1) \cdot \mathbf{M}(v), \quad (19)$$

where matrix $\mathbf{T}_3(m_3v, sg_3)$ represents revolution of the point P about the axis o_3 , matrix $\mathbf{T}_2(m_2v, sg_2)$ represents revolution of the axis o_3 about the axis o_2 and matrix $\mathbf{T}_1(m_1v, sg_1)$ represents revolution of the axis o_2 about the axis o_1 .

a) Revolution about three parallel axes

In Fig.28 is displayed surface created by revolution about three parallel axes $o_3 \parallel o_2 \parallel o_1 = t$ determined by parameters $m_1 = 4m, sg_1 = +1$, $m_2 = 4m_1, sg_2 = +1$, $m_3 = 3m_2, sg_3 = +1$, matrices $\mathbf{T}_3(m_3v, sg_3) = \mathbf{T}(-d_2, 0, 0) \cdot \mathbf{T}_z(m_3v, sg_3) \cdot \mathbf{T}(+d_2, 0, 0)$, $\mathbf{T}_2(m_2v, sg_2) = \mathbf{T}(-d_1, 0, 0) \cdot \mathbf{T}_z(m_2v, sg_2) \cdot \mathbf{T}(+d_1, 0, 0)$, $\mathbf{T}_1(m_1v, sg_1) = \mathbf{T}_z(m_1v, sg_1)$. In Fig.29 is displayed surface created by revolution about three parallel axes $o_3 \parallel o_2 \parallel o_1 = n$ determined by parameters $m_1 = 4m, sg_1 = +1$, $m_2 = 4m_1, sg_2 = +1$, $m_3 = 4m_2, sg_3 = +1$ and by transformation matrices $\mathbf{T}_1(m_1v, sg_1) = \mathbf{T}_x(m_1v, sg_1)$, $\mathbf{T}_2(m_2v, sg_2) = \mathbf{T}(0, 0, -d_1) \cdot \mathbf{T}_z(m_2v, sg_2) \cdot \mathbf{T}(0, 0, +d_1)$, $\mathbf{T}_3(m_3v, sg_3) = \mathbf{T}(0, 0, -d_2) \cdot \mathbf{T}_x(m_3v, sg_3) \cdot \mathbf{T}(0, 0, +d_2)$.

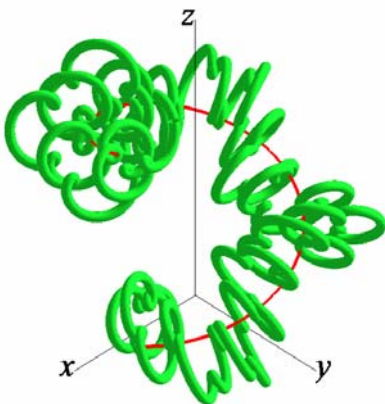


Fig. 28 : $o_3 \parallel o_2 \parallel o_1 = t$

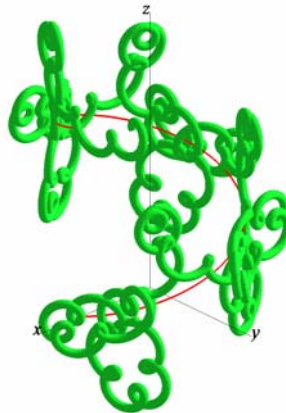


Fig. 29 : $o_3 \parallel o_2 \parallel o_1 = n$

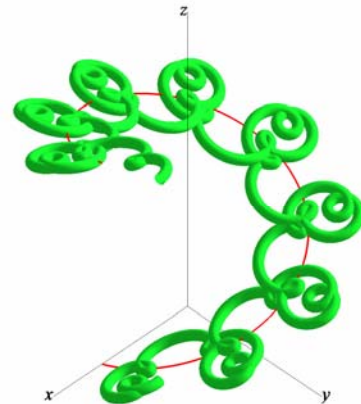
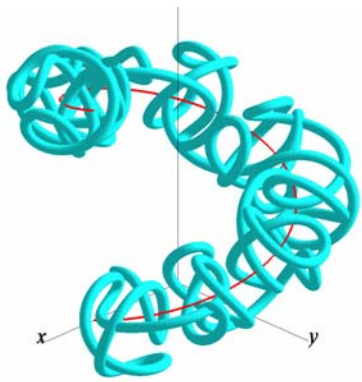
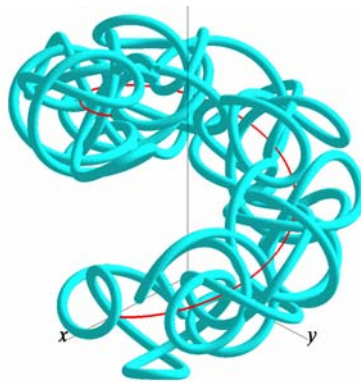
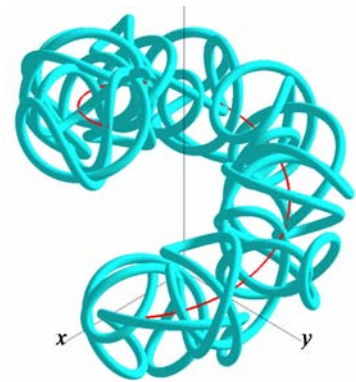


Fig. 30 : $o_3 \parallel o_2 \parallel o_1 = b$

In Fig.30 is displayed surface created by revolution about three parallel axes $o_3 \parallel o_2 \parallel o_1 = b$ determined by parameters $m_1 = 3m, sg_1 = +1$, $m_2 = 3m_1, sg_2 = +1$, $m_3 = 3m_2, sg_3 = +1$ and transformation matrices $\mathbf{T}_3(m_3v, sg_3) = \mathbf{T}(0, 0, -d_2) \cdot \mathbf{T}_y(m_3v, sg_3) \cdot \mathbf{T}(0, 0, +d_2)$, $\mathbf{T}_2(m_2v, sg_2) = \mathbf{T}(0, 0, -d_1) \cdot \mathbf{T}_y(m_2v, sg_2) \cdot \mathbf{T}(0, 0, +d_1)$, $\mathbf{T}_1(m_1v, sg_1) = \mathbf{T}_y(m_1v, sg_1)$.

b) Revolution about three perpendicular axes

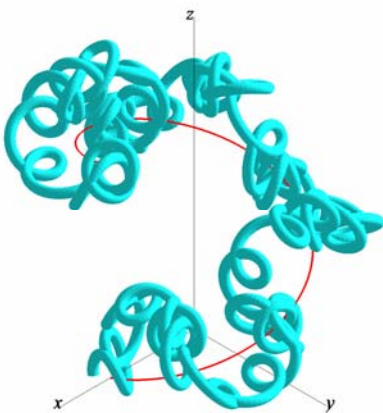
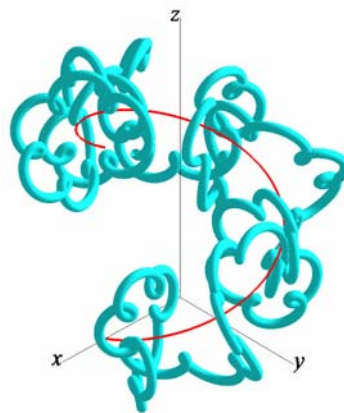
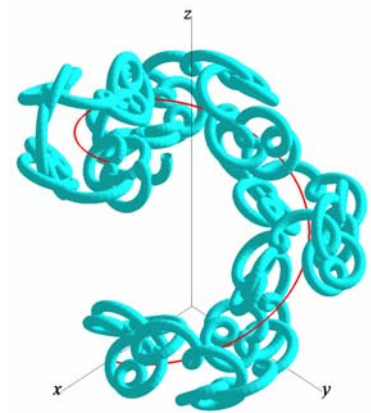
In Figs.31,32,33 are displayed surfaces created by revolution of the point $P = (d, d, 0, 1)$ about three perpendicular axes with common point $o_3 \perp o_2 \perp o_1$, which are identical with edges of the trihedron of the helix s , where parameters are the same $m_1 = 3m, sg_1 = +1$, $m_2 = 3m_1, sg_2 = +1$, $m_3 = 3m_2, sg_3 = +1$, but the order of the revolutions changes.


 Fig. 31: $o_3 = n, o_2 = b, o_1 = t$

 Fig. 32: $o_3 = n, o_2 = t, o_1 = b$

 Fig. 33: $o_3 = b, o_2 = n, o_1 = t$

c) Revolution about three skew axes

In Fig.34 are displayed surfaces created by revolution of the point $P=(0,0,0,1)$ about three skew axes $o_3/o_2/o_1$, which are parallel with edges of the trihedron of the helix s , $o_3 \parallel n$, $o_2 \parallel b$, $o_3 \parallel t$, where parameters are $m_1 = 4m, sg_1 = +1$, $m_2 = 2m_1, sg_2 = +1$, $m_3 = 6m_2, sg_3 = +1$, transformation matrices of three revolutions are

$$\begin{aligned} T_3(m_3v, sg_3) &= T(0,0,-d_3) \cdot T_x(m_3v, sg_3) \cdot T(0,0,d_3), \quad T_2(m_2v, sg_2) = T(-d_2,0,0) \cdot T_y(m_2v, sg_2) \cdot T(+d_2,0,0), \\ T_1(m_1v, sg_1) &= T(0,-d_1,0) \cdot T_z(m_1v, sg_1) \cdot T(0,+d_1,0). \end{aligned}$$


 Fig. 34: $o_3 \parallel n, o_2 \parallel b, o_3 \parallel t$

 Fig. 35: $o_3 \parallel n, o_2 = n, o_1 = b$

 Fig. 36: $o_3 \parallel n, o_2 = n, o_1 = t$

In Figs.35,36 are displayed surfaces created by revolution about the axes $o_3 \parallel n, o_2 = n, o_1 = b$ or axes $o_3 \parallel n, o_2 = n, o_1 = t$.

V. CONCLUSION

The described method of modeling of the helical- n -revolutional cyclical surfaces makes it possible to model different interest surfaces simply by changing the parameters.

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Estimation of Population Ratio in Simple Random Sampling using Variable Transformation

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Abstract - This paper proposes six new estimators of the population ratio (R) of the population means of two variables (y and x) in Simple Random Sampling (SRS) scheme, using a variable transformation of the auxiliary variable, x . Properties of the proposed estimators, including optimality conditions, are derived up to first order approximation, and conditions under which the proposed estimators perform better than the customary ratio estimator ($\hat{R} = \bar{y}/\bar{x}$) are also obtained. The results are supported with empirical illustrations, which show that some of the proposed estimators have relatively large gains in efficiency over the customary ratio estimator, $\hat{R}$ for the data set considered.

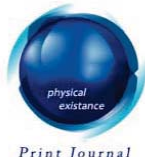
Keywords : variable transformation, ratio, product and regression-type estimators, mean squared error.

GJSFR-F Classification : MSC 2010: 62D05



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Estimation of Population Ratio in Simple Random Sampling using Variable Transformation

Onyeka, A.C.^α, Nlebedim, V.U.^σ & Izunobi, C.H.^ρ

Abstract - This paper proposes six new estimators of the population ratio (R) of the population means of two variables (y and x) in Simple Random Sampling (SRS) scheme, using a variable transformation of the auxiliary variable, x . Properties of the proposed estimators, including optimality conditions, are derived up to first order approximation, and conditions under which the proposed estimators perform better than the customary ratio estimator ($\hat{R} = \bar{y} / \bar{x}$) are also obtained. The results are supported with empirical illustrations, which show that some of the proposed estimators have relatively large gains in efficiency over the customary ratio estimator, $\hat{R}$ for the data set considered.

Keywords : variable transformation, ratio, product and regression-type estimators, mean squared error.

I. INTRODUCTION

It is a common practice, in sample surveys, to use information obtained on an auxiliary variable to improve the efficiency of estimates of the population mean and total of the study variable. In some studies, however, the ratio of the population means (or totals) of the study and auxiliary variables might be of great significance, hence the need to estimate such ratios. For instance, one might be interested in the unemployment rate, which is usually obtained as the ratio of the number of unemployed people to the number of employed individuals in a country's labour force. Other parameters that could be obtained as a ratio of two parameters include income per capita, which is the ratio of the total income of a household to the total number of members of the household; the average salary of workers in a given establishment or company, which is usually obtained as the ratio of the total salary funds of the establishment to the company's total number of employees; and the employment sex ratio of a country, which is the ratio of the number of employed men and women in the country's labour force.

The usual or customary estimator of the population ratio, $R = \bar{Y} / \bar{X}$, of the population means of two variables, y and x , under the simple random sampling scheme, is given as $\hat{R} = \bar{y} / \bar{x}$, which is the ratio of the sample means of the two variables [Cochran(1977)]. However, several authors have contributed to the problem of estimating the population ratio of two means, often utilizing information on single or more auxiliary variables. These include Singh (1965), Srivastava (1967), Srivastava et al. (1988), Upadhyaya et al. (2000), Khare and Sinha (2007), and Khare et al. (2012). In using information on one or more auxiliary characters to

estimate the population ratio, $R = \bar{Y} / \bar{X}$, the two variables, y and x , are considered as the study variables, while other variables, say z_i ($i = 1, 2, \dots, k$), are considered as auxiliary variables, known to have some strong correlation with the variables, y and x . This implies that after observing the variables, y and x , more funds would be required to obtain information on the auxiliary variables, z_i 's. If the variable y is the study variable, as it is often the case in most practical surveys, then the variable x , together with the variables, z_i 's, would all be considered as auxiliary variables, which require extra funds in order to be observed. In the present study, we restrict observations, and consequently, funding costs, to only two variables, y and x , taking the variable x as an auxiliary variable having strong correlation with the variable, y . The parameter of interest still remains the population ratio, $R = \bar{Y} / \bar{X}$, and the objective of the study is to estimate R using variable transformation of the (auxiliary) variable, x , on the assumption that the population mean ($\bar{X}$) of x is known.

II. THE PROPOSED ESTIMATORS

Let $y_i(x_i)$ be observations on two variables, and let a random sample of size n be drawn from a population of N units using simple random sampling without replacement (SRSWOR) scheme. Consider the variable transformation,

$$x_i^* = \frac{N\bar{X} - nx_i}{N - n}, \quad i = 1, 2, \dots, N, \quad (2.1)$$

The transformation (2.1) has been used by authors like Srivenkataramana (1980), Singh and Tailor (2005), Tailor and Sharma (2009), and Sharma and Tailor (2010) to improve estimates of the population mean, $\bar{Y}$, under the simple random sampling scheme, and Onyeka (2013) under the poststratified sampling scheme. The associated sample mean is obtained as:

$$\bar{x}^* = (1 + \pi)\bar{X} - \pi\bar{x}, \quad \pi = \frac{n}{N - n} \quad (2.2)$$

Using the (sample) means, $\bar{y}$, $\bar{x}$, and $\bar{x}^*$, and assuming knowledge of the population mean, $\bar{X}$, of the (auxiliary) variable, x , we propose the following six estimators of the population ratio, $R = \bar{Y} / \bar{X}$, in simple random sampling without replacement (SRSWOR) scheme:

$$\hat{R}_1 = \frac{\bar{y}}{\bar{x} - b(\bar{x}^* - \bar{X})} \quad (\text{regression-type estimator of sample mean, } \bar{x}) \quad (2.3)$$

$$\hat{R}_2 = \frac{\bar{y}}{\left(\frac{\bar{x}}{\bar{x}^*}\bar{X}\right)} = \frac{\bar{y}\bar{x}^*}{\bar{x}\bar{X}} \quad (\text{ratio-type estimator of sample mean, } \bar{x}) \quad (2.4)$$

$$\hat{R}_3 = \frac{\bar{y}}{\left(\frac{\bar{x}\bar{x}^*}{\bar{X}}\right)} = \frac{\bar{y}\bar{X}}{\bar{x}\bar{x}^*} \quad (\text{product-type estimator of sample mean, } \bar{x}) \quad (2.5)$$

$$\hat{R}_4 = \frac{\bar{y}}{\bar{x}^*} \quad (\text{transformed mean estimator, } \bar{x}^*) \quad (2.6)$$

$$\hat{R}_5 = \frac{\bar{y}}{\bar{x}^* - b(\bar{x} - \bar{X})} \quad (\text{regression-type estimator of transformed mean, } \bar{x}^*) \quad (2.7)$$

$$\hat{R}_6 = \frac{\bar{y}}{\left(\frac{\bar{x}^*}{\bar{x}}\bar{X}\right)} = \frac{\bar{y}\bar{x}}{\bar{x}^*\bar{X}} \quad (\text{ratio-type estimator of transformed mean, } \bar{x}^*) \quad (2.8)$$

where b is a suitable constant, often chosen to be very close to the population regression coefficient of y on x .

It is worth mentioning here, that Adewara et al. (2012) proposed some modified estimators of the population mean, $\bar{Y}$, involving the transformed (sample) means, $\bar{x}^*$ and $\bar{y}^*$, having the relationships:

$$\bar{X} = f\bar{x} + (1-f)\bar{x}^* \quad \text{and} \quad \bar{Y} = f\bar{y} + (1-f)\bar{y}^*, \quad f = n/N \quad (2.9)$$

This is quite worrisome in view of the fact that the transformed mean, $\bar{y}^*$, is a function of the population mean, $\bar{Y}$, which is usually unknown. If the population mean ($\bar{Y}$) of the study variable, y , is already known, then there is no need constructing estimators to estimate what is already known. Consequently, there seems to be no justification for the use of the transformed mean, $\bar{y}^*$, in estimating the population mean, $\bar{Y}$, as well as the population ratio, $R = \bar{Y}/\bar{X}$. In estimating the population ratio, in particular, the much one could do is to assume that one of the population means, (say $\bar{X}$), is known, hence the justification for the use of the transformed mean, $\bar{x}^*$. To use both the transformed means, $\bar{x}^*$ and $\bar{y}^*$, in constructing estimators of the population ratio is to suggest or assume that both the population means, $\bar{X}$ and $\bar{Y}$, are already known, which implies that the population ratio is equally known and needs not to be estimated in the first place.

$$\text{Let} \quad e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}} \quad \text{and} \quad e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}}. \quad (2.10)$$

Then,

$$E(e_0) = E(e_1) = 0 \quad (2.11)$$

$$E(e_0^2) = \frac{V(\bar{y})}{\bar{Y}^2} = \left(\frac{1-f}{n}\right) \frac{S_y^2}{\bar{Y}^2} \quad (2.12)$$

$$E(e_1^2) = \frac{V(\bar{x})}{\bar{X}^2} = \left(\frac{1-f}{n}\right) \frac{S_x^2}{\bar{X}^2} \quad (2.13)$$

and

$$E(e_0 e_1) = \frac{\text{Cov}(\bar{y}, \bar{x})}{\bar{Y}\bar{X}} = \left(\frac{1-f}{n}\right) \frac{S_{yx}}{\bar{Y}\bar{X}} \quad (2.14)$$

where S_y^2 (S_x^2) is the variance of y (x) and S_{yx} is the covariance of y and x .

To obtain the properties of the proposed estimator, $\hat{R}_1$, we first rewrite (2.3) in terms of e_0 and e_1 and expand up to first order in expected values, to obtain:

$$(\hat{R}_1 - R) = R[e_0 - (1 + \pi b)e_1 - (1 + \pi b)e_0 e_1 + (1 + \pi b)^2 e_1^2] \quad (2.15)$$

and

$$(\hat{R}_1 - R)^2 = R^2[e_0^2 + (1 + \pi b)^2 e_1^2 - 2(1 + \pi b)e_0 e_1] \quad (2.16)$$

Taking the expectations of (2.15) and (2.16), and using (2.11) – (2.14) to make the necessary substitutions, gives the bias and mean squared error of the proposed estimators, $\hat{R}_1$, up to first order approximation, respectively as:

$$B(\hat{R}_1) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) (1 + \pi b) [(1 + \pi b) R S_x^2 - S_{yx}] \quad (2.17)$$

and

$$MSE(\hat{R}_1) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [S_y^2 + (1 + \pi b)^2 R^2 S_x^2 - 2(1 + \pi b) R S_{yx}] \quad (2.18)$$

Following similar procedure, we obtain the biases and mean squared errors of the six proposed estimators, together with those of the usual or customary ratio estimator, $\hat{R} = \bar{y}/\bar{x}$, up to first order approximations as:

$$B(\hat{R}) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [R S_x^2 - S_{yx}] \quad (2.19)$$

$$B(\hat{R}_1) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) (1 + \pi b) [(1 + \pi b) R S_x^2 - S_{yx}] \quad (2.20)$$

$$B(\hat{R}_2) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) (1 + \pi) [R S_x^2 - S_{yx}] \quad (2.21)$$

$$B(\hat{R}_3) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [(1 - \pi + \pi^2) R S_x^2 - (1 - \pi) S_{yx}] \quad (2.22)$$

$$B(\hat{R}_4) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) \pi [\pi R S_x^2 + S_{yx}] \quad (2.23)$$

$$B(\hat{R}_5) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) (\pi + b) [(\pi + b) R S_x^2 + S_{yx}] \quad (2.24)$$

$$B(\hat{R}_6) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) (1 + \pi) [\pi R S_x^2 + S_{yx}] \quad (2.25)$$

and,

$$MSE(\hat{R}) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [S_y^2 + R^2 S_x^2 - 2R S_{yx}] \quad (2.26)$$

$$MSE(\hat{R}_1) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [S_y^2 + (1 + \pi b)^2 R^2 S_x^2 - 2(1 + \pi b) R S_{yx}] \quad (2.27)$$

$$MSE(\hat{R}_2) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [S_y^2 + (1 + \pi)^2 R^2 S_x^2 - 2(1 + \pi) R S_{yx}] \quad (2.28)$$

$$MSE(\hat{R}_3) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [S_y^2 + (1 - \pi)^2 R^2 S_x^2 - 2(1 - \pi) R S_{yx}] \quad (2.29)$$

$$MSE(\hat{R}_4) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [S_y^2 + \pi^2 R^2 S_x^2 + 2\pi R S_{yx}] \quad (2.30)$$

$$\text{MSE}(\hat{R}_5) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [S_y^2 + (\pi + b)^2 R^2 S_x^2 + 2(\pi + b)RS_{yx}] \quad (2.31)$$

$$\text{MSE}(\hat{R}_6) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [S_y^2 + (1 + \pi)^2 R^2 S_x^2 + 2(1 + \pi)RS_{yx}] \quad (2.32)$$

Generally, the mean squared errors of the estimators could be written as:

$$\text{MSE}(\hat{R}_d) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) [S_y^2 + \theta_d^2 R^2 S_x^2 - 2\theta_d RS_{yx}] \quad (2.33)$$

where

$$\theta_1 = (1 + \pi b), \theta_2 = (1 + \pi), \theta_3 = (1 - \pi), \theta_4 = (-\pi), \theta_5 = [-(\pi + b)], \theta_6 = [-(1 + \pi)] \quad (2.34)$$

III. EFFICIENCY COMPARISON

The efficiencies of the six proposed estimators, $\hat{R}_d$, $d = 1, 2, \dots, 6$ are first compared to that of the customary ratio estimator, $\hat{R}$, in estimating the population ratio, R , of two population means in simple random sampling scheme. The performance of the proposed estimators among themselves is also considered. Further consideration is also given to the optimum or best estimators among the proposed set of estimators.

a) Efficiency of Proposed Estimators over the Customary Ratio Estimator, $\hat{R}$

Using (2.26) and (2.33), the proposed estimators, $\hat{R}_d$, $d = 1, 2, \dots, 6$ would perform better than the customary ratio estimator, $\hat{R}$, in terms of having a smaller mean squared error if:

$$\text{or} \quad \left. \begin{array}{l} (1) \quad \theta_d < 1 \text{ and } B < R \\ (2) \quad \theta_d > 1 \text{ and } B > R \end{array} \right\} \quad (3.1)$$

where θ_d is as given in (2.34), and $B = S_{yx}/S_x^2$ is the population regression coefficient of y on x . This shows that the proposed estimators are not always more efficient than the customary ratio estimator, $\hat{R}$. For instance, the proposed estimator, $\hat{R}_2$, in (2.4) would only be more efficient than the estimator, $\hat{R}$, if and only if $B > R$, since, from (2.2) and (2.34), the quantity, $\theta_2 = 1 + \pi$ is greater than unity. This means that the customary ratio estimator, $\hat{R}$, would be more efficient than the proposed estimator, $\hat{R}_2$ for data sets in which the value of the population regression coefficient, B , is smaller than the value of the population ratio, R . However, it would be shown later in this study that the proposed estimators, under certain general optimality conditions, always perform better than the customary ratio estimator, $\hat{R}$.

b) Efficiency Comparison among the Proposed Estimators

Let $\hat{R}_j$ and $\hat{R}_k$, $j \neq k$, and $j, k = 1, 2, \dots, 6$ be any two particular estimators from the six proposed estimators in (2.3) to (2.8). Then using (2.33), the estimator, $\hat{R}_j$ would be more efficient than the estimator, $\hat{R}_k$, in terms of having a smaller mean squared error, if:

$$\text{or } \left. \begin{array}{l} (1) \quad \theta_j < \theta_k \text{ and } B < R\theta_k \\ (2) \quad \theta_j > \theta_k \text{ and } B > R\theta_k \end{array} \right\} \quad (3.2)$$

where θ_j and θ_k are obtained from θ_d , as given in (2.34). For instance, in comparing the proposed estimators, $\hat{R}_2$ and $\hat{R}_3$, we observe, from (2.34) that $\theta_2 = 1 + \pi$ and $\theta_3 = 1 - \pi$, indicating that the quantity, θ_2 , is greater than θ_3 , since $\pi = \frac{n}{N-n}$ is always positive. Consequently, and by using condition (2) of (3.2), the proposed estimator, $\hat{R}_2$ would be more efficient than the proposed estimator, $\hat{R}_3$ if and only if, $B > R(1 - \pi)$. Comparison of the efficiencies of the remaining proposed estimators could be carry out in a similar manner, using the efficiency conditions in (3.2).

c) Optimum Estimators

The optimum estimators, among the six proposed estimators, could be obtained by minimizing the mean squared error of the proposed estimators, $\hat{R}_d$, in (2.33) with respect to the quantity, θ_d defined in (2.34). Applying the least square method, this gives the optimum value of θ_d , say θ_d^0 , as

$$\theta_d^0 = \frac{B}{R} \quad (3.4)$$

with the associated minimum mean squared error of $\hat{R}_d$ obtained as:

$$\text{MSE}_{\text{opt}}(\hat{R}_d) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) (1-\rho^2) S_y^2 \quad (3.5)$$

where $\rho = S_{yx} / (S_y S_x)$ is the population correlation coefficient of the variables y and x.

Notice, from (2.3) and (2.7), that the proposed estimators, $\hat{R}_1$ and $\hat{R}_5$, particularly give us the opportunity to choose suitable values of the constant, b. This means that with both estimators, it is possible and a lot easier to meet the optimality condition (3.4) by making appropriate choice of the constant, b. Using (3.4) and (2.34), it follows that the proposed estimator, $\hat{R}_1$ would be an optimum estimator if we choose the value of b, say b_1^0 , as:

$$b_1^0 = \frac{B-R}{\pi R} \quad (3.6)$$

Similarly, the proposed estimator, $\hat{R}_5$ would be an optimum estimator if we choose the value of b, say b_5^0 , as:

$$b_5^0 = - \left(\frac{B + \pi R}{R} \right) \quad (3.7)$$

The associated minimum mean squared errors of the estimators, $\hat{R}_1$ and $\hat{R}_5$ are the same as already given in (3.5). Comparing (3.5) and (2.26), we obtain the difference between the

mean squared error of the optimum estimators and that of the customary ratio estimator, $\hat{R}$, as:

$$\Delta = \text{MSE}(\hat{R}) - \text{MSE}_{\text{opt}}(\hat{R}_d) = \frac{1}{\bar{X}^2} \left(\frac{1-f}{n} \right) (\rho S_y - R S_x)^2, \quad (3.8)$$

which is always greater than zero. Hence the optimum estimators, using the optimality condition (3.4), are always more efficient than the customary ratio estimator, $\hat{R}$, for the purpose of estimating the population ratio, R , of two population means, under the simple random sampling scheme.

IV. NUMERICAL ILLUSTRATION

The theoretical results obtained in the present study are illustrated here numerically, using the data given on page 171 of Johnston (1982). The data set is summarized as follows:

y = Percentage of hives affected by disease

x = Date of flowering of a particular summer species (number of days from January 1)

$N=10$, $n=4$, $\bar{Y}=52$, $\bar{X}=200$, $S_y^2=65.97338$, $S_x^2=84.01556$, $S_{yx}=-69.98292$

We assume, for illustration purposes, that we are interested in the ratio of the percentage of hives affected by the disease to the number of days of flowering. That is, $R = \bar{Y}/\bar{X}$. Then, the computed percentage relative efficiencies (PRE) of the six proposed estimators, $\hat{R}_d$, $d=1, 2, \dots, 6$, and the optimum estimators, $\hat{R}_d^0$, over the customary ratio estimator, $\hat{R} = \bar{y}/\bar{x}$ are displayed in Table 1.

Table 1 shows that apart from the estimator, $\hat{R}_2$, the remaining five proposed estimators are more efficient than the customary ratio estimator, $\hat{R}$, for the data under consideration, and the gains in efficiency of some of the estimators, like $\hat{R}_6$ and $\hat{R}_4$, are relatively large. Notice that the values of B and R are respectively obtained as $B = -0.83298$ and $R = 0.26$, showing that B is smaller than R . That is, $B < R$. Consequently, and using the efficiency condition (1) of (3.1), the proposed estimators would be more efficient than $\hat{R}$ only if the value of the associated θ_d is less than unity. Table 1 confirms this efficiency condition, since all the estimators whose values of θ_d are less than unity are found to perform better than the estimator, $\hat{R}$.

Table 1 : PRE of Proposed Estimators over the estimator, R

d	Estimator	π	$b = B$	θ_d	MSE	PRE
-	$\hat{R}$	0.6667		1	0.0004052	100
1	$\hat{R}_1$	0.6667	-0.83298	0.44465	0.0003123	130
2	$\hat{R}_2$	0.6667		1.66670	0.0005340	76
3	$\hat{R}_3$	0.6667		0.33330	0.0002953	137
4	$\hat{R}_4$	0.6667		-0.66670	0.0001659	244
5	$\hat{R}_5$	0.6667	-0.83298	0.16628	0.0002707	150
6	$\hat{R}_6$	0.6667		-1.66670	0.0000791	512
-	$\hat{R}_d^0$	0.6667		-3.20377	0.0000288	1407

Also to be observed from Table 1 is the fact that the first three proposed estimators, $\hat{R}_1$, $\hat{R}_2$ and $\hat{R}_3$ have smaller gains in efficiency than the last three proposed estimators, $\hat{R}_4$, $\hat{R}_5$ and $\hat{R}_6$. Notice, from (2.3) to (2.8), that while the first three proposed estimators have the sample mean, $\bar{x}$, as the lead statistic in the denominator, the transformed mean, $\bar{x}^*$, is the lead statistic in the denominator of the last three proposed estimators. Consequently, Table 1 suggests that estimators with the transformed mean, $\bar{x}^*$, as the lead statistic in the denominator are likely to be more efficient than those with the sample mean, $\bar{x}$, as the lead statistic in the denominator, when there is a strong negative correlation between the two variables, like we presently have in the data under consideration. The optimum estimators, as expected, are the most efficient estimators, in terms of having the smallest mean squared error when compared with the customary ratio estimator as well as all the proposed estimators.

V. CONCLUDING REMARKS

Here, we have proposed and considered six new estimators of the population ratio (R) of two population means in SRSWOR scheme, using a variable transformation of the auxiliary variable, x . The biases and mean squared errors of the proposed estimators were obtained up to first order approximation. Conditions under which the proposed estimators perform better than the customary ratio estimator ($\hat{R} = \bar{y}/\bar{x}$) were derived. Also obtained were the optimality conditions under which some of the proposed estimators could become the best (optimum) estimators. The results of the study were supported and illustrated numerically. The empirical illustration confirmed, among other things, both the optimality and efficiency conditions, which we had earlier obtained theoretically in the study. The empirical study revealed that relatively large gains in efficiency over the customary ratio estimator could be obtained by using some of the new estimators proposed in the present study. Again, the direction (positive or negative) of the linear relationship between the two variables plays a role in identifying some of the proposed estimators that are likely to be more efficient than the others, for a given set of data. The first three proposed estimators make use of the sample mean, $\bar{x}$ as the lead statistic in the denominator, and are likely to perform better than the last three proposed estimators, when there is a strong positive linear relationship between the two variables. When there is a strong negative correlation between the variables, the last three proposed estimators, which incidentally make use of the transformed sample mean, $\bar{x}^*$, as the lead statistic in the denominator are likely to be more efficient than the first three proposed estimators. However, the best estimators to use for any given set of data could be obtained by using the optimality conditions given in (3.4).

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By Salahuddin & Intazar Husain

P.D.M College of Engineering

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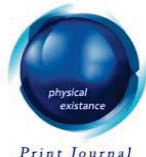
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GJSFR-F Classification : *MSC 2010: 34M30, 11T24*



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On Certain Summation Formulae Involving Gauss Theorem

Salahuddin^α & Intazar Husain^ο

Abstract - The main object of present paper is to obtain some summation formulae involving Contiguous relation, Recurrence relation, Gauss second summation theorem and Legendre duplication formula.

Keywords and Phrases : contiguous relation, recurrence relation, gauss second summation theorem, legendre duplication formula.

I. INTRODUCTION

The Pochhammer's symbol is defined by

$$(\alpha, k) = (\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + k - 1); & \text{if } k = 1, 2, 3, \dots \\ 1 & ; \text{ if } k = 0 \\ k! & ; \text{ if } \alpha = 1 \end{cases} \quad (1)$$

Generalized Gaussian Hypergeometric function of one variable is defined by

$${}_A F_B \left[\begin{matrix} a_1, a_2, \dots, a_A ; \\ b_1, b_2, \dots, b_B ; \end{matrix} z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

or

$${}_A F_B \left[\begin{matrix} (a_A) ; \\ (b_B) ; \end{matrix} z \right] \equiv {}_A F_B \left[\begin{matrix} (a_j)_{j=1}^A ; \\ (b_j)_{j=1}^B ; \end{matrix} z \right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \quad (2)$$

where the parameters $b_1, b_2, \dots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

Contiguous Relation is defined by

[Andrews p.367(8), E. D. p.52(19)]

$$c(1-z) {}_2F_1 \left[\begin{matrix} a, b ; \\ c ; \end{matrix} z \right] = c {}_2F_1 \left[\begin{matrix} a-1, b ; \\ c ; \end{matrix} z \right] - (c-b) z {}_2F_1 \left[\begin{matrix} a, b ; \\ c+1 ; \end{matrix} z \right] \quad (3)$$

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Recurrence relation

$$\Gamma(z+1) = z \Gamma(z) \quad (4)$$

Legendre's duplication formula

$$\sqrt{\pi} \Gamma(2z) = 2^{(2z-1)} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \quad (5)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)} \Gamma\left(\frac{b}{2}\right) \Gamma\left(\frac{b+1}{2}\right)}{\Gamma(b)} \quad (6)$$

$$= \frac{2^{(a-1)} \Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{a+1}{2}\right)}{\Gamma(a)} \quad (7)$$

Gauss second summation theorem [Prudnikov., 491(7.3.7.8)]

$${}_2F_1 \left[\begin{matrix} a, b ; & 1 \\ \frac{a+b+1}{2} ; & 2 \end{matrix} \right] = \frac{\Gamma\left(\frac{a+b+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(\frac{b+1}{2}\right)} \quad (8)$$

$$= \frac{2^{(b-1)} \Gamma\left(\frac{b}{2}\right) \Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma(b) \Gamma\left(\frac{a+1}{2}\right)} \quad (9)$$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov., p.491(7.3.7.8)]

$${}_2F_1 \left[\begin{matrix} a, b ; & 1 \\ \frac{a+b-1}{2} ; & 2 \end{matrix} \right] = \sqrt{\pi} \left[\frac{\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(\frac{b+1}{2}\right)} + \frac{2 \Gamma\left(\frac{a+b-1}{2}\right)}{\Gamma(a) \Gamma(b)} \right] \quad (10)$$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$${}_2F_1 \left[\begin{matrix} a, b ; & 1 \\ \frac{a+b-1}{2} ; & 2 \end{matrix} \right] = \frac{2^{(b-1)} \Gamma\left(\frac{a+b-1}{2}\right)}{\Gamma(b)} \left[\frac{\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a-1}{2}\right)} + \frac{2^{(a-b+1)} \Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{a+1}{2}\right)}{\{\Gamma(a)\}^2} + \frac{\Gamma\left(\frac{b+2}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)} \right] \quad (11)$$

II. MAIN RESULTS OF SUMMATION FORMULAE

For $a < 1$ and $a > 9$

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} a, b ; & 1 \\ \frac{a+b-9}{2} ; & 2 \end{matrix} \right] &= \frac{2^{(b-1)} \Gamma\left(\frac{a+b-9}{2}\right)}{\Gamma(b)} \left[\frac{\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a-9}{2}\right)} \left\{ \frac{(128a + 384b - 1152)}{(a-9)} - \right. \right. \\ &\quad \left. \left. - \frac{(a-b-9)(256a + 640b - 1792)}{(a-9)(a-7)} + \frac{(a-b-9)(a-b-7)(160a + 320b - 800)}{(a-9)(a-7)(a-5)} - \right. \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{(a-b-9)(a-b-7)(a-b-5)(32a+48b-96)}{(a-9)(a-7)(a-5)(a-3)} + \\
& + \frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)} \Bigg\} + \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})} \Bigg\{ \frac{(256a+256b-1792)}{(a-8)} - \\
& - \frac{(a-b-9)(416a+352b-2144)}{(a-8)(a-6)} + \frac{(a-b-9)(a-b-7)(192a+128b-640)}{(a-8)(a-6)(a-4)} - \\
& - \frac{(a-b-9)(a-b-7)(a-b-5)(22a+10b-34)}{(a-8)(a-6)(a-4)(a-2)} \Bigg\} \Bigg] \quad (12)
\end{aligned}$$

For $a < 1$ and $a > 10$

$$\begin{aligned}
{}_2F_1 \left[\begin{matrix} a, b ; \\ \frac{a+b-10}{2} ; \end{matrix} \quad \frac{1}{2} \right] &= \frac{2^{(b-1)} \Gamma(\frac{a+b-10}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-10}{2})} \left\{ \frac{(256a+768b-2560)}{(a-10)} - \right. \right. \\
& - \frac{(a-b-10)(576a+1472b-4608)}{(a-10)(a-8)} + \frac{(a-b-10)(a-b-8)(432a+912b-2592)}{(a-10)(a-8)(a-6)} - \\
& - \frac{(a-b-10)(a-b-8)(a-b-6)(120a+200b-480)}{(a-10)(a-8)(a-6)(a-4)} + \\
& + \frac{(a-b-10)(a-b-8)(a-b-6)(a-b-4)(9a+11b-18)}{(a-10)(a-8)(a-6)(a-4)(a-2)} \Bigg\} + \\
& + \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-9}{2})} \left\{ \frac{(512a+512b-4096)}{(a-9)} - \frac{(a-b-10)(960a+832b-5888)}{(a-9)(a-7)} + \right. \\
& + \frac{(a-b-10)(a-b-8)(560a+400b-2400)}{(a-9)(a-7)(a-5)} - \\
& - \frac{(a-b-10)(a-b-8)(a-b-6)(104a+56b-256)}{(a-9)(a-7)(a-5)(a-3)} + \\
& + \left. \frac{(a-b-10)(a-b-8)(a-b-6)(a-b-4)(3a+b-2)}{(a-9)(a-7)(a-5)(a-3)(a-1)} \right\} \Bigg] \quad (13)
\end{aligned}$$

For $a < 1$ and $a > 11$

$$\begin{aligned}
{}_2F_1 \left[\begin{matrix} a, b ; \\ \frac{a+b-11}{2} ; \end{matrix} \quad \frac{1}{2} \right] &= \frac{2^{(b-1)} \Gamma(\frac{a+b-11}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-11}{2})} \left\{ \frac{(512a+1536b-5632)}{(a-11)} - \right. \right. \\
& - \frac{(a-b-11)(1280a+3328b-11520)}{(a-11)(a-9)} + \frac{(a-b-11)(a-b-9)(1120a+2464b-7840)}{(a-11)(a-9)(a-7)} - \\
& - \frac{(a-b-11)(a-b-9)(a-b-7)(400a+720b-2000)}{(a-11)(a-9)(a-7)(a-5)} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(50a+70b-150)}{(a-11)(a-9)(a-7)(a-5)(a-3)} - \\
& - \frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \Bigg\} + \\
& + \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-10}{2})} \Bigg\{ \frac{(1024a+1024b-9216)}{(a-10)} - \frac{(a-b-11)(2176a+1920b-15488)}{(a-10)(a-8)} + \\
& + \frac{(a-b-11)(a-b-9)(1536a+1152b-8064)}{(a-10)(a-8)(a-6)} - \\
& - \frac{(a-b-11)(a-b-9)(a-b-7)(400a+240b-1360)}{(a-10)(a-8)(a-6)(a-4)} + \\
& + \frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(28a+12b-44)}{(a-10)(a-8)(a-6)(a-4)(a-2)} \Bigg\} \Bigg] \quad (14)
\end{aligned}$$

The above summation theorems can be easily verified by using computer algebra system programming languages, like Maple, Matlab, or Mathematica.

III. DERIVATIONS OF SUMMATION FORMULAE (12) TO (14)

Derivation of (12): Substituting $c = \frac{a+b-9}{2}$ and $z = \frac{1}{2}$ in equation (3), we get

$$\begin{aligned}
& \left(\frac{a+b-9}{4} \right) {}_2F_1 \left[\begin{matrix} a, b \\ \frac{a+b-9}{2} \end{matrix} ; \quad \frac{1}{2} \right] = \left(\frac{a+b-9}{2} \right) {}_2F_1 \left[\begin{matrix} a-1, b \\ \frac{a+b-9}{2} \end{matrix} ; \quad \frac{1}{2} \right] - \\
& - \left(\frac{a-b-9}{4} \right) {}_2F_1 \left[\begin{matrix} a, b \\ \frac{a+b-7}{2} \end{matrix} ; \quad \frac{1}{2} \right] \\
& {}_2F_1 \left[\begin{matrix} a, b \\ \frac{a+b-9}{2} \end{matrix} ; \quad \frac{1}{2} \right] = 2 \cdot {}_2F_1 \left[\begin{matrix} a-1, b \\ \frac{a+b-9}{2} \end{matrix} ; \quad \frac{1}{2} \right] - \left(\frac{a-b-9}{a+b-9} \right) {}_2F_1 \left[\begin{matrix} a, b \\ \frac{a+b-7}{2} \end{matrix} ; \quad \frac{1}{2} \right]
\end{aligned}$$

Now involving the derived from Asish et al, we get

$$\begin{aligned}
& {}_2F_1 \left[\begin{matrix} a, b \\ \frac{a+b-9}{2} \end{matrix} ; \quad \frac{1}{2} \right] = \frac{2^{(b-1)} \Gamma(\frac{a+b-9}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})} \left\{ \frac{(128a+384b-1152)}{(a-9)} - \right. \right. \\
& - \frac{(a-b-9)(224a+544b-1568)}{(a-9)(a-7)} + \frac{(a-b-9)(a-b-7)(112a+208b-560)}{(a-9)(a-7)(a-5)} \\
& - \left. \frac{(a-b-9)(a-b-7)(a-b-5)(14a+18b-42)}{(a-9)(a-7)(a-5)(a-3)} \right\} + \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})} \left\{ \frac{(256a+256b-1792)}{(a-8)} - \right. \\
& - \left. \frac{(a-b-9)(352a+288b-1824)}{(a-8)(a-6)} + \frac{(a-b-9)(a-b-7)(120a+72b-408)}{(a-8)(a-6)(a-4)} - \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{(a-b-9)(a-b-7)(a-b-5)(6a+2b-10)}{(a-8)(a-6)(a-4)(a-2)} \Bigg] - \\
& - \left(\frac{a-b-9}{a+b-9} \right) \frac{2^{(b-1)} \Gamma(\frac{a+b-7}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-7}{2})} \left\{ \frac{(32a+96b-224)}{(a-7)} - \frac{(a-b-7)(48a+112b-240)}{(a-7)(a-5)} + \right. \right. \\
& + \frac{(a-b-7)(a-b-5)(18a+30b-54)}{(a-7)(a-5)(a-3)} - \frac{(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-7)(a-5)(a-3)(a-1)} \Bigg\} + \\
& + \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-6}{2})} \left\{ \frac{(64a+64b-320)}{(a-6)} - \frac{(a-b-7)(72a+56b-232)}{(a-6)(a-4)} + \right. \\
& \left. \left. + \frac{(a-b-7)(a-b-5)(16a+8b-24)}{(a-6)(a-4)(a-2)} \right\} \right] \\
& = \frac{2^{(b-1)} \Gamma(\frac{a+b-9}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})} \left\{ \frac{(128a+384b-1152)}{(a-9)} - \frac{(a-b-9)(224a+544b-1568)}{(a-9)(a-7)} + \right. \right. \\
& + \frac{(a-b-9)(a-b-7)(112a+208b-560)}{(a-9)(a-7)(a-5)} - \\
& - \frac{(a-b-9)(a-b-7)(a-b-5)(14a+18b-42)}{(a-9)(a-7)(a-5)(a-3)} \Bigg\} + \\
& + \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})} \left\{ \frac{(256a+256b-1792)}{(a-8)} - \frac{(a-b-9)(352a+288b-1824)}{(a-8)(a-6)} + \right. \\
& + \frac{(a-b-9)(a-b-7)(120a+72b-408)}{(a-8)(a-6)(a-4)} - \\
& - \frac{(a-b-9)(a-b-7)(a-b-5)(6a+2b-10)}{(a-8)(a-6)(a-4)(a-2)} \Bigg\] - \\
& - \frac{2^{(b-1)} \Gamma(\frac{a+b-9}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})} \left\{ \frac{(a-b-9)(32a+96b-224)}{(a-9)(a-7)} - \right. \right. \\
& - \frac{(a-b-9)(a-b-7)(48a+112b-240)}{(a-9)(a-7)(a-5)} + \\
& + \frac{(a-b-9)(a-b-7)(a-b-5)(18a+30b-54)}{(a-9)(a-7)(a-5)(a-3)} - \\
& - \frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)} \Bigg\} + \\
& + \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})} \left\{ \frac{(a-b-9)(64a+64b-320)}{(a-8)(a-6)} - \frac{(a-b-9)(a-b-7)(72a+56b-232)}{(a-8)(a-6)(a-4)} + \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{(a-b-9)(a-b-7)(a-b-5)(16a+8b-24)}{(a-8)(a-6)(a-4)(a-2)} \Bigg\} \\
& = \frac{2^{(b-1)} \Gamma(\frac{a+b-9}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})} \left\{ \frac{(128a+384b-1152)}{(a-9)} - \right. \right. \\
& - \frac{(a-b-9)(256a+640b-1792)}{(a-9)(a-7)} + \frac{(a-b-9)(a-b-7)(160a+320b-800)}{(a-9)(a-7)(a-5)} - \\
& - \frac{(a-b-9)(a-b-7)(a-b-5)(32a+48b-96)}{(a-9)(a-7)(a-5)(a-3)} + \\
& + \left. \frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)} \right\} + \\
& + \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})} \left\{ \frac{(256a+256b-1792)}{(a-8)} - \frac{(a-b-9)(416a+352b-2144)}{(a-8)(a-6)} + \right. \\
& + \frac{(a-b-9)(a-b-7)(192a+128b-640)}{(a-8)(a-6)(a-4)} - \\
& \left. \left. - \frac{(a-b-9)(a-b-7)(a-b-5)(22a+10b-34)}{(a-8)(a-6)(a-4)(a-2)} \right\} \right]
\end{aligned}$$

Thus , we prove the result (12)

Similarly, we can prove the other results.

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Semi-Invariant Submanifolds of Nearly Hyperbolic Cosymplectic Manifold

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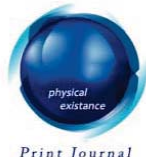
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GJSFR-F Classification : MSC 2010: 20F67, 90C34



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Semi-Invariant Submanifolds of Nearly Hyperbolic Cosymplectic Manifold

Mobin Ahmad^α & Kashif Ali^σ

Abstract - We consider a nearly hyperbolic cosymplectic manifold and study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold. We also study parallel distributions on nearly hyperbolic cosymplectic manifold and find the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold.

Keywords and phrases : semi-invariant submanifolds, nearly hyperbolic cosymplectic manifold, integrability conditions, parallel distribution.

I. INTRODUCTION

The notion of CR-submanifolds of a Kaehler manifold as generalization of invariant and anti-invariant submanifolds was initiated by A. Bejancu in [7]. A semi-invariant submanifold is the extension of the concept of a CR-submanifold of a Kaehler manifold to submanifolds of almost contact manifolds. The study of Semiinvariant submanifolds of Sasakian manifolds was initiated by Bejancu-Papaghuic in [9]. The same concept was studied under the name contact CR-submanifold by Yano-Kon in [16] and K. Matsumoto in [14]. The study of semi-invariant submanifolds in almost contact manifold was enriched by several geometers (see, [1], [2], [3], [4], [5], [6], [11], [14]). On the other hand, almost hyperbolic (f, g, η, ξ) -structure was defined and studied by Upadhyay and Dube in [15]. Joshi and Dube studied semi-invariant submanifolds of an almost r-contact hyperbolic metric manifolds in [12]. In this paper, we study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold.

II. PRELIMINARIES

Let $\bar{M}$ be an n -dimensional almost hyperbolic contact metric manifold with almost hyperbolic contact metric structure- (ϕ, ξ, η, g) , where a tensor ϕ of type $(1,1)$, a vector field ξ , called structure vector field and η , the dual 1-form of ξ satisfying the followings

$$\phi^2 X = X + \eta(X)\xi, \quad (2.1)$$

$$g(X, \xi) = \eta(X), \eta(\xi) = -1, \quad (2.2)$$

$$\phi(\xi) = 0, \eta\phi = 0, \quad (2.3)$$

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$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \quad (2.4)$$

for any X, Y tangent to $\bar{M}$ [15]. In this case

$$g(\phi X, Y) = -g(\phi Y, X). \quad (2.5)$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact metric structure (ϕ, ξ, η, g) is said to be nearly hyperbolic cosymplectic manifold [10] if

$$(\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_Y X) = 0, \quad (2.6)$$

$$\bar{\nabla}_X \xi = 0 \quad (2.7)$$

for all X, Y tangent to $\bar{M}$.

The Nijenhuis tensor $N(X, Y)$ of a nearly hyperbolic cosymplectic manifold $\bar{M}$ is defined as

$$N(X, Y) = (\bar{\nabla}_{\phi X} \phi)Y - (\bar{\nabla}_{\phi Y} \phi)X - \phi(\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_Y \phi)X \quad (2.8)$$

for any $X, Y \in T\bar{M}$.

From (2.6), we have

$$(\bar{\nabla}_{\phi X} \phi)Y = -(\bar{\nabla}_Y \phi)\phi X. \quad (2.9)$$

Also, we have

$$(\bar{\nabla}_Y \phi)\phi X = (\bar{\nabla}_Y \eta)(X)\xi + \phi(\bar{\nabla}_Y \phi)X. \quad (2.10)$$

From (2.9) and (2.10), we get

$$(\bar{\nabla}_{\phi X} \phi)Y = (\bar{\nabla}_Y \eta)(X)\xi + \phi(\bar{\nabla}_Y \phi)X. \quad (2.11)$$

Using (2.11) in (2.8), we get

$$N(X, Y) = 4\phi(\bar{\nabla}_Y \phi)X + 2g(\phi X, Y)\xi \quad (2.12)$$

for $X, Y \in TM$.

The paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic cosymplectic manifold. In section 3, we study some properties semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold. In section 4, we discuss the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold. In section 5, we study parallel horizontal distribution on nearly hyperbolic Kenmotsu manifold.

III. SEMI-INVARIANT SUBMANIFOLDS

Let M be a submanifold immersed in $\bar{M}$. We assume that the vector field ξ is tangent to M . Denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M . Then M is called a semi-invariant submanifold [8] of $\bar{M}$ if there exist two differentiable distributions D and $D^\perp$ on M satisfying.

- (i) $TM = D \oplus D^\perp \oplus \{\xi\}$, where D , $D^\perp$ and $\{\xi\}$ are mutually orthogonal to each other.
- (ii) The distribution D is invariant by ϕ , that is, $\phi D_X = D_X$ for each $X \in M$,
- (iii) The distribution $D^\perp$ is anti-invariant by ϕ , that is, $\phi D_X^\perp \subset T_X M^\perp$ for each $X \in M$,

R_{ref.}

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where TM and $T^\perp M$ be the Lie algebra of vector fields tangential to M and normal to M respectively. Let the Riemannian metric induced on M is denoted by the same symbol g and ∇ be the induced Levi-Civita connection on M , then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (3.1)$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \quad (3.2)$$

for any $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla^\perp$ is a connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X, Y), N) \quad (3.3)$$

for any $x \in M$ and $X \in T_x M$. We write

$$X = PX + QX, \quad (3.4)$$

where $PX \in D$ and $QX \in D^\perp$.

Similarly, for N normal to M we have

$$\phi N = BN + CN, \quad (3.5)$$

where BN (resp. CN) is the tangential component (resp. normal component) of ϕN .

Lemma 3.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$, then

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y]$$

for all $X, Y \in D$.

Proof. By Gauss formula (3.1), we have

$$\bar{\nabla}_X Y - \bar{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y]. \quad (3.6)$$

Also, by covariant differentiation we get

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X, Y]. \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y]. \quad (3.8)$$

Adding (2.6) and (3.8), we get

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y]$$

for all $X, Y \in D$.

Hence Lemma is proved.

Lemma 3.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$, then

$$2(\bar{\nabla}_X \phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y^\perp \phi X - \phi[X, Y] \quad (3.9)$$

for all $X, Y \in D^\perp$.

Proof. By Weingarten formula (3.2), we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y^\perp \phi X$$

for any $X, Y \in D^\perp$.

Also, by covariant differentiation, we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X, Y]. \quad (3.10)$$

From (3.9) and (3.10), we get

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y^\perp \phi X - \phi[X, Y]. \quad (3.11)$$

Adding (2.6) and (3.11), we get

$$2(\bar{\nabla}_X \phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y^\perp \phi X - \phi[X, Y]$$

for all $X, Y \in D^\perp$.

Hence Lemma is proved.

Lemma 3.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then

$$2(\bar{\nabla}_X \phi)Y = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]$$

for all $X \in D$ and $Y \in D^\perp$.

Proof. Using Gauss and Weingarten formulas, we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X). \quad (3.12)$$

Also, by covariant differentiation, we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X, Y]. \quad (3.13)$$

From (3.12) and (3.13), we get

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]. \quad (3.14)$$

Adding (2.6) and (3.14), we obtain

$$2(\bar{\nabla}_X \phi)Y = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]$$

for all $X \in D$ and $Y \in D^\perp$.

Hence Lemma is proved.

Lemma 3.4. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y = \phi P(\nabla_X Y) + \phi P(\nabla_Y X), \quad (3.15)$$

$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - QA_{\phi QY}X - QA_{\phi QX}Y = 2Bh(X, Y), \quad (3.16)$$

$$\begin{aligned} h(Y, \phi PX) + h(X, \phi PY) + \nabla_X^\perp \phi QY + \nabla_Y^\perp \phi QX = & -2\eta(Y)\phi QX - 2\eta(X)\phi QY + \phi Q(\nabla_X Y) \\ & + \phi Q(\nabla_Y X) + 2Ch(X, Y), \end{aligned} \quad (3.17)$$

$$\eta(\nabla_X \phi PY) + \eta(\nabla_Y \phi PX) - \eta(A_{\phi QY}X) - \eta(A_{\phi QX}Y) = 0 \quad (3.18)$$

for any $X, Y \in T\bar{M}$.

Proof. Differentiating (3.4) covariantly and using (3.1) and (3.2), we get

$$\bar{\nabla}_Y \phi X = (\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(X, Y).$$

Also,

$$\begin{aligned} \bar{\nabla}_Y \phi X = & P\nabla_X \phi PY + Q\nabla_Y \phi PX + \eta(\nabla_Y \phi PX)\xi + h(Y, \phi PX) + \nabla_{\phi QX}^\perp Y \\ & - PA_{\phi QX}Y - QA_{\phi QX}Y - \eta(A_{\phi QX}Y)\xi. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(X, Y) = & P\nabla_X \phi PY + Q\nabla_Y \phi PX + \eta(\nabla_Y \phi PX)\xi \\ & + h(Y, \phi PX) + \nabla_{\phi QX}^\perp Y - PA_{\phi QX}Y - QA_{\phi QX}Y - \eta(A_{\phi QX}Y)\xi. \end{aligned} \quad (3.19)$$

Interchanging X and Y , we have

$$\begin{aligned} (\bar{\nabla}_X \phi)Y + \phi(\nabla_X Y) + \phi h(Y, X) = & P\nabla_Y \phi PX + Q\nabla_X \phi PY + \eta(\nabla_X \phi PY)\xi \\ & + h(X, \phi PY) + \nabla_{\phi QY}^\perp X - PA_{\phi QY}X - QA_{\phi QY}X - \eta(A_{\phi QY}X)\xi. \end{aligned} \quad (3.20)$$

Adding (3.19) and (3.20), we have

$$\begin{aligned} (\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi(\nabla_X Y) + 2\phi h(X, Y) = & P\nabla_X \phi PY + P\nabla_Y \phi PX \\ & + Q\nabla_Y \phi PX + Q\nabla_X \phi PY + \eta(\nabla_Y \phi PX)\xi + \eta(\nabla_X \phi PY)\xi + h(Y, \phi PX) + h(X, \phi PY) \\ & + \nabla_{\phi QX}^\perp Y + \nabla_{\phi QY}^\perp X - PA_{\phi QX}Y - PA_{\phi QY}X - QA_{\phi QX}Y - QA_{\phi QY}X - \eta(A_{\phi QX}Y)\xi \\ & - \eta(A_{\phi QY}X)\xi. \end{aligned} \quad (3.21)$$

By virtue of (2.6) and (3.21), we obtain

$$\begin{aligned} \phi P(\nabla_Y X) + \phi P(\nabla_X Y) + \phi Q \nabla_Y X + \phi Q \nabla_X Y + 2Bh(X, Y) + 2Ch(X, Y) &= P \nabla_Y \phi P X \\ &+ P \nabla_X \phi P Y + Q \nabla_Y \phi P X + Q \nabla_X \phi P Y + \eta(\nabla_Y \phi P X) \xi + \eta(\nabla_X \phi P Y) \xi + h(Y, \phi P X) \\ &+ h(X, \phi P Y) + \nabla_{\phi Q X}^\perp Y + \nabla_{\phi Q Y}^\perp X - P A_{\phi Q X} Y - P A_{\phi Q Y} X - Q A_{\phi Q X} Y - Q A_{\phi Q Y} X \\ &- \eta(A_{\phi Q X} Y) \xi - \eta(A_{\phi Q Y} X) \xi. \end{aligned}$$

Comparing horizontal, vertical and normal components we get the desired result.

Hence the Lemma is proved.

Definition 3.5. The horizontal distribution D is said to be parallel [10] on M if $\nabla_X Y \in D$ for all vector field $X, Y \in D$.

Theorem 3.6. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. If the horizontal distribution D is parallel, then

$$h(X, \phi Y) = h(Y, \phi X)$$

for all $X, Y \in D$.

Proof. Let $X, Y \in D$ and D is parallel then $\nabla_X \phi Y \in D$ and $\nabla_Y \phi X \in D$. From (3.12), we have

$$h(Y, \phi X) + h(X, \phi Y) = 2\phi h(X, Y). \quad (3.22)$$

Replacing X by ϕX in (3.22) and using (2.1), we get

$$h(Y, X) + h(\phi X, \phi Y) = 2\phi h(\phi X, Y). \quad (3.23)$$

Again replacing Y by ϕY in (3.22) and using (2.1), we get

$$h(\phi Y, \phi X) + h(X, Y) = 2\phi h(X, \phi Y). \quad (3.24)$$

By virtue of (3.23) and (3.24), we have

$$\phi h(\phi X, Y) = \phi h(X, \phi Y). \quad (3.25)$$

Operating ϕ on both sides of (3.25), we get

$$h(\phi X, Y) = h(X, \phi Y).$$

Hence the theorem is proved.

Definition 3.7. A semi-invariant submanifold is said to be mixed totally geodesic [8] if $h(X, Y) = 0$ for all $X \in D$ and $Y \in D^\perp$.

Theorem 3.8. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then M is a mixed totally geodesic if and only if $A_{NX} \in D$ for all $X \in D$.

Proof. Let $A_{NX} \in D$ for all $X \in D$.

Now, $g(h(X, Y), N) = g(A_{NX}, Y) = 0$ for $Y \in D^\perp$, which is equivalent to $h(X, Y) = 0$.

Hence M is totally mixed geodesic.

Conversely, Let M is totally mixed geodesic, that is $h(X, Y) = 0$ for $X \in D$ and $Y \in D^\perp$.

Now, $g(h(X, Y), N) = g(A_{NX}, Y)$ gives that $g(A_{NX}, Y) = 0$. Consequently, we have $A_{NX} \in D$ for all $Y \in D^\perp$. Hence the theorem is proved.

IV. INTEGRABILITY CONDITIONS FOR DISTRIBUTIONS

Theorem 4.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then the distribution $D \oplus \langle \xi \rangle$ is integrable if

$$h(\phi X, Z) + h(\phi Z, X) = 0 \quad (4.1)$$

for any $X, Y, Z \in D \oplus \langle \xi \rangle$.

Proof. The torsion tensor $S(X, Y)$ of an almost contact structure (ϕ, ξ, η, g) Using Gauss formula in (2.12), we have

$$N(X, Y) = 4\phi(\nabla_Y \phi X - \phi \nabla_Y X - \phi h(Y, X) + h(Y, \phi X)). \quad (4.2)$$

If $D \oplus \langle \xi \rangle$ is integrable, then $N(X, Y) = 0$. Hence

$$4\phi \nabla_Y \phi X - 4\nabla_Y X - 4\eta(\nabla_Y X)\xi - 4h(Y, X) + 4\phi h(Y, \phi X) = 0. \quad (4.3)$$

Comparing normal parts from both sides of (4.3), we get

$$\phi Q(\nabla_Y \phi X) - h(Y, X) = Ch(Y, \phi X) = 0. \quad (4.4)$$

Replacing Y by $\phi Z, Z \in D$ in (4.4), we have

$$\phi Q(\nabla_{\phi Z} \phi X) - h(\phi Z, X) + Ch(\phi Y, \phi X) = 0. \quad (4.5)$$

Interchanging X and Z in (4.5), we obtain

$$\phi Q(\nabla_{\phi X} \phi Z) - h(\phi X, Z) + Ch(\phi X, \phi Y) = 0. \quad (4.6)$$

Subtracting (4.6) from (4.5), we get

$$\phi Q[\phi X, \phi Z] - h(\phi X, Z) + h(\phi Z, X) = 0. \quad (4.7)$$

Since $D \oplus \langle \xi \rangle$ is integrable so that $[\phi X, \phi Z] \in D \oplus \langle \xi \rangle$ for $X, Z \in D$. Consequently (4.7) gives

$$h(\phi X, Z) + h(\phi Z, X) = 0.$$

Hence the theorem is proved.

Proposition 4.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then

$$A_{\phi Y} Z - A_{\phi Z} Y = \phi P[Y, Z]$$

for any $Y, Z \in D^\perp$.

Proof. Let $X \in \chi(M)$ and $Y, Z \in D^\perp$. From (3.1) and (3.2), we have

$$2g(A_{\phi Z}Y, X) = g(h(Y, X), \phi Z) + g(h(X, Y), \phi Z).$$

Using (3.3) and (3.1), we get

$$2g(A_{\phi Z}Y, X) = -g(\bar{\nabla}_Y \phi X, Z) - g(\bar{\nabla}_X \phi Y, Z) + g((\bar{\nabla}_Y \phi)X + (\bar{\nabla}_X \phi)Y, Z).$$

Using (2.6) in above equation, we have

$$2g(A_{\phi Z}Y, X) = -g(\phi \bar{\nabla}_Y Z, X) + g(A_{\phi Y}Z, X). \quad (4.8)$$

Transvecting X from both sides of (4.8), we have

$$2A_{\phi Z}Y = \phi \bar{\nabla}_Y Z + A_{\phi Y}Z.$$

Interchanging Y and Z , we have

$$2A_{\phi Y}Z = \phi \bar{\nabla}_Z Y + A_{\phi Z}Y.$$

Subtracting above two equations, we get

$$A_{\phi Z}Y - A_{\phi Y}Z = \frac{1}{3}\phi[Y, Z].$$

Comparing the tangential parts from both sides of above equation, we get

$$A_{\phi Z}Y - A_{\phi Y}Z = \frac{1}{3}\phi P[Y, Z], \quad (4.9)$$

where $[Y, Z]$ is Lie bracket.

Hence the proposition is proved.

Theorem 4.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then the distribution $D^\perp$ is integrable if and only if

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 \quad (4.10)$$

for all $Y, Z \in D^\perp$.

Proof. Suppose that the distribution $D^\perp$ is integrable. Then $[Y, Z] \in D^\perp$ for any $Y, Z \in D^\perp$. Therefore, $P[Y, Z] = 0$. From (4.9), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = 0.$$

Conversely, let (4.10) holds good. Then by virtue of (4.9), we get

$$\phi P[Y, Z] = 0$$

for all $Y, Z \in D^\perp$. Since $\text{rank } \phi = 2n$, therefore, either $P[Y, Z] = 0$ or $P[Y, Z] = k\xi$.

But $P[Y, Z] = k\xi$ is not possible as P being a projection operator on D . Hence $P[Y, Z] = 0$.

This implies that $[Y, Z] \in D^\perp$ for all $Y, Z \in D^\perp$. Thus $D^\perp$ is integrable.

Hence the theorem is proved.

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Strictly Practical Stabilization of Impulsive Functional Differential Equations by using Lyapunov Functions

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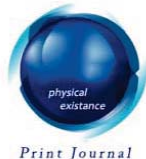
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1. Dilbaj Singh, S.K. Srivastava, Stability criteria of impulsive differential equations with comparison results, *Advances in Differential Equations and Control Processes*, Vol. 10, pp. 171-182, 2012.

Strictly Practical Stabilization of Impulsive Functional Differential Equations by using Lyapunov Functions

Sapna Rani ^α & Dilbaj Singh ^σ

Abstract - In this paper, we extend the concept of strict practical stability to impulsive functional differential equations by using Lyapunov functions and Razumikhin technique. As practical stability does not give us much information about the rate of decay of solution so we develop the idea for strict practical stability of functional differential equations with impulsive effect and obtained some conditions for strict practical uniform stability for functional differential equations with impulse by using piecewise continuous Lyapunov functions and Razumikhin technique.

Keywords : *strict practical stability, impulsive differential equations, lyapunov function.*

1. INTRODUCTION

Various physical processes undergo abrupt changes of state at certain moments of time between intervals of continuous equation. The duration of these changes is often negligible in comparison with that of the entire evolution process and thus the abrupt changes can be well approximated in terms of the instantaneous changes of state i.e. Impulses [7]. On the other hand Functional differential equations are important in scientific and technical professions and they are used to represent a rate of change of time varying phenomenon [5]. When both functional differential equations and Impulses are Involved, Impulsive functional differential system becomes a natural framework of mathematical modelling of varying physical phenomenon [5]. Impulsive functional differential systems are different from ordinary differential systems in the sense that the state undergo abrupt changes at certain moments and the derivation of the state variable depends not only on present state but also on past state. Stability is highly important in all physical application [1,2,3]. A stable equilibrium represents a behaviour usually which cannot be changed. Several stability criteria are obtained by many authors which shows that impulses do contribute to the stabilization of functional differential equations [2,3,5].

Theory of stability in the sense of Lyapunov is now well known and is widely used in concrete problems of real world. The desirable feature is to know the size of region of stability so that we can judge whether or not a given system is sufficiently stable to function properly and may be able to see how to improve its stability. On the other hand the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. So we need a notion of stability that is more suitable than stability and such concept is practical stability [4,6,8,9].

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In this paper we establish stability result which provides sufficient conditions to maintain uniform strict practical stability of the trivial solution of a functional differential equation with impulse. The Lyapunov's second method of functions called Lyapunov function is employed in this work.

This paper is organized as: In section 2, we introduce some basic definitions and notations. In section 3, based upon Lyapunov functions and Razumikhin method, some conditions for strict uniform practical stability are obtained. Finally in section 4, some concluding remarks are given.

II. PRELIMINARIES

Consider the following impulsive functional differential system:

$$\dot{x}(t) = f(t, x_t), \quad t \geq t_0, \quad t \neq \tau_k \quad (2.1)$$

$$\Delta x(t) = x(\tau_k) - x(\tau_k^-) = I_k(x(\tau_k^-)), \quad t = \tau_k, \quad k = 1, 2, \dots$$

Where $f \in C([R^+ \times D, R^n])$, D is an open set in $PC([- \tau, 0], R^n)$, where $\tau > 0$, $PC([- \tau, 0], R^n) = \{ \varphi : [- \tau, 0] \rightarrow R^n \}$, $\varphi(t)$ is continuous everywhere except a finite number of points $\hat{t}$ at which $\varphi(\hat{t}^+)$ and $\varphi(\hat{t}^-)$ exist and $\varphi(\hat{t}^+) = \varphi(\hat{t}^-)$. $f(t, 0) = 0$ for all $t \in R^+$, $I_k(0) = 0$, for all $k \in \mathbb{Z}$, $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_k \rightarrow \infty$ for $k \rightarrow \infty$ and $x(t^+) = \lim_{s \rightarrow t^+} x(s)$, $x(t^-) = \lim_{s \rightarrow t^-} x(s)$. For each $t \geq t_0$, $x_t \in D$ is defined by $x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$. For $\varphi \in PC([- \tau, 0], R^n)$, $\|\varphi\|_1$ is defined by $\|\varphi\|_1 = \sup_{-\tau \leq s \leq 0} \|\varphi\|$, $\|\varphi\|_2$ is defined by $\|\varphi\|_2 = \inf_{-\tau \leq s \leq 0} \|\varphi\|$. We can see that $x(t) = 0$ is a solution of (2.1) which we call the zero solution or trivial solution.

For a given $\sigma \geq 0$ and $\varphi \in PC([- \tau, 0], R^n)$, the initial value problem of equation (2.1) is

$$\dot{x}(t) = f(t, x_t), \quad \sigma \geq t_0, \quad t \neq \tau_k \quad (2.2)$$

$$\Delta x(t) = x(\tau_k) - x(\tau_k^-) = I_k(x(\tau_k^-)), \quad t = \tau_k, \quad k = 1, 2, \dots \text{ Where } x_\sigma = \varphi.$$

Throughout this paper we let the following hypothesis hold:

(H₁) For each function $x(s) : [\sigma - \tau, \infty] \rightarrow R^n$, $\sigma \geq t_0$, which continuous everywhere except a finite number of points τ_k at which $x(\tau_k^+)$ and $x(\tau_k^-)$ exist and $x(\tau_k^+) = x(\tau_k^-)$, where $f(t, x_t)$ is continuous for almost all $t \in [\sigma, \infty)$ and at the discontinuous f is the right continuous.

(H₂) $f(t, \varphi)$ is lipschitzian in φ in each compact set in $PC([- \tau, 0], R^n)$.

(H₃) The functions $I_k : R^n \rightarrow R^n$, $k = 1, 2, 3, \dots$ are such that for any $H > 0$, there exist a $\rho > 0$ such that if,

$x \in s(\rho) = \{x \in R^n : \|x\| < \rho\}$ implies that $\|x + I_k(x)\| < H$.

Under these hypothesis a unique solution of problem (2.2) exist throughout (σ, φ) .

Let

$K = \{a \in [R^+, R^+] : a(t) \text{ is monotone strictly increasing and } a(0) = 0\}$

$$K_1 = \{w \in [R^+, R^+]: w(t) \in K \text{ and } 0 < w(s) < s, s > 0\}$$

$$PC_1(\rho) = \{\varphi \in PC([- \tau, 0], R^n): \|\varphi\|_1 < \rho\}$$

$$PC_2(\theta) = \{\varphi \in PC([- \tau, 0], R^n): \|\varphi\|_2 > \theta\}$$

We have the following definitions:

Definition: The trivial solution of (2.1) is said to be

(A₁) Strict practical stable, if for any $\sigma \geq t_0$ There exist (λ_1, A_1) , $\lambda_1 \leq A_1$ such that $\varphi \in PC_1(\lambda_1)$ implies $\|x(t; \sigma, \varphi)\| < A_1$, $t \geq \sigma$, and for every $0 < \lambda_2 \leq \lambda_1$, there exist $0 < A_2 \leq \lambda_2$ such that $\varphi \in PC_2(\lambda_2)$ implies $\|x(t; \sigma, \varphi)\| > A_2$, $t \geq \sigma$.

(A₂) Strict Practically Uniformly Stable, if (A₁) holds for all $t \in R^+$.

Definition: The function $V: [t_0, \infty) \times S(\rho) \rightarrow R^+$ belongs to class v_0 if

- I. The function V is continuous on each of the sets $[\tau_{k-1}, \tau_k) \times S(\rho)$ and for all, $t \geq t_0$, $V(t, 0) = 0$.
- II. $V(t, x)$ is locally lipschitzian in $x \in S(\rho)$.
- III. For each $k = 1, 2, \dots$ there exist finite limits.

$$\lim_{(t,y) \rightarrow (\tau_k^-, x)} V(t, y) = V(\tau_k^-, x)$$

$$\lim_{(t,y) \rightarrow (\tau_k^+, x)} V(t, y) = V(\tau_k^+, x)$$

With $V(\tau_k^+, x) = V(\tau_k, x)$ satisfied.

Definition: Let $V \in v_0$, for $(t, x) \in [\tau_{k-1}, \tau_k) \times S(\rho)$, D^+V is defined as:

$$D^+V(t, x(t)) = \lim_{\delta \rightarrow 0^+} \sup \frac{1}{\delta} \{V(t + \delta, x(t + \delta)) - V(t, x(t))\}.$$

IV. MAIN RESULT

Now we consider the strict practical stability of the Impulsive functional differential equation (2.1) with following results:

Theorem: Assume that

- (i) There exist (λ_1, A_1) , $0 < \lambda_1 \leq A_1$ and $V_1 \in v_0$, such that $b_1(\|x(t)\|) \leq V_1(t, x(t)) \leq a_1(\|x(t)\|)$, $a_1, b_1 \in K$
- (ii) For any solution $x(t)$ of (2.1), $V_1(t + s, x(t + s)) \leq V_1(t, x(t))$ for $s \in [-\tau, 0]$, implies that

$$D^+V_1(t, x(t)) \leq 0.$$

Also for all $K \in Z^+$ and $x \in S(\rho)$

$$V_1(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1 + d_k)V_1(\tau_k, x(\tau_k^-)), \text{ where } d_k \geq 0 \text{ and } \sum_{k=1}^{\infty} d_k < \infty$$

- (iii) For any $0 < \lambda_2 \leq \lambda_1$ and $V_2 \in v_0$

$$b_2(\|x(t)\|) \leq V_2(t, x(t)) \leq a_2(\|x(t)\|), \quad a_2, b_2 \in K$$

(iv) For any solution $x(t)$ of (2.1), $V_2(t + s, x(t + s)) \geq V_2(t, x(t))$ for $s \in [-\tau, 0]$, implies

$$D^+V_2(t, x(t)) \geq 0.$$

Also for all $K \in \mathbb{Z}^+$ and $x \in S(\rho)$

$$V_2(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \geq (1 - c_k) V_2(\tau_k, x(\tau_k^-)),$$

Where $0 \leq c_k < 1$ and $\sum_{k=1}^{\infty} c_k < \infty$

Then the trivial solution of (2.1) is strict practical uniformly stable.

Proof: Since $\sum_{k=1}^{\infty} d_k < \infty$ and $\sum_{k=1}^{\infty} c_k < \infty$.

It follows that, $\prod_{k=1}^{\infty} (1 + d_k) = M$ and $\prod_{k=1}^{\infty} (1 - c_k) = N$, Obviously $1 \leq M < \infty$, $0 < N \leq 1$

Let $0 < A_1 < \rho$ and $\sigma \geq t_0$ be given and $\sigma \in [\tau_k, \tau_{k+1}]$ for some $k \in \mathbb{Z}$, Such that $Ma_1(\lambda_1) < b_1 A_1$

Then we claim that $\varphi \in PC_1(\lambda_1)$ implies $\|\varphi(t)\| < A_1$, $t \geq \sigma$

Obviously for any $t \in [\tau, \sigma]$, there exist $\theta \in [\tau, 0]$, such that

$$V_1(t, x(t)) = V_1(\sigma + \theta, x(\sigma + \theta)) \leq a_1(\|x(\sigma + \theta)\|) =$$

$$a_1\|x_\sigma(\theta)\| = a_1\|\varphi(\theta)\| \leq a_1(\lambda_1)$$

Then, we claim that

$$V_1(t, x(t)) \leq a_1, \quad \sigma \leq t < \tau_k \quad (3.1)$$

If the inequality (3.1), does not hold, then there exist a $\hat{t} \in (\sigma, \tau_k)$ such that

$$V_1(\hat{t}, x(\hat{t})) > a_1(\lambda_1) \geq V_1(\sigma, x(\sigma))$$

which implies that there exist a $\check{t} \in (\sigma, \hat{t}]$, such that

$$D^+V_1(\check{t}, x(\check{t})) > 0 \quad (3.2)$$

and

$$V_1(\check{t} + s, x(\check{t} + s)) \leq V_1(\hat{t}, x(\hat{t})), \text{ where } s \in [-\tau, 0]$$

by condition (ii), which implies that $D^+V_1(\check{t}, x(\check{t})) \leq 0$. This contradicts inequality (3.2)

So inequality (3.1) holds.

From condition (ii), we have

$$V_1(\tau_k, x(\tau_k)) = V_1(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1 + d_k) V_1(\tau_k, x(\tau_k^-)) \leq (1 + dk) a_1(\lambda_1)$$

Next, we claim that

$$V_1(t, x(t)) \leq (1 + dk) a_1(\lambda_1), \quad \tau_k \leq t \leq \tau_{k+1} \quad (3.3)$$

If inequality (3.3) does not hold then, there exist $\hat{s} \in (\tau_k, \tau_{k+1})$, such that

$$V_1(\hat{s}, x(\hat{s})) \geq (1 + d_k) a_1(\lambda_1) \geq V_1(\tau_k, x(\tau_k))$$

which implies that there exist an $\check{s} \in (\tau_k, \tau_{k+1})$, such that

$$D^+V_1(\check{s}, x(\check{s})) > 0 \quad (3.4)$$

And

$$V_1(\check{s} + s, x(\check{s} + s)) \leq V_1(\hat{s}, x(\hat{s})), \text{ where } s \in [-\tau, 0]$$

by condition (ii), which implies that $D^+V_1(\hat{s}, x(\hat{s})) \leq 0$, This Contradicts inequality (3.4) so inequality (3.3) holds.

And from condition (ii), we have

$$\begin{aligned} V_1(\tau_{k+1}, x(\tau_{k+1})) &= V_1(\tau_{k+1}, x(\tau_{k+1}^-) + I_k(x(\tau_{k+1}^-))) \leq (1 + \\ d_k) V_1(\tau_{k+1}, x(\tau_{k+1}^-)) &\leq (1 + d_k) (1 + d_k) a_1(\lambda_1). \end{aligned}$$

By a simple induction, we can easily prove that in general form for, $m = 0, 1, 2, 3, \dots$

$$V_1(t, x(t)) \leq (1 + d_{k+m}) \dots \leq (1 + d_k a_1(\lambda_1)), \text{ Where } \tau_{k+m} \leq t \leq \tau_{k+m+1}$$

Which together with inequality (3.1) and condition (i), we have

$$b_1(\|x(t)\|) \leq V_1(t, x(t)) \leq M a_1(\lambda_1) < b_1(A_1), t \geq \sigma$$

Thus we have

$$\|x(t)\| < A_1, t \geq \sigma$$

Now, let $0 < \lambda_2 \leq \lambda_1$ and choose $0 < A_2 < \lambda_2$, such that $a_2(A_2) < N b_2(\lambda_2)$.

Next, we claim that $\phi \in PC_2(\lambda_2)$ implies $\|\phi(t)\| > A_2, t \geq \sigma$

If it holds, then $\phi \in PC_1(\lambda_1) \cap PC_2(\lambda_2)$ implies

$$A_2 < \|\phi(t)\| < A_1, t \geq \sigma$$

Obviously for any, $t \in [\sigma - \tau, \sigma]$, there exist a $\theta \in [-\tau, 0]$, such that

$$\begin{aligned} V_2(t, x(t)) &= V_2(\sigma + \theta, x(\sigma + \theta)) \geq b_2(\|x(\sigma + \theta)\|) = b_2(\|\phi(\theta)\|) \\ &= b_2(\|\phi(\theta)\|) \geq b_2(\lambda_2) \end{aligned}$$

Then, we claim that

$$V_2(t, x(t)) \geq b_2(\lambda_2), \sigma \leq t < \tau_k \quad (3.5)$$

If inequality (3.5) does not hold, then there exist $\bar{t} \in (\sigma, \tau_k)$, such that

$$V_2(\bar{t}, x(\bar{t})) < b_2(\lambda_2) \leq V_2(\sigma, x(\sigma))$$

Which implies that there exist a $t_1 \in (\sigma, \bar{t})$, such that

$$D^+V_2(t_1, x(t_1)) < 0 \quad (3.6)$$

And

$$V_2(t_1 + s, x(t_1 + s)) \geq V_2(t_1, x(t_1)), \quad s \in [-\tau, 0]$$

By condition (iv), we have, $D^+V_2(t_1, x(t_1)) \geq 0$, This contradicts inequality (3.6) So, inequality (3.5) holds.

From condition (iv), we have

$$V_2(\tau_k, x(\tau_k)) = V_2(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \geq (1 - c_k) V_2(\tau_k, x(\tau_k^-)) \geq (1 - c_k) b_2(\lambda_2)$$

Next, we claim that

$$V_2(t, x(t)) \geq (1 - c_k) b_2(\lambda_2), \quad \tau_k \leq t < \tau_{k+1} \quad (3.7)$$

If the inequality (3.7) does not hold then there exist an $\bar{r} \in (\tau_k, \tau_{k+1})$, such that

$$V_2(\bar{r}, x(\bar{r})) < (1 - c_k) b_2(\lambda_2) \leq V_2(\tau_k, x(\tau_k))$$

Which implies that there exist an $\check{r} \in (\tau_k, \bar{r})$, such that

$$D^+V_2(\check{r}, x(\check{r})) < 0 \quad (3.8)$$

And

$$V_2(\check{r} + s, x(\check{r} + s)) \geq V_2(\check{r}, x(\check{r})), \quad s \in [-\tau, 0]$$

By condition (iv), we have, $D^+V_2(\check{r}, x(\check{r})) \geq 0$. Which contradicts inequality (3.8) So, inequality (3.7) holds

And from condition (iv), we have

$$\begin{aligned} V_2(\tau_{k+1}, x(\tau_{k+1})) &= V_2(\tau_{k+1}, x(\tau_{k+1}^-) + I_k(x(\tau_{k+1}^-))) \geq (1 - c_k) V_2(\tau_{k+1}, x(\tau_{k+1}^-)) \\ &\geq (1 - c_{k+1})(1 - c_k) b_2(\lambda_2). \end{aligned}$$

And by a simple induction we can prove that, in general, for, $m = 0, 1, 2, 3, \dots$

$$V_2(t, x(t)) \geq (1 - c_{k+m}) \dots \dots \dots \leq (1 - c_k) b_2(\lambda_2), \quad \tau_{k+m} \leq t \leq \tau_{k+m+1}$$

Which together with inequality (3.5) and condition (iii), we have

$$a_2(\|x(t)\|) \leq V_2(t, x(t)) \leq N b_2(\lambda_2) > a_2(A_2), \quad t \geq \sigma$$

Thus we have, $\|x(t)\| > A_2, t \geq \sigma$

Thus the zero solution of (2.1) is strict practical uniformly stable.

The proof of theorem is complete.

IV. CONCLUSION

In this paper, we investigated the strict practical stability criteria in the form of theorem for impulsive functional differential equations, which is more useful as compared to practical stability. It gives rate of decay of the solution, so it is finer concept which can give us more precise information. In future we can modify this theorem to get less restricted conditions to verify strict practical stability.

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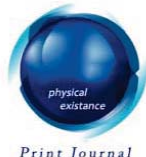
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GJSFR-F Classification : 2010 MSC No: 33C60 , 33C70 , 33D15 , 33D50 , 33D60.



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Computation of a Summation Formula Clung To Recurrence Relation

Salahuddin^α, M. P. Chaudhary^σ & Upendra Kumar Pandit^ρ

Abstract - In this paper we have established a summation formula clung to contiguous relation and recurrence relation.

Keywords : gaussian hypergeometric function, contiguous function, recurrence relation of gamma function, bailey summation theorem and legendre duplication formula.

I. BASIC INTRODUCTION

The Pochhammer symbol or **generalized factorial function** or **shifted factorial** or **falling factorial** is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & ; n = 0 \\ a(a+1)(a+2)\dots(a+n-1) & ; n = 1, 2, 3, \dots \end{cases} \quad (1)$$

where $a \neq 0, -1, -2, \dots$ and the notation Γ stands for Gamma function.

$$(b)_{-n} = \frac{\Gamma(b-n)}{\Gamma(b)} = \frac{(-1)^n}{(1-b)_n} ; \quad (2)$$

where $b \neq \dots -3, -2, -1, 0, 1, 2, 3, \dots$ and $n = 1, 2, 3, \dots$

If $m = 1, 2, 3, 4, \dots$ and $n = 0, 1, 2, 3, \dots$, then

$$(b)_{mn} = m^{mn} \left(\frac{b}{m}\right)_n \left(\frac{b+1}{m}\right)_n \dots \left(\frac{b+m-2}{m}\right)_n \left(\frac{b+m-1}{m}\right)_n \quad (3)$$

Generalized Gaussian hypergeometric function of one variable is followed as

$${}_A F_B \left[\begin{matrix} a_1, a_2, \dots, a_A & ; \\ b_1, b_2, \dots, b_B & ; \end{matrix} \quad z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_A)_k z^k}{(b_1)_k (b_2)_k \dots (b_B)_k k!}$$

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or

$${}_A F_B \left[\begin{matrix} (a_A) & ; \\ (b_B) & ; \end{matrix} \middle| z \right] \equiv {}_A F_B \left[\begin{matrix} (a_j)_{j=1}^A & ; \\ (b_j)_{j=1}^B & ; \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \quad (4)$$

where the parameters $b_1, b_2, \dots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

Contiguous Relation[E. D. p.51(10), Andrews p.363(9.16)] is defined as follows

$$(a-b) {}_2F_1 \left[\begin{matrix} a, b & ; \\ c & ; \end{matrix} \middle| z \right] = a {}_2F_1 \left[\begin{matrix} a+1, b & ; \\ c & ; \end{matrix} \middle| z \right] - b {}_2F_1 \left[\begin{matrix} a, b+1 & ; \\ c & ; \end{matrix} \middle| z \right] \quad (5)$$

Recurrence relation of gamma function is defined as follows

$$\Gamma(z+1) = z \Gamma(z) \quad (6)$$

Legendre duplication formula[Bells & Wong p.26(2.3.1)] is defined as follows

$$\sqrt{\pi} \Gamma(2z) = 2^{(2z-1)} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \quad (7)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)} \Gamma\left(\frac{b}{2}\right) \Gamma\left(\frac{b+1}{2}\right)}{\Gamma(b)} \quad (8)$$

$$= \frac{2^{(a-1)} \Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{a+1}{2}\right)}{\Gamma(a)} \quad (9)$$

Bailey summation theorem [Prudnikov, p.491(7.3.7.8)] is defined as follows

$${}_2F_1 \left[\begin{matrix} a, 1-a & ; \\ c & ; \end{matrix} \middle| \frac{1}{2} \right] = \frac{\Gamma\left(\frac{c}{2}\right) \Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\frac{c+a}{2}\right) \Gamma\left(\frac{c+1-a}{2}\right)} = \frac{\sqrt{\pi} \Gamma(c)}{2^{c-1} \Gamma\left(\frac{c+a}{2}\right) \Gamma\left(\frac{c+1-a}{2}\right)} \quad (10)$$

II. MAIN RESULT OF SUMMATION FORMULA

$$\begin{aligned} & {}_2F_1 \left[\begin{matrix} a, & -a-51 & ; \\ c & & ; \end{matrix} \middle| \frac{1}{2} \right] \\ &= \frac{\sqrt{\pi} \Gamma(c)}{2^{c+51}} \left[\frac{1}{\Gamma\left(\frac{c-a}{2}\right) \Gamma\left(\frac{c+a+51}{2}\right)} \right] \left\{ +3921562936321638831406344160935936000000 \right. \\ &\quad \left. -7402156813239853267416133723756953600000a \right. \\ &\quad \left. +4629598150639117497437482135995432960000a^2 \right\} \end{aligned}$$

$$\begin{aligned}
& -1320609504346225933240581617829869568000a^3 \\
& +181193044645895982439149503813419315200a^4 \\
& -9311856861379155575957484232594867200a^5 \\
& -319305701534575139018589169352966400a^6 + 43290665140850487112626960992505600a^7 \\
& +145952653265054471925093988984000a^8 - 87719174108599202375406755755200a^9 \\
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\end{aligned}$$

$$\begin{aligned}
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\end{aligned}$$

$$\begin{aligned}
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\end{aligned}$$

$$\begin{aligned}
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& - 6067926395459215226369951249203200a^3c^8 \\
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& - 331378512757105965465600a^8c^{10} - 1597994314386127257600a^9c^{10} \\
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& - 12797219635200a^{13}c^{10} - 35846553600a^{14}c^{10} + 2350559102017734118725983508889600c^{11} \\
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& + 16508310593497412701519872a^5c^{11} + 29185648162482631386857472a^6c^{11} \\
& + 139415419763233167310848a^7c^{11} - 6872246298970496630784a^8c^{11}
\end{aligned}$$

$$\begin{aligned}
& -68461810010010353664a^9c^{11} + 313126051801006080a^{10}c^{11} + 5905660917252096a^{11}c^{11} \\
& + 14157389955072a^{12}c^{11} - 46535344128a^{13}c^{11} - 130351104a^{14}c^{11} \\
& + 140753672802031073521882365952000c^{12} - 26142919734994299344681277849600ac^{12} \\
& + 1161846332581787801054923980800a^2c^{12} + 19816175844567736562968166400a^3c^{12} \\
& - 1514680816526070538928128000a^4c^{12} - 13007910157465601064960000a^5c^{12} \\
& + 667491473107756064768000a^6c^{12} + 7452079025079848140800a^7c^{12} \\
& - 83890949279322931200a^8c^{12} - 1363318831300608000a^9c^{12} - 1212846415872000a^{10}c^{12} \\
& + 44208576921600a^{11}c^{12} + 144472473600a^{12}c^{12} + 7033255557317565417731034972160c^{13} \\
& - 1062714470690285719045451808768ac^{13} + 32779414532156325397071396864a^2c^{13} \\
& + 1002269237403965038709440512a^3c^{13} - 35211188622667072624394240a^4c^{13} \\
& - 593591726750187523276800a^5c^{13} + 10030581135956355645440a^6c^{13} \\
& + 182420416817402609664a^7c^{13} - 343475889878532096a^8c^{13} - 15530359717232640a^9c^{13} \\
& - 48184903925760a^{10}c^{13} + 136026390528a^{11}c^{13} + 444530688a^{12}c^{13} \\
& + 293814830383077351070105600000c^{14} - 35630582416423246085750784000ac^{14} \\
& + 687943867344091157102592000a^2c^{14} + 33783509590480862576640000a^3c^{14} \\
& - 571007455827728793600000a^4c^{14} - 15474236362054041600000a^5c^{14} \\
& + 79181320218083328000a^6c^{14} + 2727329130086400000a^7c^{14} + 5936389816320000a^8c^{14} \\
& - 97161707520000a^9c^{14} - 381026304000a^{10}c^{14} + 10259229421653434996201881600c^{15} \\
& - 981838721987742945377255424ac^{15} + 9437807196628709814566912a^2c^{15} \\
& + 831105645582877238231040a^3c^{15} - 5375071988357110169600a^4c^{15} \\
& - 269825240089991577600a^5c^{15} - 203567309276577792a^6c^{15} + 25409729750630400a^7c^{15} \\
& + 104736510443520a^8c^{15} - 259097886720a^9c^{15} - 1016070144a^{10}c^{15} \\
& + 298698289887056716890112000c^{16} - 22087743903606947866214400ac^{16} \\
& + 34749165208324354867200a^2c^{16} + 15256504595570570035200a^3c^{16} \\
& + 2601854431140249600a^4c^{16} - 3210494869241856000a^5c^{16} - 12890056359936000a^6c^{16} \\
& + 136026390528000a^7c^{16} + 666796032000a^8c^{16} + 7216309814626765703741440c^{17} \\
& - 401332490274906285539328ac^{17} - 1994694819151673819136a^2c^{17} \\
& + 207841950178055553024a^3c^{17} + 943705218464612352a^4c^{17} - 25157520819486720a^5c^{17} \\
& - 145384545976320a^6c^{17} + 320062095360a^7c^{17} + 1568931840a^8c^{17} \\
& + 143537708493187317760000c^{18} - 5795478578754119270400ac^{18} \\
& - 58868619943503462400a^2c^{18} + 2046025394552832000a^3c^{18} + 15071438176256000a^4c^{18} \\
& - 117356101632000a^5c^{18} - 767033344000a^6c^{18} + 2323444065633933721600c^{19}
\end{aligned}$$

$$\begin{aligned}
& -64925192127422398464ac^{19} - 916234973881565184a^2c^{19} + 13778265172869120a^3c^{19} \\
& + 124580748328960a^4c^{19} - 247065477120a^5c^{19} - 1614807040a^6c^{19} \\
& + 30092976877207552000c^{20} - 543645346509619200ac^{20} - 9210673653350400a^2c^{20} \\
& + 56825059737600a^3c^{20} + 557108428800a^4c^{20} + 304216262815252480c^{21} \\
& - 3200279126212608ac^{21} - 59990496772096a^2c^{21} + 108238209024a^3c^{21} + 1061158912a^4c^{21} \\
& + 2311397048320000c^{22} - 11807804620800ac^{22} - 231525580800a^2c^{22} + 12408428953600c^{23} \\
& - 20535312384ac^{23} - 402653184a^2c^{23} + 41943040000c^{24} + 67108864c^{25} \} + \\
& + \frac{1}{\Gamma(\frac{c-a+1}{2}) \Gamma(\frac{c+a+52}{2})} \left\{ -17464069942802715386614602906796032000000a \right. \\
& + 20686287355803117929950207741475389440000a^2 \\
& - 8937827735095218966400376828381233152000a^3 \\
& + 1778187523142914385364228271879685836800a^4 \\
& - 152958349994552708911714321312984796160a^5 \\
& + 845807912612769929240251601955023616a^6 \\
& + 625073884704101388272682931412331456a^7 - 16351780395261526276780909546342272a^8 \\
& - 1387715913059429859324257558607888a^9 + 29422980952667711908287591614240a^{10} \\
& + 2197272271496937147099418055796a^{11} - 9154621385702607267312543112a^{12} \\
& - 2009055895946659394444049063a^{13} - 22907465907142301442089065a^{14} \\
& + 678068948453250922512126a^{15} + 18820430104162303836518a^{16} \\
& + 69913513840255423767a^{17} - 3106032871839381055a^{18} - 48230575964749464a^{19} \\
& - 173954956930612a^{20} + 2022903890007a^{21} + 24154252265a^{22} + 87862086a^{23} + 2678a^{24} \\
& - 663a^{25} - a^{26} + 17464069942802730897824646237782016000000c \\
& - 54007367613351708706039996361205350400000ac \\
& + 41767493571718136116945131402825154560000a^2c \\
& - 13573255377452899961200986142034724864000a^3c \\
& + 2054671717945610860650665253770997657600a^4c \\
& - 118357661481661180348642083685374950400a^5c \\
& - 3187810538604850822031784475679443200a^6c \\
& + 537679455896348207294860188265593600a^7c + 215615678779331976571132002878400a^8c \\
& - 1106434442250655994775072987969600a^9c - 3900521188164301838303830970800a^{10}c \\
& + 1299352759576913233189722301200a^{11}c + 18242603793449626590118783300a^{12}c \\
& - 681341880584185413666746400a^{13}c - 19313434798930849743184200a^{14}c
\end{aligned}$$

$$\begin{aligned}
& +7547544375035398047600a^{15}c + 6009125692951825000700a^{16}c \\
& +75428625908577427200a^{17}c - 93989456930626400a^{18}c - 9694744240381200a^{19}c \\
& -81018694656900a^{20}c - 120521200800a^{21}c + 1807184600a^{22}c + 10342800a^{23}c \\
& +16900a^{24}c + 33321080257548649966218600320933560320000c^2 \\
& -61112319783077679498758852113706385408000ac^2 \\
& +35239168592443682121597364920255189811200a^2c^2 \\
& -8927594857872117885236854981968830177280a^3c^2 \\
& +1026664214177111730613351645069572913152a^4c^2 \\
& -33937321295214246160668222485847019008a^5c^2 \\
& -2718208603981464439015598061682548864a^6c^2 \\
& +173468343367992388669809486729151872a^7c^2 \\
& +4388963074547519323475859214777568a^8c^2 - 304879454658965983527375203119392a^9c^2 \\
& -7323613290082452492196274779416a^{10}c^2 + 238107120577467928543770726024a^{11}c^2 \\
& +8316109579875619326471215666a^{12}c^2 - 31210924941366650068334928a^{13}c^2 \\
& -4030678750850031042823684a^{14}c^2 - 42456192021451570839048a^{15}c^2 \\
& +434601027474585700014a^{16}c^2 + 12315977161258028544a^{17}c^2 + 74364791296267472a^{18}c^2 \\
& -350917550607624a^{19}c^2 - 6666815293138a^{20}c^2 - 28956944016a^{21}c^2 - 11176308a^{22}c^2 \\
& +206856a^{23}c^2 + 338a^{24}c^2 + 28282653946454862828385645890424012800000c^3 \\
& -37847109234486369007419794151897563136000ac^3 \\
& +17134579371035444556951735602324491468800a^2c^3 \\
& -3434873986298546756997832642768581427200a^3c^3 \\
& +294489047200929889091063975811094118400a^4c^3 \\
& -2987298299292112459085731949157580800a^5c^3 \\
& -940576167322783522108200581862758400a^6c^3 \\
& +25431434993231939895777252470553600a^7c^3 \\
& +1651317174366087827218764566988800a^8c^3 - 34397369483011709151574125734400a^9c^3 \\
& -2028663240492155118701115926400a^{10}c^3 + 8646400883231480087479051200a^{11}c^3 \\
& +1360478905870410824014753600a^{12}c^3 + 12722508323197889416200000a^{13}c^3 \\
& -312903211189702937918400a^{14}c^3 - 6976348681997275152000a^{15}c^3 \\
& -20901902309034032000a^{16}c^3 + 683604505843459200a^{17}c^3 + 7884558877395200a^{18}c^3 \\
& +20265613368000a^{19}c^3 - 149805656000a^{20}c^3 - 1061860800a^{21}c^3 - 1892800a^{22}c^3
\end{aligned}$$

$$\begin{aligned}
& +14403008496352774801189788846473084928000c^4 \\
& -15096022481936553413589608250481689231360ac^4 \\
& +5498501813068048184727758084694272114688a^2c^4 \\
& -875089957529533052576034038524658614272a^3c^4 \\
& +53643746953681259243251636646277341184a^4c^4 \\
& +745901964781113133197915990274824192a^5c^4 \\
& -184010830303513208010308817367427584a^6c^4 \\
& +818367161622811724817415382305536a^7c^4 \\
& +295069311238367211284859400869888a^8c^4 + 46878810145326952668499942656a^9c^4 \\
& -265547644209905563443206559264a^{10}c^4 - 2695516436663412096911209488a^{11}c^4 \\
& +102893805882101328651547536a^{12}c^4 + 2201913986591405648562000a^{13}c^4 \\
& -2561830314196148179184a^{14}c^4 - 455189373853931951520a^{15}c^4 \\
& -4154321981773140320a^{16}c^4 + 5748965064434592a^{17}c^4 + 290840371773952a^{18}c^4 \\
& +1529982133680a^{19}c^4 + 1104543440a^{20}c^4 - 10618608a^{21}c^4 - 18928a^{22}c^4 \\
& +4987324600822239464281096535340692275200c^5 \\
& -4231337690172994726513975846616589926400ac^5 \\
& +1254635850649267987637572392143899852800a^2c^5 \\
& -157710034258206885829138153197028147200a^3c^5 \\
& +6341912119170355136606289232861593600a^4c^5 \\
& +285724555386011051713443871587993600a^5c^5 \\
& -22516557698790506384456059080371200a^6c^5 \\
& -325958511808137888230326702617600a^7c^5 + 30409655946495926079802976524800a^8c^5 \\
& +478117034612337247952376988800a^9c^5 - 18449758131511605131140689600a^{10}c^5 \\
& -446430201516143293713264000a^{11}c^5 + 2530496214522729526673600a^{12}c^5 \\
& +157223273868315560505600a^{13}c^5 + 1120193390121640348800a^{14}c^5 \\
& -11946206978927020800a^{15}c^5 - 211771076704387200a^{16}c^5 - 790629695856000a^{17}c^5 \\
& +3422210792000a^{18}c^5 + 31855824000a^{19}c^5 + 62462400a^{20}c^5 \\
& +1258557502795465748945746499311572615168c^6 \\
& -879365308715679527902577076638293426176ac^6 \\
& +213226342068718901576910141848419172352a^2c^6 \\
& -20888741739044083190415260963003547648a^3c^6 \\
& +445350239425183456899743815003930624a^4c^6
\end{aligned}$$

$$\begin{aligned}
& +47697044249582889704241686615706624a^5c^6 \\
& -1732992323545178952756409713975808a^6c^6 \\
& -64083551616583168014156782390784a^7c^6 + 1805944697836994302445981656832a^8c^6 \\
& +63374804317943610253947686592a^9c^6 - 527451127952075305840671264a^{10}c^6 \\
& -32744884604360817430301760a^{11}c^6 - 171000547255446995262176a^{12}c^6 \\
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& +416416a^{20}c^6 + 241830426344143445812620039165050880000c^7 \\
& -140430911900394092446841334069303705600ac^7 \\
& +27834251545029924172972638770980454400a^2c^7 \\
& -2075507711455508219786606192335257600a^3c^7 \\
& +7224541537977508352876554072883200a^4c^7 \\
& +5076141928013605253433604772659200a^5c^7 \\
& -71161905489860225212016854630400a^6c^7 - 6269520441917164052706604646400a^7c^7 \\
& +42254801435457396442227916800a^8c^7 + 4426714807409895718863974400a^9c^7 \\
& +18703722525593885784576000a^{10}c^7 - 1329251649972418610380800a^{11}c^7 \\
& -17162615802950082150400a^{12}c^7 + 65746142983045324800a^{13}c^7 \\
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& -26753393366626560251904a^{11}c^8 - 679369622518788810752a^{12}c^8 \\
& -2194454423158373376a^{13}c^8 + 36737957249190912a^{14}c^8 + 310595376199680a^{15}c^8 \\
& +445617469440a^{16}c^8 - 2184399360a^{17}c^8 - 4759040a^{18}c^8 \\
& +4404068776039340967770266733117440000c^9 \\
& -1783261310081474157882210340306944000ac^9 \\
& +232113864710755881803031756996608000a^2c^9
\end{aligned}$$

$$\begin{aligned}
& -8644066342858541965878847832064000a^3c^9 \\
& -32011077785639506612985319424000a^4c^9 + 19869691087509808561877532672000a^5c^9 \\
& + 304557493863427189495126016000a^6c^9 - 16022453066187820842670080000a^7c^9 \\
& - 271058114268117146164480000a^8c^9 + 4574312832443797487616000a^9c^9 \\
& + 115502439778380182528000a^{10}c^9 + 56878831656751104000a^{11}c^9 \\
& - 14511749989248512000a^{12}c^9 - 102960741531648000a^{13}c^9 + 132455504896000a^{14}c^9 \\
& + 3300870144000a^{15}c^9 + 8090368000a^{16}c^9 + 433049472563672995745231040948469760c^{10} \\
& - 146246331256312980451798744905547776ac^{10} \\
& + 15168496614286038219626167337746432a^2c^{10} \\
& - 334297602497318758645044192608256a^3c^{10} \\
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& + 22833231529388666525619544064a^6c^{10} - 407031430380012660742840320a^7c^{10} \\
& - 13106081077916016150737920a^8c^{10} + 31814940216382283710464a^9c^{10} \\
& + 3054844045550978650112a^{10}c^{10} + 18015970550045884416a^{11}c^{10} \\
& - 153108176727554048a^{12}c^{10} - 1798208426606592a^{13}c^{10} - 3353554620416a^{14}c^{10} \\
& + 13203480576a^{15}c^{10} + 32361472a^{16}c^{10} + 35054645522562299555398518046720000c^{11} \\
& - 9835785679506365388250671375974400ac^{11} \\
& + 798354333336317066524270736179200a^2c^{11} \\
& - 6732318236841339725022599577600a^3c^{11} - 1298325671635335690566880460800a^4c^{11} \\
& + 16031712704360699496923136000a^5c^{11} + 1024278892308122232589516800a^6c^{11} \\
& - 3828065311029971489587200a^7c^{11} - 389444718791600912793600a^8c^{11} \\
& - 1853146085309787340800a^9c^{11} + 47307691166058086400a^{10}c^{11} \\
& + 481411625154969600a^{11}c^{11} - 97959940915200a^{12}c^{11} - 1512398641600a^{13}c^{11} \\
& - 42364108800a^{14}c^{11} + 2355274163715658446956074002022400c^{12} \\
& - 545823206218701245204546608693248ac^{12} + 33794155269834290973635734667264a^2c^{12} \\
& + 140113723880388288346516881408a^3c^{12} - 50947210104854667731378241536a^4c^{12} \\
& + 34417799286102523775877120a^5c^{12} + 31335119871918556766765056a^6c^{12} \\
& + 145912724832886638821376a^7c^{12} - 7435186673784827789312a^8c^{12} \\
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& + 15324651503616a^{12}c^{12} - 50413289472a^{13}c^{12} - 141213696a^{14}c^{12} \\
& + 132091082500530653053056974848000c^{13} - 25080727018788218156955048345600ac^{13}
\end{aligned}$$

$$\begin{aligned}
& +1141009818986979339736134451200a^2c^{13} + 18804311236219616172716851200a^3c^{13} \\
& -1497358459614442599553433600a^4c^{13} - 12541248552098962096128000a^5c^{13} \\
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& -955308033798859563215320252416ac^{14} + 30129275830597180093530898432a^2c^{14} \\
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& -783056757312556655274098688ac^{16} + 7740346964796905746333696a^2c^{16} \\
& +667638067387310968406016a^3c^{16} - 4401822778388480786432a^4c^{16} \\
& -218306735322712178688a^5c^{16} - 160045814544924672a^6c^{16} + 20633616468541440a^7c^{16} \\
& +85040625745920a^8c^{16} - 210517032960a^9c^{16} - 825556992a^{10}c^{16} \\
& +222870204968868473995264000c^{17} - 16665542994690293052211200ac^{17} \\
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\end{aligned}$$

$$\begin{aligned}
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& -11123294208ac^{24} - 218103808a^2c^{24} + 21810380800c^{25} + 33554432c^{26} \Big\} \quad (11)
\end{aligned}$$

III. DERIVATION OF MAIN FORMULA

Substituting $b = -a - 51$, $z = \frac{1}{2}$ in given result (5), we get

$$\begin{aligned}
& (2a + 51) {}_2F_1 \left[\begin{matrix} a, & -a - 51 \\ c & \end{matrix} ; \frac{1}{2} \right] \\
& = a {}_2F_1 \left[\begin{matrix} a + 1, & -a - 51 \\ c & \end{matrix} ; \frac{1}{2} \right] + (a + 51) {}_2F_1 \left[\begin{matrix} a, & -a - 50 \\ c & \end{matrix} ; \frac{1}{2} \right]
\end{aligned}$$

Now involving the result which is established in Ref[5], we can establish the main result.

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11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.



12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. Pick a good study spot: To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.



27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. Think technically: Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium through which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

INFORMAL GUIDELINES OF RESEARCH PAPER WRITING

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.



Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

- Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

- Use standard writing style including articles ("a", "the," etc.)
- Keep on paying attention on the research topic of the paper
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- Use present tense to report well accepted
- Use past tense to describe specific results
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- Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

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Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript-- must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

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- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
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- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
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Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.



- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
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- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

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This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
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- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

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- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently. You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.



Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
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The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
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- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.



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Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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