Online ISSN: 2249-4626
Print ISSN: 0975-5896

Global Journal

of Science Frontier Research: F

Mathematics and Decision Sciences



Some Further Developments

Estimation of Population Ratio

Highlights

Semi-Invariant Submanifolds

Strictly Practical Stabilization

Discovering Thoughts, Inventing Future

VOLUME 13 ISSUE 4 VERSION 1.0

© 2001-2013 by Global Journal of Science Frontier Research , USA

Volume 13 Issue 14 was originally issued as F4

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences
Volume 13 Issue 4 (Ver. 1.0)

Open Association of Research Society

© Global Journal of Science Frontier Research .2013.

All rights reserved.

This is a special issue published in version 1.0 of "Global Journal of Science Frontier Research." By Global Journals Inc.

All articles are open access articles distributed under "Global Journal of Science Frontier Research"

Reading License, which permits restricted use. Entire contents are copyright by of "Global Journal of Science Frontier Research" unless otherwise noted on specific articles.

No part of this publication may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopy, recording, or any information storage and retrieval system, without written permission.

The opinions and statements made in this book are those of the authors concerned. Ultraculture has not verified and neither confirms nor denies any of the foregoing and no warranty or fitness is implied.

Engage with the contents herein at your own risk.

The use of this journal, and the terms and conditions for our providing information, is governed by our Disclaimer, Terms and Conditions and Privacy Policy given on our website http://globaljournals.us/terms-and-condition/menu-id-1463/

By referring / using / reading / any type of association / referencing this journal, this signifies and you acknowledge that you have read them and that you accept and will be bound by the terms thereof.

All information, journals, this journal, activities undertaken, materials, services and our website, terms and conditions, privacy policy, and this journal is subject to change anytime without any prior notice.

Incorporation No.: 0423089
License No.: 42125/022010/1186
Registration No.: 430374
Import-Export Code: 1109007027
Employer Identification Number (EIN):
USA Tax ID: 98-0673427

Global Journals Inc.

*(A Delaware USA Incorporation with "Good Standing"; **Reg. Number: 0423089**)*

Sponsors: Open Association of Research Society
Open Scientific Standards

Publisher's Headquarters office

Global Journals Inc., Headquarters Corporate Office, Cambridge Office Center, II Canal Park, Floor No. 5th, ***Cambridge (Massachusetts)***, Pin: MA 02141 United States
USA Toll Free: +001-888-839-7392
USA Toll Free Fax: +001-888-839-7392

Offset Typesetting

Open Association of Research Society, Marsh Road, Rainham, Essex, London RM13 8EU
United Kingdom.

Packaging & Continental Dispatching

Global Journals, India

Find a correspondence nodal officer near you

To find nodal officer of your country, please email us at *local@globaljournals.org*

eContacts

Press Inquiries: *press@globaljournals.org*
Investor Inquiries: *investers@globaljournals.org*
Technical Support: *technology@globaljournals.org*
Media & Releases: *media@globaljournals.org*

Pricing (Including by Air Parcel Charges):

For Authors:
22 USD (B/W) & 50 USD (Color)
Yearly Subscription (Personal & Institutional):
200 USD (B/W) & 250 USD (Color)

Editorial Board Members (HON.)

John A. Hamilton,"Drew" Jr.,
Ph.D., Professor, Management
Computer Science and Software Engineering
Director, Information Assurance Laboratory
Auburn University

Dr. Henry Hexmoor
IEEE senior member since 2004
Ph.D. Computer Science, University at Buffalo
Department of Computer Science
Southern Illinois University at Carbondale

Dr. Osman Balci, Professor
Department of Computer Science
Virginia Tech, Virginia University
Ph.D.and M.S.Syracuse University, Syracuse, New York
M.S. and B.S. Bogazici University, Istanbul, Turkey

Yogita Bajpai
M.Sc. (Computer Science), FICCT
U.S.A.Email:
yogita@computerresearch.org

Dr. T. David A. Forbes
Associate Professor and Range Nutritionist
Ph.D. Edinburgh University - Animal Nutrition
M.S. Aberdeen University - Animal Nutrition
B.A. University of Dublin- Zoology

Dr. Wenying Feng
Professor, Department of Computing & Information Systems
Department of Mathematics
Trent University, Peterborough,
ON Canada K9J 7B8

Dr. Thomas Wischgoll
Computer Science and Engineering,
Wright State University, Dayton, Ohio
B.S., M.S., Ph.D.
(University of Kaiserslautern)

Dr. Abdurrahman Arslanyilmaz
Computer Science & Information Systems Department
Youngstown State University
Ph.D., Texas A&M University
University of Missouri, Columbia
Gazi University, Turkey

Dr. Xiaohong He
Professor of International Business
University of Quinnipiac
BS, Jilin Institute of Technology; MA, MS, PhD,. (University of Texas-Dallas)

Burcin Becerik-Gerber
University of Southern California
Ph.D. in Civil Engineering
DDes from Harvard University
M.S. from University of California, Berkeley & Istanbul University

Dr. Bart Lambrecht
Director of Research in Accounting and FinanceProfessor of Finance
Lancaster University Management School
BA (Antwerp); MPhil, MA, PhD (Cambridge)

Dr. Carlos García Pont
Associate Professor of Marketing
IESE Business School, University of Navarra
Doctor of Philosophy (Management), Massachusetts Institute of Technology (MIT)
Master in Business Administration, IESE, University of Navarra
Degree in Industrial Engineering, Universitat Politècnica de Catalunya

Dr. Fotini Labropulu
Mathematics - Luther College
University of ReginaPh.D., M.Sc. in Mathematics
B.A. (Honors) in Mathematics
University of Windso

Dr. Lynn Lim
Reader in Business and Marketing
Roehampton University, London
BCom, PGDip, MBA (Distinction), PhD, FHEA

Dr. Mihaly Mezei
ASSOCIATE PROFESSOR
Department of Structural and Chemical Biology, Mount Sinai School of Medical Center
Ph.D., Etvs Lornd University
Postdoctoral Training,
New York University

Dr. Söhnke M. Bartram
Department of Accounting and FinanceLancaster University Management SchoolPh.D. (WHU Koblenz)
MBA/BBA (University of Saarbrücken)

Dr. Miguel Angel Ariño
Professor of Decision Sciences
IESE Business School
Barcelona, Spain (Universidad de Navarra)
CEIBS (China Europe International Business School).
Beijing, Shanghai and Shenzhen
Ph.D. in Mathematics
University of Barcelona
BA in Mathematics (Licenciatura)
University of Barcelona

Philip G. Moscoso
Technology and Operations Management
IESE Business School, University of Navarra
Ph.D in Industrial Engineering and Management, ETH Zurich
M.Sc. in Chemical Engineering, ETH Zurich

Dr. Sanjay Dixit, M.D.
Director, EP Laboratories, Philadelphia VA Medical Center
Cardiovascular Medicine - Cardiac Arrhythmia
Univ of Penn School of Medicine

Dr. Han-Xiang Deng
MD., Ph.D
Associate Professor and Research
Department Division of Neuromuscular Medicine
Davee Department of Neurology and Clinical NeuroscienceNorthwestern University
Feinberg School of Medicine

Dr. Pina C. Sanelli
Associate Professor of Public Health
Weill Cornell Medical College
Associate Attending Radiologist
NewYork-Presbyterian Hospital
MRI, MRA, CT, and CTA
Neuroradiology and Diagnostic Radiology
M.D., State University of New York at Buffalo,School of Medicine and Biomedical Sciences

Dr. Roberto Sanchez
Associate Professor
Department of Structural and Chemical Biology
Mount Sinai School of Medicine
Ph.D., The Rockefeller University

Dr. Wen-Yih Sun
Professor of Earth and Atmospheric SciencesPurdue University Director
National Center for Typhoon and Flooding Research, Taiwan
University Chair Professor
Department of Atmospheric Sciences, National Central University, Chung-Li, TaiwanUniversity Chair Professor
Institute of Environmental Engineering, National Chiao Tung University, Hsin-chu, Taiwan.Ph.D., MS The University of Chicago, Geophysical Sciences
BS National Taiwan University, Atmospheric Sciences
Associate Professor of Radiology

Dr. Michael R. Rudnick
M.D., FACP
Associate Professor of Medicine
Chief, Renal Electrolyte and Hypertension Division (PMC)
Penn Medicine, University of Pennsylvania
Presbyterian Medical Center, Philadelphia
Nephrology and Internal Medicine
Certified by the American Board of Internal Medicine

Dr. Bassey Benjamin Esu
B.Sc. Marketing; MBA Marketing; Ph.D Marketing
Lecturer, Department of Marketing, University of Calabar
Tourism Consultant, Cross River State Tourism Development Department
Co-ordinator , Sustainable Tourism Initiative, Calabar, Nigeria

Dr. Aziz M. Barbar, Ph.D.
IEEE Senior Member
Chairperson, Department of Computer Science
AUST - American University of Science & Technology
Alfred Naccash Avenue – Ashrafieh

President Editor (HON.)

Dr. George Perry, (Neuroscientist)
Dean and Professor, College of Sciences
Denham Harman Research Award (American Aging Association)
ISI Highly Cited Researcher, Iberoamerican Molecular Biology Organization
AAAS Fellow, Correspondent Member of Spanish Royal Academy of Sciences
University of Texas at San Antonio
Postdoctoral Fellow (Department of Cell Biology)
Baylor College of Medicine
Houston, Texas, United States

Chief Author (HON.)

Dr. R.K. Dixit
M.Sc., Ph.D., FICCT
Chief Author, India
Email: authorind@computerresearch.org

Dean & Editor-in-Chief (HON.)

Vivek Dubey(HON.)
MS (Industrial Engineering),
MS (Mechanical Engineering)
University of Wisconsin, FICCT
Editor-in-Chief, USA
editorusa@computerresearch.org

Sangita Dixit
M.Sc., FICCT
Dean & Chancellor (Asia Pacific)
deanind@computerresearch.org

Suyash Dixit
(B.E., Computer Science Engineering), FICCTT
President, Web Administration and
Development , CEO at IOSRD
COO at GAOR & OSS

Er. Suyog Dixit
(M. Tech), BE (HONS. in CSE), FICCT
SAP Certified Consultant
CEO at IOSRD, GAOR & OSS
Technical Dean, Global Journals Inc. (US)
Website: www.suyogdixit.com
Email:suyog@suyogdixit.com

Pritesh Rajvaidya
(MS) Computer Science Department
California State University
BE (Computer Science), FICCT
Technical Dean, USA
Email: pritesh@computerresearch.org

Luis Galárraga
J!Research Project Leader
Saarbrücken, Germany

Contents of the Volume

i. Copyright Notice
ii. Editorial Board Members
iii. Chief Author and Dean
iv. Table of Contents
v. From the Chief Editor's Desk
vi. Research and Review Papers

1. Some Further Developments in the Infinite Product Representation of Elementary Functions. *1-27*
2. Comparison of Numerical Schemes for Shallow Water Equation. *29-46*
3. Helical-One, Two, Three-Revolutional Cyclical Surfaces. *47-56*
4. Estimation of Population Ratio in Simple Random Sampling using Variable Transformation. *57-65*
5. On Certain Summation Formulae Involving Gauss Theorem. *67-72*
6. Semi-Invariant Submanifolds of Nearly Hyperbolic Cosymplectic Manifold. *73-81*
7. Strictly Practical Stabilization of Impulsive Functional Differential Equations by using Lyapunov Functions. *83-89*
8. Computation of a Summation Formula Clung To Recurrence Relation. *91-104*

vii. Auxiliary Memberships
viii. Process of Submission of Research Paper
ix. Preferred Author Guidelines
x. Index

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 4 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Some Further Developments in the Infinite Product Representation of Elementary Functions

By Viktor Reshniak

Middle Tennessee State University

Abstract - An innovatory approach has been recently proposed for the derivation of infinite product representation of elementary functions. The approach is based on the comparison of different alternative forms of Green's functions for boundary-value problems stated for the two-dimensional Laplace equation. A number of new infinite product representations of elementary functions was actually derived within the scope of that approach. The present study continues the trend: it aims at an analysis of the approach and exploring ways for its extending to some other problem statements that might also be efficiently treated.

Keywords : *green's functions; infinite products; elementary functions.*

GJSFR-F Classification : *MSC 2010: 08B25 , 26A09*



Strictly as per the compliance and regulations of :

© 2013. Viktor Reshniak. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

eJournal
physical existance Print Journal
epaper
open standard
3d virtual journal

1. C. Caratheodory, Conformal representation, Cambridge university press, 1969.

Some Further Developments in the Infinite Product Representation of Elementary Functions

Viktor Reshniak

Abstract - An innovatory approach has been recently proposed for the derivation of infinite product representation of elementary functions. The approach is based on the comparison of different alternative forms of Green's functions for boundary-value problems stated for the two-dimensional Laplace equation. A number of new infinite product representations of elementary functions was actually derived within the scope of that approach. The present study continues the trend: it aims at an analysis of the approach and exploring ways for its extending to some other problem statements that might also be efficiently treated.

Keywords : green's functions; infinite products; elementary functions.

I. Introduction

In a series of recent works (see, for example, [6] and [7]), an innovatory approach was proposed to the derivation of infinite product representation of elementary functions. The approach is based on the multiplicity of forms of Green's functions for some boundary-value problems. To introduce the key idea of the approach, we turn to one of the simplest problem settings of that kind. That is the Dirichlet problem

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0, \qquad (x,y)\in\Omega \tag{1}$$

$$u(x,0)=u(x,b)=0, \qquad \lim_{x\to\pm\infty} u(x,y)<\infty \tag{2}$$

posed in the infinite strip $\Omega=\{-\infty<x<\infty, 0<y<b\}$.
The classical representation

$$G(x,y;\xi,\eta)=\frac{1}{4\pi}\ln\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}, \qquad \omega=\frac{\pi}{b} \tag{3}$$

of the Green's function for the setting in (1)-(2) can be obtained by one of the two standard methods available in the field. Indeed, either the conformal mapping method [1, 2] algorithm, or the method of eigenfunction expansion [4, 5] procedure, with subsequent summation of trigonometric series that represents the Green's function, appear successful.

Author : Computational Science Program Middle Tennessee State University Murfreesboro, TN 37132, USA.
E-mail : vr2m@mtmail.mtsu.edu

© 2013 Global Journals Inc. (US)

In [6], it was recalled that if the method of images routine (that is also considered workable, but efficient in a limited number of problem settings) is used instead, then we arrive at an alternative to (3) expression

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\prod_{n=-\infty}^{\infty}\sqrt{\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2}} \tag{4}$$

Since the forms in (3) and (4) are equivalent, we obtain the multi-variable identity

$$\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}=\prod_{n=-\infty}^{\infty}\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2} \tag{5}$$

which can be used as a starting point for the derivation of a number of infinite product representations of elementary functions.

Upon some trivial algebra, the identity in (5) can, for example, be transformed in the infinite product representation

$$\sin t=\frac{2t}{\pi}\prod_{k=1}^{\infty}\left[1+\frac{4t^2-\pi^2}{(1-4k^2)\pi^2}\right]$$

of the trigonometric sine function. The latter can be considered as an equivalent alternative form to the classical Euler infinite product expansion

$$\sin t=t\prod_{k=1}^{\infty}\left(1-\frac{t^2}{k^2\pi^2}\right)$$

of the sine function.

A score of new as well as alternative to existing infinite product representations of elementary functions was presented in [7] by the approach just described. Note, that the author of [7] has claimed that research in the area is still open. Accepting the challenge, we will show in the next section, that the approach also appears efficient in a few other cases.

II. Further Extension of the Approach

a) Semi-infinite Strip Region

We start with the mixed boundary-value problem

$$u(0,y)=\frac{\partial u(a,y)}{\partial x}=0 \tag{6}$$

$$u(x,0)=0,\qquad \lim_{y\to\infty}u(x,y)<\infty \tag{7}$$

posed for the two-dimensional Laplace equation in the semi-infinite strip region $\Omega=\{0<x<a,0<y<\infty\}$.

Following the technique proposed in [6] and [7], two equivalent representations of the Green's function for the problem in (6)-(7) can be found. The classical approach of eigenfunction expansion leads to a closed analytical form of its solution. On the other

Ref.

6. Yu. A. Melnikov, A new approach to the representation of some trigonometric and hyperbolic functions by infinite products, J. Math. Anal. Appl., 344, 1, 2008, 521-534.

© 2013 Global Journals Inc. (US)

hand, another classical approach – the method of images - provides an infinite product expression of the required Green's function.

Method of eigenfunction expansion. Let the Poisson equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -f(x,y), \qquad (x,y) \in \Omega \tag{8}$$

be subject to the boundary conditions in (6)-(7).
Once the solution of the stated problem is found as the integral

$$u(x,y) = \iint_{\Omega} G(x,y;\xi,\eta) f(\xi,\eta) d\Omega(\xi,\eta), \qquad (x,y) \in \Omega \tag{9}$$

the kernel function $G\,(\,x,y;\xi,\eta\,)$ of the above represents the Green's function to the boundary value problem in (6)-(7).

According to the theory of Fourier series, when the eigenfunctions are known, the solution of the given boundary-value problem can be obtained by the superposition [5]

$$u(x,y) = \sum_{k=1}^{\infty} Y_k(y) X_k(x) \tag{10}$$

where X_k is a complete system of eigenfunctions, orthogonal in [0,a]

$$X_k = \sin \lambda_k x,$$
$$\lambda_k = \frac{\pi(2k-1)}{2a}, \quad k = 1, 2, ...$$

Functions $Y_k(y)$ are to be determined from the Fourier series expansion of the right-hand side part in (8)

$$f(x,y) = \sum_{k=1}^{\infty} f_k(y) X_k(x), \qquad f_k(y) = \frac{2}{a} \int_0^a f(\xi,y) X_k(\xi) d\xi \tag{11}$$

resulting in

$$\frac{d^2 Y_k(y)}{dy^2} - \lambda_k^2 Y_k^2(y) = -\frac{2}{a} \int_0^a f(\xi,y) \sin \lambda_k \xi \, d\xi \tag{12}$$

$$Y_k(0) = 0 \tag{13}$$

$$\lim_{y \to \infty} |Y_k(y)| < \infty \tag{14}$$

Clearly, solution of the problem in (12)-(14) has the form

$$Y_k(y) = \int_0^{\infty} f_k(\eta) g_k(y;\eta) d\eta \tag{15}$$

5. R. Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, Pearson, 2012.

© 2013 Global Journals Inc. (US)

where $g_k(y;\eta)$ is the Green's function for the homogeneous equation corresponding to (12) with boundary conditions in (13) and (14) ([4],[7])

$$g_k(y;\eta) = \frac{1}{2\lambda_k}\begin{cases} e^{\lambda_k(y-\eta)} - e^{-\lambda_k(y+\eta)}, & y \le \eta \\ e^{\lambda_k(\eta-y)} - e^{-\lambda_k(y+\eta)}, & y \ge \eta \end{cases}$$

Therefore the series in (10), along with (15) solves the problem posed in (6)-(8)

$$u(x,y) = \int_0^a \int_0^\infty \left[\frac{2}{a}\sum_{k=1}^{\infty} g_k(y;\eta)\ \sin\lambda_k\xi\ \sin\lambda_k x\right] f(\xi,\eta)d\xi d\eta$$

In light of (9) the kernel function of the above expression represents the Green's function to the problem in (6)-(7) posed for the Laplace equation in Ω

$$G(x,y;\xi,\eta) = \frac{2}{a}\sum_{k=1}^{\infty} g_k(y;\eta)\ \sin\lambda_k\xi\ \sin\lambda_k x \tag{16}$$

Upon the trigonometric identity

$$\sin\lambda_k\xi\ \sin\lambda_k x = \frac{1}{2}\left[\cos\lambda_k(\xi - x) - \cos\lambda_k(\xi + x)\right]$$

and (see, for example [3]) the summation formula

$$\sum_{k=1}^{\infty}\frac{p^{2k-1}\cos(2k-1)x}{2k-1} = \frac{1}{4}\ln\frac{1+2p\cos x+p^2}{1-2p\cos x+p^2}, \qquad 0<x<2\pi, \quad p^2\le 1$$

the expression in (16) transforms into

$$\begin{aligned} G(x,y;\xi,\eta) = \frac{1}{4\pi}\Bigg(&\ln\frac{1+2e^{\omega(y-\eta)}\cos\ \omega(\xi-x)+e^{2\omega(y-\eta)}}{1-2e^{\omega(y-\eta)}\cos\ \omega(\xi-x)+e^{2\omega(y-\eta)}} \\ &-\ln\frac{1+2e^{\omega(y-\eta)}\cos\ \omega(\xi+x)+e^{2\omega(y-\eta)}}{1-2e^{\omega(y-\eta)}\cos\ \omega(\xi+x)+e^{2\omega(y-\eta)}} \\ &-\ln\frac{1+2e^{-\omega(y+\eta)}\cos\ \omega(\xi-x)+e^{-2\omega(y+\eta)}}{1-2e^{-\omega(y+\eta)}\cos\ \omega(\xi-x)+e^{-2\omega(y+\eta)}} \\ &+\ln\frac{1+2e^{-\omega(y+\eta)}\cos\ \omega(\xi+x)+e^{-2\omega(y+\eta)}}{1-2e^{-\omega(y+\eta)}\cos\ \omega(\xi+x)+e^{-2\omega(y+\eta)}}\Bigg) \end{aligned} \tag{17}$$

where

$$\omega = \frac{\pi}{2a}$$

After introducing the variables

$$\alpha = \frac{\pi}{2a}(y-\eta); \quad \beta = -\frac{\pi}{2a}(y+\eta); \quad \gamma = \frac{\pi}{2a}(x-\xi); \quad \theta = \frac{\pi}{2a}(\xi+x) \tag{18}$$

Ref.

4. Dean G. Duffy, Green's Functions with Applications, Chapman and Hall/CRC, 2001.

© 2013 Global Journals Inc. (US)

the form in (17) reads

$$G(\alpha,\beta,\gamma,\theta)=\frac{1}{4\pi}\ln\ \frac{\left(1+2e^{\alpha}\cos\ \gamma+e^{2\alpha}\right)}{\left(1-2e^{\alpha}\cos\ \gamma+e^{2\alpha}\right)}\frac{\left(1+2e^{\beta}\cos\ \theta+e^{2\beta}\right)}{\left(1-2e^{\beta}\cos\ \theta+e^{2\beta}\right)}$$
$$\times\frac{\left(1-2e^{\alpha}\cos\ \theta+e^{2\alpha}\right)}{\left(1+2e^{\alpha}\cos\ \theta+e^{2\alpha}\right)}\frac{\left(1-2e^{\beta}\cos\ \gamma+e^{2\beta}\right)}{\left(1+2e^{\beta}\cos\ \gamma+e^{2\beta}\right)}\Bigg) \tag{19}$$

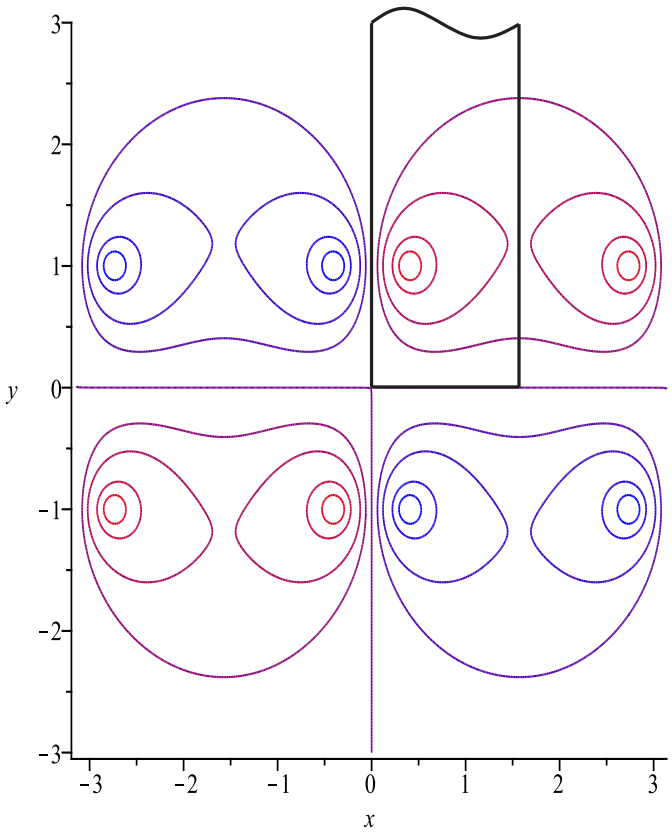


Figure 1 : Field generated in the semi-infinite strip $0 \leq x \leq \pi/2,\ y \geq 0$ by a source located at $\xi = \pi/8,\ \eta = 1$

By problem statement, the observation point (x, y) and the source point (ξ, η) are interior to the semi-infinite strip Ω. It means that parameters $\alpha, \beta, \gamma, \theta$ range as

$$-\infty < \alpha < \infty, \qquad -\infty < \beta < 0, \qquad -\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}, \qquad 0 \leq \theta \leq \pi \tag{20}$$

given that α and γ are not equal to zero at the same time.

But on the other hand the function in (19) is analytic everywhere except, of course, at the points of singularity, as it can be seen in Figure 1. Hence, the parameter ranges in (20) can be extended to the entire region of analyticity of (19).

Method of images. The Green's function for the two-dimensional Laplace equation has the form ([5, 7])

Ref

5. R. Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, Pearson, 2012.

© 2013 Global Journals Inc. (US)

$$G(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln r(x,y,\xi,\eta) + \mu(x,y;\xi,\eta)$$

where $r(x,y,\xi,\eta)$ is the distance between the field point and the source point, and $\mu(x,y;\xi,\eta)$ is a harmonic in Ω function.

According to the method of images, the regular component $\mu(x,y;\xi,\eta)$ is sought as a superposition of responses at a field point P from singularities Q_j^* placed outside the region Ω

$$\mu(x,y) = \sum_{j=1}^{m} \pm\frac{1}{2\pi}\ln|P - Q_j^*|$$

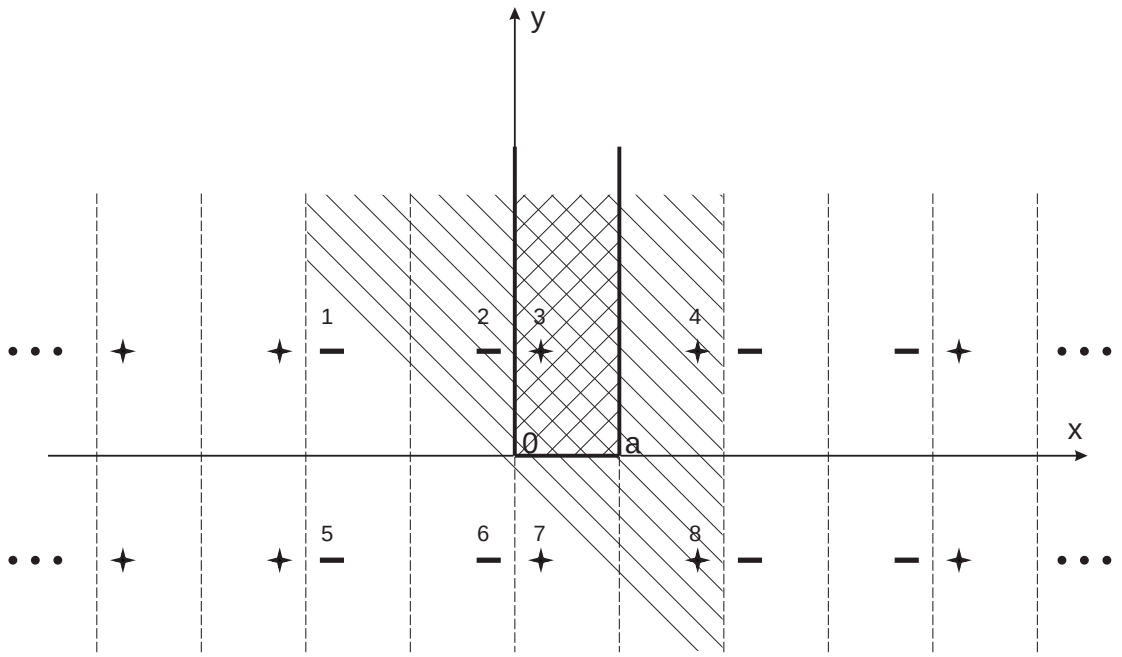


Figure 2 : Sequence of sources and sinks that arises in the method of images applied to the problem in (6)-(7)

The sign of each term in the above as well as the positions of the corresponding singularities are chosen to satisfy boundary conditions of the given boundary-value problem.

Figure 2 illustrates this approach to the problem in (6)-(7). Unit sources are labeled with the plussign and sinks with the minus sign. Geometry of Ω leads to the infinite sequence of singularities, where the shaded strip of width $4a$ is repeated periodically in the direction of the x-axis. The singularities have the following coordinates.

$$(\xi_1^0 = -(2a-\xi);\ \eta_1^0 = \eta) \qquad (\xi_5^0 = -(2a-\xi);\ \eta_5^0 = -\eta)$$

$$(\xi_2^0 = -\xi;\ \eta_2^0 = \eta) \qquad (\xi_6^0 = -\xi;\ \eta_6^0 = -\eta)$$

$$(\xi_3^0 = \xi;\ \eta_3^0 = \eta) \qquad (\xi_7^0 = \xi;\ \eta_7^0 = -\eta)$$

$$(\xi_4^0 = 2a-\xi;\ \eta_4^0 = \eta) \qquad (\xi_8^0 = 2a-\xi;\ \eta_8^0 = -\eta)$$

© 2013 Global Journals Inc. (US)

Hence, the total effect on the field point P from the infinite sequence of suitably chosen sinks and sources Q_j^* is described by the following Green's function

$$G(x,y;\xi,\eta) = \sum_{k=-\infty}^{\infty} G^k(x,y;\xi,\eta)$$

$$= \frac{1}{2\pi}\sum_{k=0}^{\infty} \ln \sqrt{\frac{\Big((x-\xi_1^k)^2+(y-\eta_1^k)^2\Big)\Big((x-\xi_2^k)^2+(y-\eta_2^k)^2\Big)}{\Big((x-\xi_3^k)^2+(y-\eta_3^k)^2\Big)\Big((x-\xi_4^k)^2+(y-\eta_4^k)^2\Big)}}$$

$$\times \sqrt{\frac{\Big((x-\xi_7^k)^2+(y-\eta_7^k)^2\Big)\Big((x-\xi_8^k)^2+(y-\eta_8^k)^2\Big)}{\Big((x-\xi_5^k)^2+(y-\eta_5^k)^2\Big)\Big((x-\xi_6^k)^2+(y-\eta_6^k)^2\Big)}} \tag{21}$$

with ξ_n^k representing the coordinates of singularities

$$\xi_n^k = \xi_n^0 + 4ak, \qquad k = \pm 1, \pm 2, ...$$

The form in (21) reads, in terms of the variables introduced in (18), as

$$G(\alpha,\beta,\gamma,\theta) = \frac{1}{4\pi}\ln \prod_{k=-\infty}^{\infty}\left(\frac{\Big((\gamma+\pi(1-2k))^2+\alpha^2\Big)\Big((\theta-2\pi k)^2+\alpha^2\Big)}{\Big((\gamma+\pi(1-2k))^2+\beta^2\Big)\Big((\theta-2\pi k)^2+\beta^2\Big)} \right.$$

$$\left. \times \frac{\Big((\theta-\pi(1+2k))^2+\beta^2\Big)\Big((\gamma-2\pi k)^2+\beta^2\Big)}{\Big((\theta-\pi(1+2k))^2+\alpha^2\Big)\Big((\gamma-2\pi k)^2+\alpha^2\Big)} \right) \tag{22}$$

Infinite Products and Elementary Functions. As it follows directly from the equivalence of the representations of (19) and (22), the identity

$$\frac{\big(1+2e^{\alpha}\cos\,\gamma+e^{2\alpha}\big)}{\big(1-2e^{\alpha}\cos\,\gamma+e^{2\alpha}\big)}\frac{\big(1-2e^{\beta}\cos\,\gamma+e^{2\beta}\big)}{\big(1+2e^{\beta}\cos\,\gamma+e^{2\beta}\big)}$$

$$\times\frac{\big(1+2e^{\beta}\cos\,\theta+e^{2\beta}\big)}{\big(1-2e^{\beta}\cos\,\theta+e^{2\beta}\big)}\frac{\big(1-2e^{\alpha}\cos\,\theta+e^{2\alpha}\big)}{\big(1+2e^{\alpha}\cos\,\theta+e^{2\alpha}\big)}$$

$$= \prod_{k=-\infty}^{\infty}\left(\frac{\Big((\gamma+\pi(1-2k))^2+\alpha^2\Big)\Big((\theta-2\pi k)^2+\alpha^2\Big)}{\Big((\gamma+\pi(1-2k))^2+\beta^2\Big)\Big((\theta-2\pi k)^2+\beta^2\Big)} \right.$$

$$\left. \times \frac{\Big((\theta-\pi(1+2k))^2+\beta^2\Big)\Big((\gamma-2\pi k)^2+\beta^2\Big)}{\Big((\theta-\pi(1+2k))^2+\alpha^2\Big)\Big((\gamma-2\pi k)^2+\alpha^2\Big)} \right) \tag{23}$$

© 2013 Global Journals Inc. (US)

is valid at the region of analiticity of the function in (19).

The above identity involves four arbitrary parameters, and by a suitable choice of those some interesting representations of elementary functions can be obtained. To start with, we write its left-hand side in the form

$$\frac{(\cos\gamma + A_2)(\cos\gamma - A_1)(\cos\theta + A_1)(\cos\theta - A_2)}{(\cos\gamma - A_2)(\cos\gamma + A_1)(\cos\theta - A_1)(\cos\theta + A_2)}, \tag{24}$$

where

$$A_1 = \frac{1 + e^{2\beta}}{2e^{\beta}}, \quad A_2 = \frac{1 + e^{2\alpha}}{2e^{\alpha}}$$

This shows that it cannot be converted to an expression consisting of a single trigonometric function for any real value of the parameters α, β, γ, θ. It's also obvious from (24), that to avoid trivial results we should put some constraints on these parameters, specifically $\alpha \neq \pm\beta$, $\gamma \neq \pm\theta$.

It is worth noting that the trigonometric terms in the expression (24) depend only on the parameters γ and θ, and their successful choice allows that expression to be dependent only on the exponential terms.

As an illustration, let us consider the following parameter values: $\gamma = 0$, $\theta = \frac{\pi}{2}$, $\beta = 2u$, $\alpha = 2v$, which converts the expression in (23) into

$$\frac{\tanh^2 u}{\tanh^2 v} = \prod_{k=-\infty}^{\infty} \frac{\left(\pi^2(1+2k)^2 + 4v^2\right)\left(\pi^2 k^2 + u^2\right)}{\left(\pi^2(1+2k)^2 + 4u^2\right)\left(\pi^2 k^2 + v^2\right)} \tag{25}$$

Note that the above representation has already been derived in [6] and [7] upon considering a different problem.

If the parameter v in (25) is taken to infinity then we arrive at

$$\tanh^2 u = \prod_{k=-\infty}^{\infty} \frac{4\left(\pi^2 k^2 + u^2\right)}{\pi^2(1+2k)^2 + 4u^2} \tag{26}$$

When the parameter u is taken to infinity, the following expansion arises

$$\coth^2 v = \prod_{k=-\infty}^{\infty} \frac{\pi^2(1+2k)^2 + 4v^2}{4\left(\pi^2 k^2 + v^2\right)} \tag{27}$$

Relations in (26) and (27) have already been derived in [7] and their convergence has already been investigated.

Recalling the interrelation between hyperbolic and trigonometric functions

$$\tanh(x) = -i\ \tan(ix)$$

$$\coth(x) = i\ \cot(ix)$$

Ref.
7. Yuri A. Melnikov, Green's functions and Infinite Products: Bridging the Divide, Birkhauser-Springer, New York - London, 2011.

© 2013 Global Journals Inc. (US)

the new infinite product representations

$$\tan^2 u = \prod_{k=-\infty}^{\infty} \frac{4\left(\pi^2 k^2 - u^2\right)}{4u^2 - \pi^2(1+2k)^2} \tag{28}$$

$$\cot^2 v = \prod_{k=-\infty}^{\infty} \frac{4v^2 - \pi^2(1+2k)^2}{4\left(\pi^2 k^2 - v^2\right)} \tag{29}$$

can be obtained.

To prove the convergence of the identity in (28), we rewrite it in the form

$$\tan^2 u = \frac{4u^2}{4u^2 - \pi^2} \prod_{k=1}^{\infty} \frac{16\left(\pi^2 k^2 - u^2\right)^2}{\left(4u^2 - \pi^2(1+2k)^2\right)\left(4u^2 - \pi^2(1-2k)^2\right)}$$

$$= \frac{4u^2}{4u^2 - \pi^2} \prod_{k=1}^{\infty} \left(1 + \pi^2 \frac{\pi^2(8k^2-1) + 8u^2}{\left(4u^2 - \pi^2(1+2k)^2\right)\left(4u^2 - \pi^2(1-2k)^2\right)}\right) \tag{30}$$

Since the numerator in the second additive component in the product is a second degree polynomial in k, while the denominator is a polynomial of degree four the expansion in (30) converges uniformly for all $u \neq (1 \pm 2n)\pi/2$ at the rate of $1/k^2$.

A similar procedure shows the uniform convergence of the representation in (29)

$$\cot^2 v = \frac{\pi^2 - 4v^2}{4v^2} \prod_{k=1}^{\infty} \left(1 - \frac{\pi^2}{16}\left[\frac{\pi^2(8k^2+1) + 8v^2}{(\pi^2 k^2 - v^2)^2}\right]\right)$$

for all $v \neq \pm\pi n$ at the rate of $1/k^2$.

The representations in (28) and (29) deliver

$$\tan u = \pm \prod_{k=-\infty}^{\infty} \sqrt{\frac{4\left(\pi^2 k^2 - u^2\right)}{4u^2 - \pi^2(1+2k)^2}} \tag{31}$$

where the minus sign is valid for $u \in (-\frac{\pi}{2}; 0)$ while the plus sign holds for $u \in [0; \frac{\pi}{2})$.

$$\cot v = \pm \prod_{k=-\infty}^{\infty} \sqrt{\frac{4v^2 - \pi^2(1+2k)^2}{4\left(\pi^2 k^2 - v^2\right)}} \tag{32}$$

where the plus sign is valid for $u \in (0; \frac{\pi}{2})$ while the minus corresponds to $u \in [\frac{\pi}{2}; \pi)$.

Figure 3 provides a sense of the convergence of the representations in (31).

Two other expansions can be obtained using the standard identities

© 2013 Global Journals Inc. (US)

$$\sec u = \pm\sqrt{1+\tan^2 u} = \pm\sqrt{1+\prod_{k=-\infty}^{\infty}\frac{4\left(\pi^2k^2-u^2\right)}{4u^2-\pi^2(1+2k)^2}} \tag{33}$$

where the plus sign is valid for $u \in [-\frac{\pi}{2};\frac{\pi}{2})$, while the minus sign stays for $u \in [\frac{\pi}{2};\frac{3\pi}{2}]$, and

$$\csc v = \pm\sqrt{1+\cot^2 v} = \pm\sqrt{1+\prod_{k=-\infty}^{\infty}\frac{4v^2-\pi^2(1+2k)^2}{4\left(\pi^2k^2-v^2\right)}} \tag{34}$$

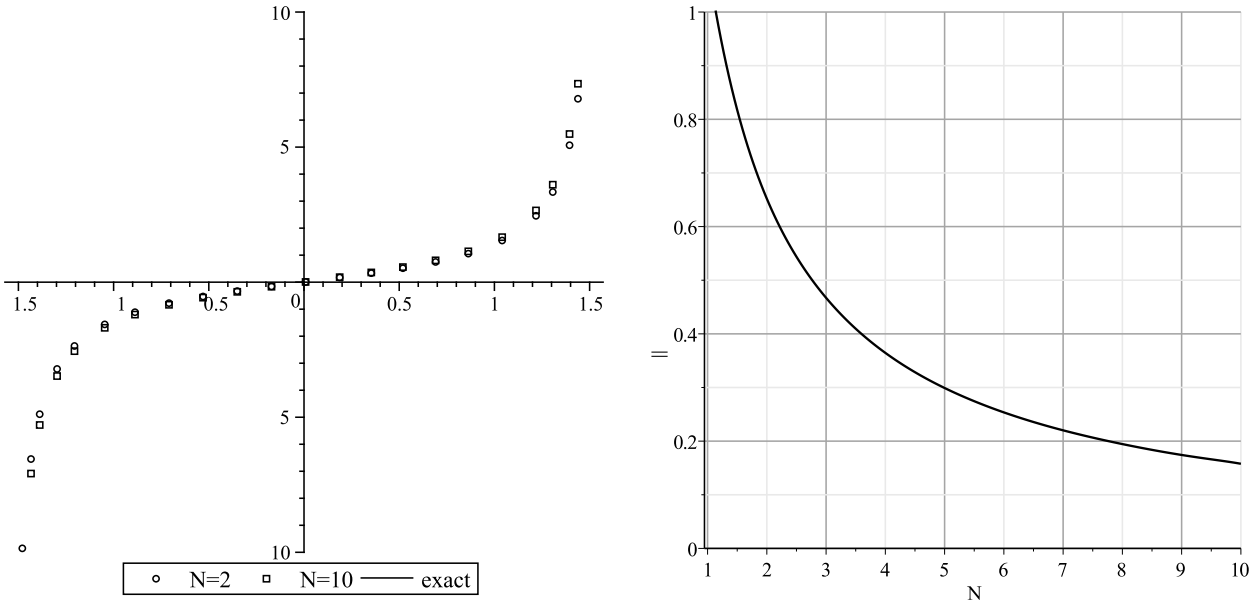
where the plus sign stays for $u \in [0;\pi)$ and the minus sign stays for $u \in [\pi;2\pi]$.

The convergence rate of the representation in (33) is illustrated in Figure 4.

Similarly to the work done for the problem stated in (6)-(7), we will consider another mixed boundary-value problem

$$u(0,y) = \frac{\partial u(a,y)}{\partial x} = 0 \tag{35}$$

$$\frac{\partial u(x,0)}{\partial x} = 0, \qquad \lim_{y\to\infty} u(x,y) < \infty \tag{36}$$



(a) Convergence with the 2nd and 10th partial products.

(b) L^2-norm of the error in $[-0.95\frac{\pi}{2}, 0.95\frac{\pi}{2}]$; N is a number of partial products

Figure 3 : Convergence of the representation in (31)

posed for the two-dimensional Laplace equation in the semi-infinite strip region $\Omega = \{0 < x < a, 0 < y < \infty\}$.

© 2013 Global Journals Inc. (US)

Method of eigenfunction expansion. Upon implementing the routine applied to the problem in (6)-(7) and using the same sequence of eigenfunctions, one obtains the Green's function for the problem in (35)-(36) in the form

$$G(x,y;\xi,\eta) = \frac{2}{a}\sum_{k=1}^{\infty} g_k(y;\eta)\ \sin\lambda_k\xi\ \sin\lambda_k x$$

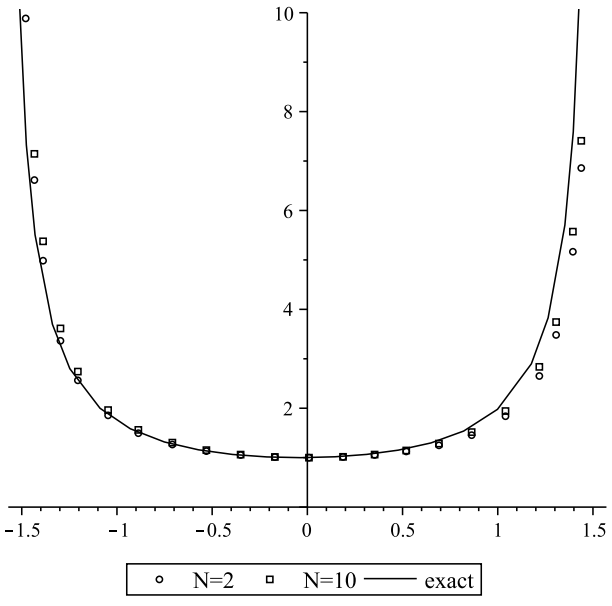
$$g_k(y;\eta) = \frac{1}{2\lambda_k}\begin{cases} e^{\lambda_k(y-\eta)} + e^{-\lambda_k(y+\eta)}, & y \le \eta \\ e^{\lambda_k(\eta-y)} + e^{-\lambda_k(y+\eta)}, & y \ge \eta \end{cases}$$

which reduces to

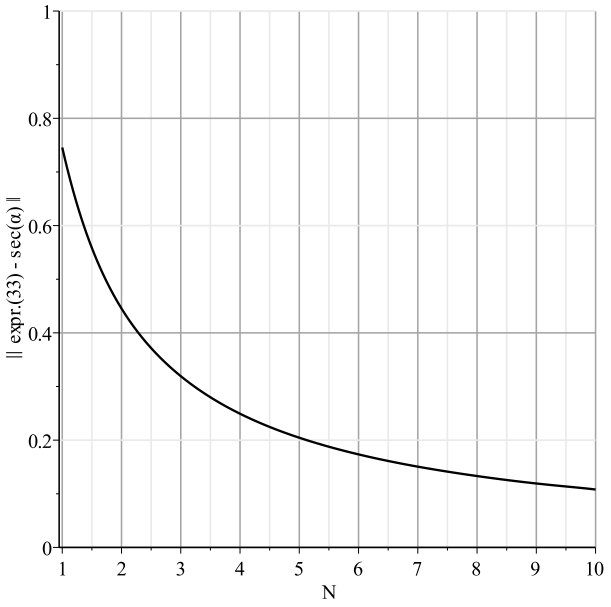
$$G(\alpha,\beta,\gamma,\theta) = \frac{1}{4\pi}\ln \quad \frac{\left(1+2e^{\alpha}\cos\ \gamma + e^{2\alpha}\right)}{\left(1-2e^{\alpha}\cos\ \gamma + e^{2\alpha}\right)}\frac{\left(1+2e^{\beta}\cos\ \gamma + e^{2\beta}\right)}{\left(1-2e^{\beta}\cos\ \gamma + e^{2\beta}\right)}$$

$$\times\frac{\left(1-2e^{\alpha}\cos\ \theta + e^{2\alpha}\right)}{\left(1+2e^{\alpha}\cos\ \theta + e^{2\alpha}\right)}\frac{\left(1-2e^{\beta}\cos\ \theta + e^{2\beta}\right)}{\left(1+2e^{\beta}\cos\ \theta + e^{2\beta}\right)}\Bigg) \tag{37}$$

where we use the variables introduced earlier in (18).



(a) Convergence with the 2nd and 10th partial products.



(b) L^2-norm of the error in $[-0.95\frac{\pi}{2}, 0.95\frac{\pi}{2}\ ]$; N is a number of partial products

Figure 4 : Convergence of the representation in (33)

Method of images. Distribution of singularities used in the method of images for the problem in (35)-(36) is shown in Figure 5. This gives rise to the infinite product version of the Green's function, written in terms of the variables introduced in (18), as

© 2013 Global Journals Inc. (US)

$$G(x,y;\xi,\eta)=\frac{1}{4\pi}\ln\prod_{k=-\infty}^{\infty}\frac{\Big((\gamma+\pi(1-2k))^2+\alpha^2\Big)\Big((\theta-2\pi k)^2+\alpha^2\Big)}{\Big((\theta-\pi(1+2k))^2+\beta^2\Big)\Big((\gamma-2\pi k)^2+\beta^2\Big)}$$

$$\frac{\Big((\gamma+\pi(1-2k))^2+\beta^2\Big)\Big((\theta-2\pi k)^2+\beta^2\Big)}{\Big((\theta-\pi(1+2k))^2+\alpha^2\Big)\Big((\gamma-2\pi k)^2+\alpha^2\Big)}$$

which being compared with (37) constitutes another identity valid for $\alpha\neq\pm\beta$, $\gamma\neq\pm\theta$. Assuming $\gamma=0$, $\theta=\frac{\pi}{2}$, $\beta=2u$, $\alpha=2v$, one arrives at

$$\coth^2 u\ \coth^2 v=\prod_{k=-\infty}^{\infty}\frac{\Big(\pi^2(1+2k)^2+4v^2\Big)\Big(\pi^2(1+2k)^2+4u^2\Big)}{\Big(\pi^2k^2+u^2\Big)\Big(\pi^2k^2+v^2\Big)}$$

Note that it has already been obtained earlier (see (25)) in this presentation.

b) Exterior of Circles

The key idea of the developments in the previous sections is to obtain an alternative to the classical expression for a Green's function as an appropriate arrangement of sinks and sources. For some regions such arrangements are periodic and described by infinite sequences of symmetrically placed images.

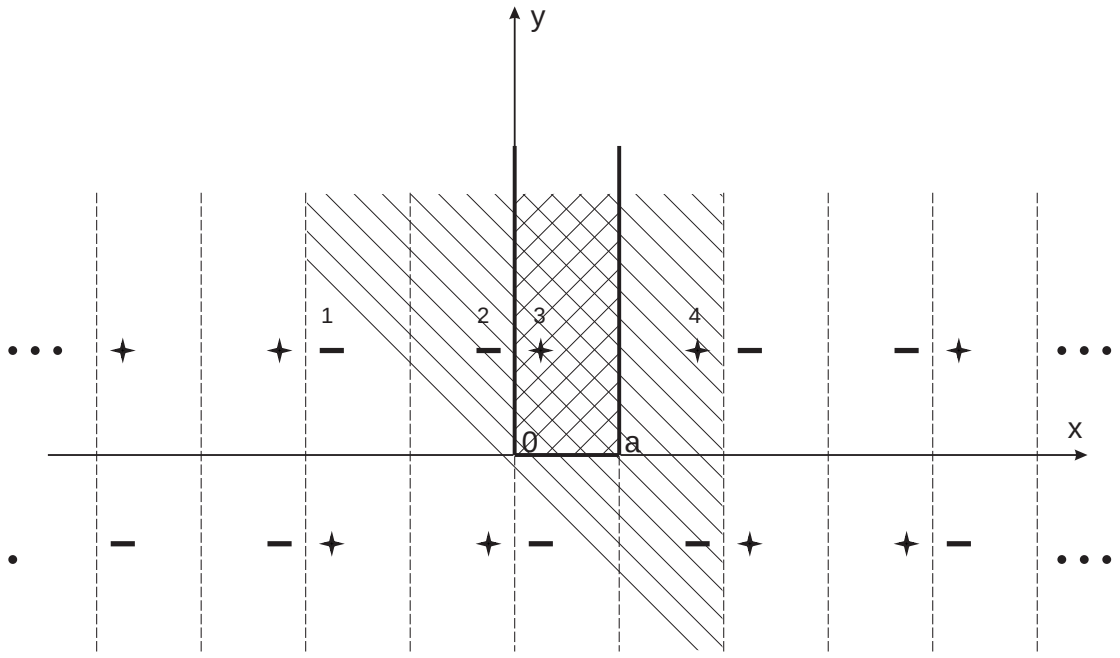


Figure 5 : Sequence of sources and sinks that arises in the method of images for the problem in (35)-(36)

© 2013 Global Journals Inc. (US)

This approach was applied, for instance, in [8] to construct the potential due to a line charge in the infinite strip or in [2, 8] to construct the potential induced by a single line charge in the rectangular prism or in the circular ring. The same approach was used in [6, 7] to find a potential field induced by the Green's function to the boundary-value problem posed for the Laplace equation in infinite and semi-infinite strip region with different combinations of boundary conditions. Results obtained in the earlier works served as a basis for construction of infinite product representations of elementary functions. The present work is the logical continuation of [6, 7], and the expansions (31) - (32) and (33) - (34) complement the results derived there.

Working on further developments of the discussed approach, we have probably to turn to different geometries or different equations. One of such cases is described below.

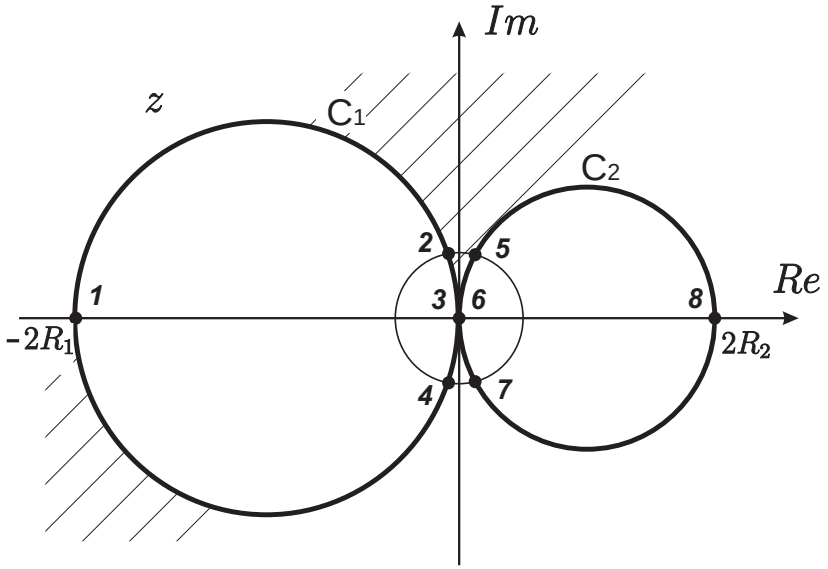
Conformal mapping. Conformal mapping gives us a tool to derive Green's functions of the Dirichlet problem posed for the Laplace equation in a simply-connected region. If there is known a function $w(z,\zeta)$ which conformally maps the given region in the z-plane onto the interior of the unit disc in the w-plane with point ζ mapped on the center of the disc, the corresponding Green's function is presented ([1, 2, 7]) in terms of $w(z,\zeta)$ as

$$G(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln|\omega(z,\zeta)| \tag{38}$$

where

$$z = x + iy$$
$$\zeta = \xi + i\eta$$

As an example, consider the exterior of two circles having external contact (Figure 6). Function $f(z) = 1/z$ transforms this region into the infinite strip ([1, 2]) while the latter can be mapped onto the interior of a unit circle. Omitting cumbersome but trivial algebra, we just present the function that maps the exterior of two circles with external contact onto the interior of a unit circle as



2. Z. Nehari, Conformal mapping, McGraw-Hill, New York, 1952.

© 2013 Global Journals Inc. (US)

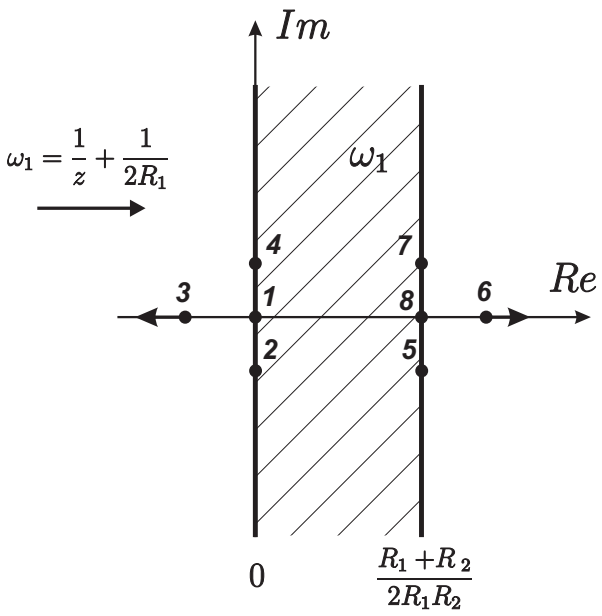


Figure 6 : Conformal mapping of the exterior of two circles with external contact onto the infinite strip

$$\omega(z,\zeta) = \frac{\exp\left(i\lambda\left(\frac{1}{z}+\frac{1}{2R_1}\right)\right) - \exp\left(i\lambda\left(\frac{1}{\zeta}+\frac{1}{2R_1}\right)\right)}{\exp\left(i\lambda\left(\frac{1}{z}+\frac{1}{2R_1}\right)\right) - \exp\left(\overline{i\lambda\left(\frac{1}{\zeta}+\frac{1}{2R_1}\right)}\right)}$$

where

$$\lambda = 2\pi\frac{R_1R_2}{R_1+R_2}$$

The modulus of the above reads as

$$|\omega(z,\zeta)| = \sqrt{\frac{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\lambda(\alpha-\beta)}{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\lambda\left(\alpha+\beta+\frac{1}{R_1}\right)}}$$

where

$$\alpha = \frac{x}{x^2+y^2}, \quad \beta = \frac{\xi}{\xi^2+\eta^2}, \quad \gamma = \frac{y}{x^2+y^2}, \quad \theta = \frac{\eta}{\xi^2+\eta^2} \tag{39}$$

Hence, in light of (38), the Green's function to the Dirichlet problem for the region depicted in Figure 6 appears as

$$G(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\frac{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda\left[\alpha+\beta+\frac{1}{R_1}\right]\right)}{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda(\alpha-\beta)\right)} \tag{40}$$

© 2013 Global Journals Inc. (US)

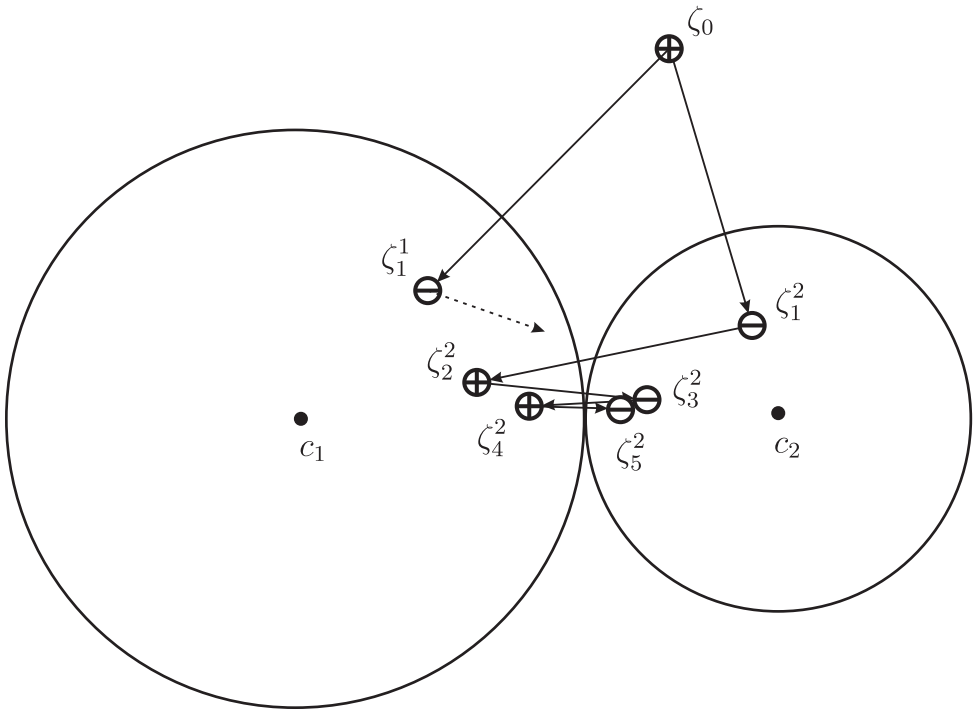


Figure 7: Sequence of sources and sinks that arises in the method of images

Method of images. In the case when boundaries of the region are formed by straight lines, images are placed symmetrically with respect to boundaries in a straightforward manner. But every straight line can be considered as a circumference with infinite radius. Thus the method of images can also be generalized to regions formed by circular arcs. In order to implement this idea, recall that an inversion with respect to a circle with radius R centered at a point c is given by the formula

$$z_1 = f(z_0) = \frac{R^2}{z_0 - c} + c \tag{41}$$

where the points c, z_0, z_1 lie in the same plane.

Thus the Green's function to the Dirichlet problem posed for the Laplace equation in the exterior of a circle is given by the expression

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln \frac{|z - z_0|}{|z - z_1|} + \mu(x, y; \xi, \eta) \tag{42}$$

The compensatory function $\mu(\, x, y; \xi, \eta\,)$ can easily be derived from the fact that the argument of the logarithm in (42) has a constant value on the circumference of a circle and is equal to $\frac{|z_0 - c|}{R}$.

Thus the formula (42) converts to

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{|z - z_1||z_0 - c|}{|z - z_0|R}$$

© 2013 Global Journals Inc. (US)

In the case of two circles, each inversion of the form in (41) with respect to the first circle perturbs the potential on the circumference of the second circle and thus, should be compensated by another inversion with respect to the perturbed circle and so on. Consequently to satisfy the Dirichlet boundary conditions we should use the infinite sequence of inversions (Figure 7). This generates the potential field

$$F(z,\zeta_n^j) = \frac{1}{2\pi}\Bigg(\ln\frac{|\zeta_1^1 - z||\zeta_0^1 - c_1|}{|\zeta_0^1 - z|R_1} + \ln\frac{|\zeta_1^2 - z||\zeta_0^2 - c_2|}{|\zeta_0^2 - z|R_2}$$
$$- \ln\frac{|\zeta_2^1 - z||\zeta_1^1 - c_2|}{|\zeta_1^1 - z|R_2} - \ln\frac{|\zeta_2^2 - z||\zeta_1^2 - c_1|}{|\zeta_1^2 - z|R_1}$$
$$+ \ln\frac{|\zeta_3^1 - z||\zeta_2^1 - c_1|}{|\zeta_2^1 - z|R_1} + \ln\frac{|\zeta_3^2 - z||\zeta_2^2 - c_2|}{|\zeta_2^2 - z|R_2} - \ldots\Bigg)$$

Clearly, all the singularities appear twice in the above expression and hence the Green's function has the following infinite product form

$$G(x,y;\xi,\eta) = \frac{1}{2}F(z;\zeta_n^j)$$
$$= \frac{1}{4\pi}\ln\prod_{k=0}^{\infty}\frac{|\zeta_{2k+1}^1 - z|^2|\zeta_{2k+1}^2 - z|^2|\zeta_{2k}^1 - c_1||\zeta_{2k}^2 - c_2|}{|\zeta_{2k}^1 - z||\zeta_{2k}^2 - z||\zeta_{2k+2}^1 - z||\zeta_{2k+2}^2 - z||\zeta_{2k+1}^1 - c_2||\zeta_{2k+1}^2 - c_1|} \tag{43}$$

where $z = x+iy$ is a field point, c_1, c_2, R_1, R_2 are centres and radii of the circles, $\zeta_0^1 = \zeta_0^2 = z_0 = \xi + i\eta$ is a source point and ζ_k^j are points at which singularities are placed

$$\zeta_k^1 = \begin{cases} \dfrac{R_2^2}{\overline{\zeta_{k-1}^1 - c_2}} + c_2, & \text{if } k \text{ is even} \\ \dfrac{R_1^2}{\overline{\zeta_{k-1}^1 - c_1}} + c_1, & \text{if } k \text{ is odd} \end{cases}$$
$$\zeta_k^2 = \begin{cases} \dfrac{R_1^2}{\overline{\zeta_{k-1}^2 - c_1}} + c_1, & \text{if } k \text{ is even} \\ \dfrac{R_2^2}{\overline{\zeta_{k-1}^2 - c_2}} + c_2, & \text{if } k \text{ is odd} \end{cases} \tag{44}$$

In our case, $c_1 = -R_1, c_2 = R_2$ and the expressions in (44) transform to

$$\zeta_k^1 = \begin{cases} \dfrac{z_0}{\dfrac{k}{2}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)z_0 + 1}, & \text{if } k \text{ is even} \\ \dfrac{-\overline{z}_0}{\dfrac{1}{2}\left(\dfrac{k+1}{R_1} + \dfrac{k-1}{R_2}\right)\overline{z}_0 + 1}, & \text{if } k \text{ is odd} \end{cases}$$

© 2013 Global Journals Inc. (US)

$$\zeta_k^2 = \begin{cases} \dfrac{-z_0}{\dfrac{k}{2}\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right)z_0 - 1}, & \text{if } k \text{ is even} \\ \dfrac{\overline{z}_0}{\dfrac{1}{2}\left(\dfrac{k-1}{R_1}+\dfrac{k+1}{R_2}\right)\overline{z}_0 - 1}, & \text{if } k \text{ is odd} \end{cases}$$

When k approaches infinity both values ζ_k^1 and ζ_k^2 tend to zero, implying that the limit of the general term in (43) equals unity. That is

$$\lim_{k\to\infty}\left(\frac{|\zeta_{2k+1}^1 - z|^2|\zeta_{2k+1}^2 - z|^2|\zeta_{2k}^1 + R_1||\zeta_{2k}^2 - R_2|}{|\zeta_{2k}^1 - z||\zeta_{2k}^2 - z||\zeta_{2k+2}^1 - z||\zeta_{2k+2}^2 - z||\zeta_{2k+1}^1 - R_2||\zeta_{2k+1}^2 + R_1|}\right) = 1$$

and thus the infinite product in (43) is absolutely and uniformly convergent.

Infinite Products and Elementary Functions. Obviously, in (40) and (43) one finds just alternative representations of the same Green's function, and we arrive at the multivariable identity

$$\frac{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda\left[\alpha+\beta+\dfrac{1}{R_1}\right]\right)}{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda(\alpha-\beta)\right)}$$

$$= \prod_{k=0}^{\infty}\frac{|\zeta_{2k+1}^1 - z|^2|\zeta_{2k+1}^2 - z|^2|\zeta_{2k}^1 + R_1||\zeta_{2k}^2 - R_2|}{|\zeta_{2k}^1 - z||\zeta_{2k}^2 - z||\zeta_{2k+2}^1 - z||\zeta_{2k+2}^2 - z||\zeta_{2k+1}^1 - R_2||\zeta_{2k+1}^2 + R_1|} \tag{45}$$

Complex variables z and z_0 can be written in terms of real parameters α, β, γ, θ, which were introduced earlier in (39), as

$$z_0 = \xi + i\eta = \frac{\beta + i\theta}{\beta^2 + \theta^2}$$

$$z = x + iy = \frac{\alpha + i\gamma}{\beta^2 + \theta^2}$$

Thus, the expression in (45) has six arbitrary parameters: α, β, γ, θ, R_1 and R_2. If $\lambda = 1$, then $R_2 = R_1/(2\pi R_1 - 1)$ and the number of independent parameters reduces to five.

Trigonometric functions. For any fixed value of the parameters γ and θ the expression in (45)involves only trigonometric functions. Particularly, when $\gamma = 0$ and $\theta = 0$ the left-hand side of (45) represents the following elementary function

$$\frac{1 - \cos\left(\alpha+\beta+\dfrac{1}{R_1}\right)}{1 - \cos(\alpha-\beta)} \tag{46}$$

© 2013 Global Journals Inc. (US)

If $R_1 = \dfrac{1}{n\pi}$ $\left(R_2 = \dfrac{1}{\pi(2-n)}, n = 0, 1, ...\right)$ the above expression converts to

$$\frac{1-(-1)^n \cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} \tag{47}$$

Similarly, when $R_1 = \dfrac{2}{\pi(1+2n)}$ $\left(R_2 = \dfrac{2}{\pi(3-2n)}, n = 0, 1, ...\right)$, the function (46) converts to

$$\frac{1+(-1)^n \sin(\alpha+\beta)}{1-\cos(\alpha-\beta)}$$

So the functions (47) and (48) have the following infinite product representations

$$\frac{1-(-1)^n \cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} = \prod_{k=0}^{\infty} \frac{\Big[\alpha+\beta+\pi(2k+n)\Big]^2 \Big[\alpha+\beta-\pi(2(k+1)-n)\Big]^2}{\Big|(\alpha-\beta)^2-4\pi^2k^2\Big|\Big|(\alpha-\beta)^2-4\pi^2(k+1)^2\Big|} \tag{49}$$

and

$$\frac{1+(-1)^n \sin(\alpha+\beta)}{1-\cos(\alpha-\beta)} = \prod_{k=0}^{\infty} \frac{\Big[2(\alpha+\beta)+\pi(1+4k+2n)\Big]^2 \Big[2(\alpha+\beta)-\pi(3+4k-2n)\Big]^2}{16\Big|(\alpha-\beta)^2-4\pi^2k^2\Big|\Big|(\alpha-\beta)^2-4\pi^2(k+1)^2\Big|} \tag{50}$$

Different interrelations between the parameters α and β are possible. For instance, when $\beta = \alpha \pm (2p+1)\pi$ $(p = 0, 1, 2, ...)$, the expressions in (49) and (50) read

$$\frac{1+(-1)^n \cos(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\Big[\Big(2\alpha+\pi\big(2p+n\big)\Big)^2-\pi^2(2k+1)^2\Big]^2}{\pi^4\Big|(2p+1)^2-4k^2\Big|\Big|(2p+1)^2-4(k+1)^2\Big|}, \tag{51}$$

$$\frac{1+(-1)^n \cos(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\Big[\Big(2\alpha-\pi\big(2(p+1)-n\big)\Big)^2-\pi^2(2k+1)^2\Big]^2}{\pi^4\Big|(2p+1)^2-4k^2\Big|\Big|(2p+1)^2-4(k+1)^2\Big|} \tag{52}$$

and

$$\frac{1-(-1)^n \sin(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\Big[\Big(4\alpha+\pi\big(2(2p+n)+1\big)\Big)^2-4\pi^2(2k+1)^2\Big]^2}{16\pi^4\Big|(2p+1)^2-4k^2\Big|\Big|(2p+1)^2-4(k+1)^2\Big|},$$

$$\frac{1-(-1)^n \sin(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\Big[\Big(4\alpha-\pi\big(2(2p-n)+3\big)\Big)^2-4\pi^2(2k+1)^2\Big]^2}{16\pi^4\Big|(2p+1)^2-4k^2\Big|\Big|(2p+1)^2-4(k+1)^2\Big|}$$

which, in turn, provide us with infinite product representations of the sine and cosine functions

© 2013 Global Journals Inc. (US)

$$\cos\alpha = (-1)^n \left(-1+2\prod_{k=0}^{\infty} \frac{\left[\left(\alpha+\pi\left(2p+n\right)\right)^2 - \pi^2(2k+1)^2\right]^2}{\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{53}$$

$$\cos\alpha = (-1)^n \left(-1+2\prod_{k=0}^{\infty} \frac{\left[\left(\alpha-\pi(2(p+1)-n)\right)^2 - \pi^2(2k+1)^2\right]^2}{\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{54}$$

and

$$\sin\alpha = (-1)^n \left(1-2\prod_{k=0}^{\infty} \frac{\left[\left(2\alpha+\pi\left(2(2p+n)+1\right)\right)^2 - 4\pi^2(2k+1)^2\right]^2}{16\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{55}$$

$$\sin\alpha = (-1)^n \left(1-2\prod_{k=0}^{\infty} \frac{\left[\left(2\alpha-\pi\left(2(2p-n)+3\right)\right)^2 - 4\pi^2(2k+1)^2\right]^2}{16\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{56}$$

where $n, p = 0, 1, 2, ...$ For example, in the c ase of n= 1 and p = 0 the identities in (53)-(56) reduce to

$$\cos\alpha = 1+2\prod_{k=0}^{\infty} \frac{\left((\alpha+\pi)^2 - \pi^2(2k+1)^2\right)^2}{\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{57}$$

$$\cos\alpha = 1+2\prod_{k=0}^{\infty} \frac{\left((\alpha-\pi)^2 - \pi^2(2k+1)^2\right)^2}{\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{58}$$

and

$$\sin\alpha = -1-2\prod_{k=0}^{\infty} \frac{\left((2\alpha+3\pi)^2 - 4\pi^2(k+1)^2\right)^2}{16\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{59}$$

$$\sin\alpha = -1-2\prod_{k=0}^{\infty} \frac{\left((2\alpha-\pi)^2 - 4\pi^2(k+1)^2\right)^2}{16\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{60}$$

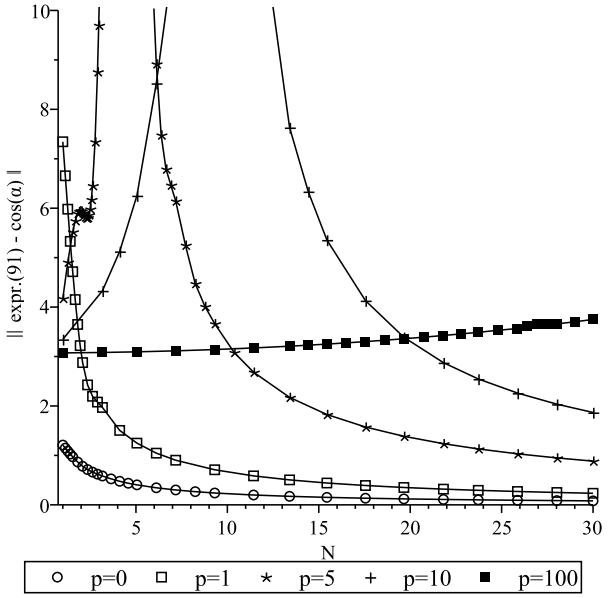
Figures 8 and 9 illustrate the sensitivity of the convergence of the representation in (53) to the choice of the parameters n and p. It is seen that the parameter values have the dramatical influence on the rate of convergence. The identities in (54)-(56) exhibit a similar behaviour.

By applying the power reduction formula to the expressions in (51) and (52) several new infinite product expansions for the sine and cosine functions can be obtained.

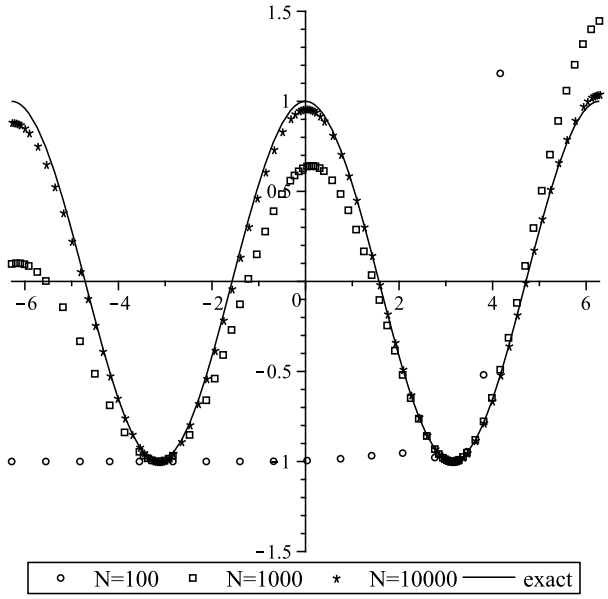
© 2013 Global Journals Inc. (US)

If n in (51) and (52) is an even number, i.e. $n = 2n$, then the expansions of the cosine function can be written as

$$\cos\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|4\Big(\alpha+\pi\big(p+n\big)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{61}$$



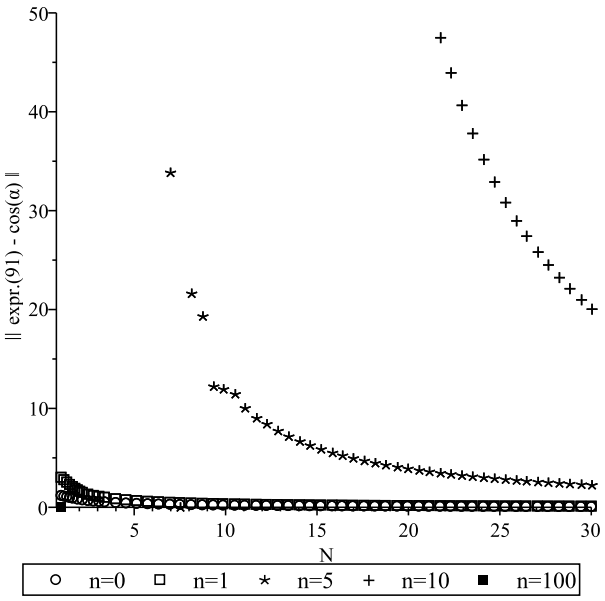
(a) L^2-norm of the error in $[0, 2\pi]$; N is a number of partial products.



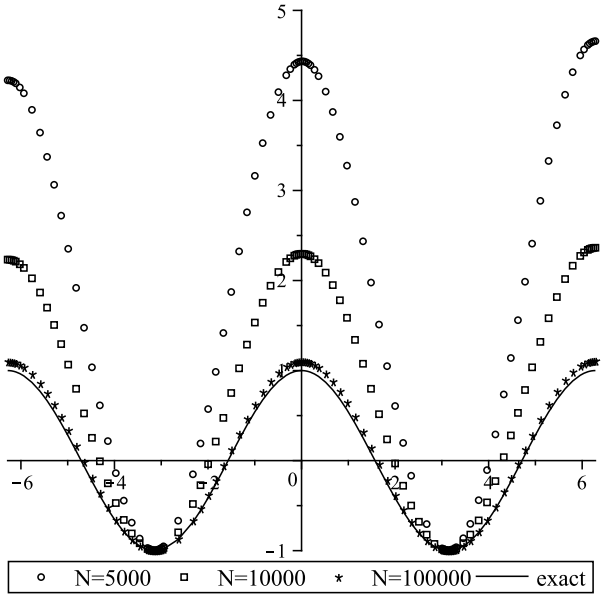
(b) Convergence of the infinite product expansion when $n = 0$, $p = 100$

Figure 8 : Convergence of the representation in (53) for different values of the parameter p; parameter $n = 0$ is fixed

© 2013 Global Journals Inc. (US)



(a) L^2-norm of the error in $[0, 2\pi]$; N is a number of partial products



(b) Convergence of the infinite product expansion when $p{=}\ 0,\ n = 100$

Figure 9 : Convergence of the representation in (53) for different values of the parameter n; parameter $p{=}\ 0$ is fixed

$$\cos\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|4\Big(\alpha-\pi(p-n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{62}$$

where the plus sign corresponds to $\alpha\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right)$ and the minus sign corresponds to $\alpha\in\left[\frac{\pi}{2};\frac{3\pi}{2}\right]$.

© 2013 Global Journals Inc. (US)

Similarly, when n in (51) and (52) is an odd number, i.e. $n= 2n+ 1$, then the expansion of the sine function can be written as

$$\sin\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi\big(2(p+n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{63}$$

$$\sin\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi(2(p-n)+1)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{64}$$

where the plus sign corresponds to $\alpha\in[0;\pi)$ and the minus sign corresponds to $\alpha\in[\pi;2\pi]$.

The infinite product representations of the tangent function can also be obtained from the expansions in (61)-(64). Those are

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi\big(2(p+n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha+\pi\big(p+n\big)\Big)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi\big(2(p+n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha-\pi(p-n+1)\Big)^2-\pi^2(2k+1)^2\right|}$$

and

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi(2(p-n)+1)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha+\pi\big(p+n\big)\Big)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi(2(p-n)+1)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha-\pi(p-n+1)\Big)^2-\pi^2(2k+1)^2\right|}$$

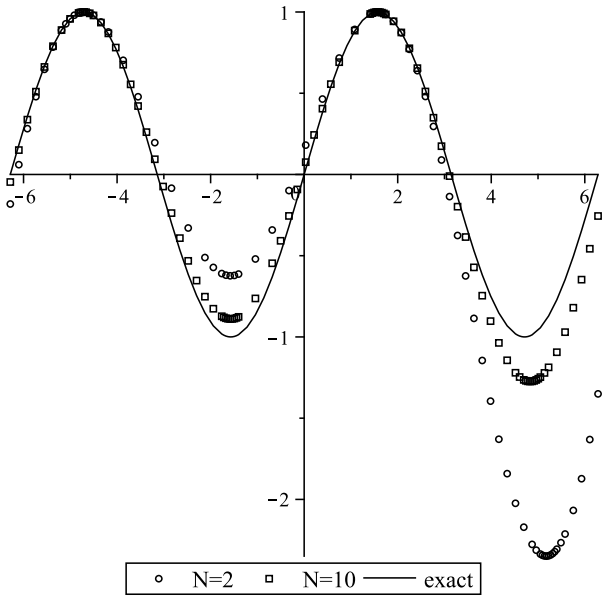
where the plus sign corresponds to $\alpha\in\left[-\frac{\pi}{2};0\right)$ and the minus sign corresponds to $\alpha\in\left[0;\frac{\pi}{2}\right]$.

So far we assumed $\beta=\alpha\pm(2p+1)\pi$. Different representations can be derived for the relation between α and β as

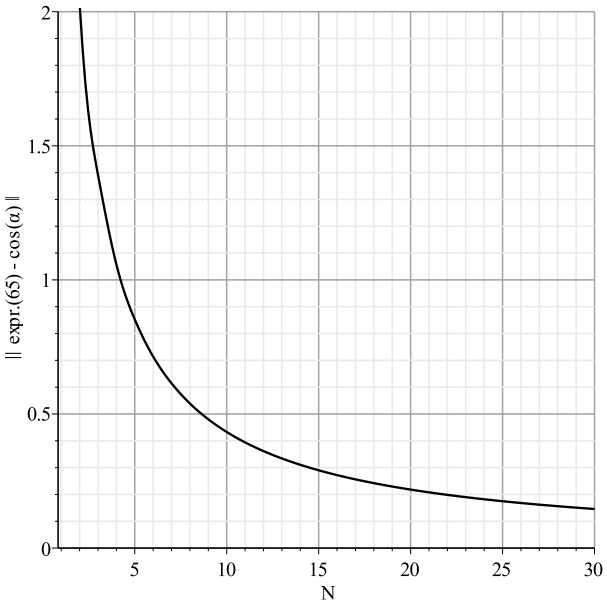
$$\beta=\alpha\pm(2p+1)\frac{\pi}{2},\qquad p=0,1,2,...$$

yielding in the following

© 2013 Global Journals Inc. (US)



(a) Convergence with the 2nd and 10th partial products.



(b) $L2$-norm of the error in $[0, 2\pi]$; N is a number of partial products

Figure 10 : Convergence of the representation in (65)

$$\sin\alpha = (-1)^{n+p}\left(-1+\prod_{k=0}^{\infty}\frac{\left[\left(2\alpha+\pi(2(p+n)-1)\right)^2-4\pi^2(2k+1)^2\right]^2}{\pi^4\left|(1+2p)^2-16k^2\right|\left|(1+2p)^2-16(k+1)^2\right|}\right) \tag{65}$$

$$\sin\alpha = (-1)^{n+p}\left(1-\prod_{k=0}^{\infty}\frac{\left[\left(2\alpha-\pi(2(p-n)+3)\right)^2-4\pi^2(2k+1)^2\right]^2}{\pi^4\left|(1+2p)^2-16k^2\right|\left|(1+2p)^2-16(k+1)^2\right|}\right) \tag{66}$$

© 2013 Global Journals Inc. (US)

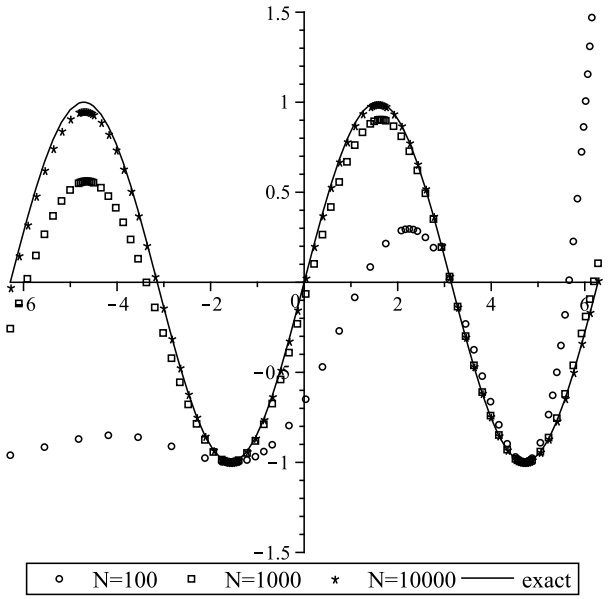
and

$$\cos\alpha = (-1)^{n+p}\left(-1+\prod_{k=0}^{\infty}\frac{16\Big[\Big(\alpha+\pi(p+n)\Big)^2-\pi^2(2k+1)^2\Big]^2}{\pi^4\Big|(1+2p)^2-16k^2\Big|\Big|(1+2p)^2-16(k+1)^2\Big|}\right) \tag{67}$$

$$\cos\alpha = (-1)^{n+p}\left(1-\prod_{k=0}^{\infty}\frac{16\Big[\Big(\alpha-\pi(p-n+1)\Big)^2-\pi^2(2k+1)^2\Big]^2}{\pi^4\Big|(1+2p)^2-16k^2\Big|\Big|(1+2p)^2-16(k+1)^2\Big|}\right) \tag{68}$$

The convergence rate of the expansion in (65) with $n=1$, $p=0$ is illustrated in Figure 10. The sensitivity of the convergence in (65) to the choice of the parameters n and p is illustrated in Figure 11. The identities in (66)-(68) exhibit the similar behaviour.

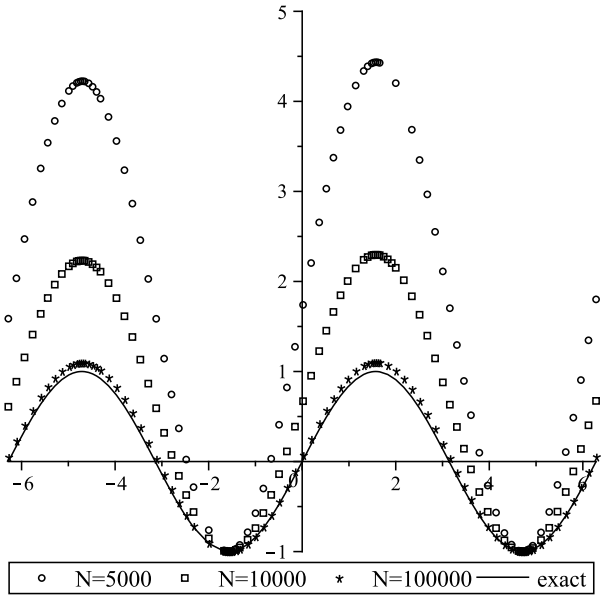
By applying the power reduction formula to the identities in (67)-(68) one can derive alternative representations of the sine and cosine functions. When n and p are even numbers, i.e. $n=2n$ and $p=2p$, the expansions of the cosine and sine functions can be obtained in the form



(a) Convergence of the infinite product expansion when $p=0$, $n=100$; N is a number of partial products

© 2013 Global Journals Inc. (US)

Notes



(b) Convergence of the infinite product expansion when $p=0$, $n=100$; N is a number of partial products

Figure 11 : Convergence of the representation in (65) for different values of the parameters n and $p=0$

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+2\pi(p+n)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{69}$$

and

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-\pi(2p-2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{70}$$

When n is odd and p is even, i.e. $n=2n+1$ and $p=2p$, one arrives at

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+\pi(2p+2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{71}$$

and

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-2\pi(p-n)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{72}$$

When n is even and p is odd, i.e. $n=2\ n$ and $p=2p+1$, we arrive at

© 2013 Global Journals Inc. (US)

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\Big(2\alpha+\pi(2p+2n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{73}$$

and

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\Big(2\alpha-2\pi\left((p-n)+1\right)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{74}$$

Finally, if both n and p are odd, i.e. $n=2n+1$ and $p=2p+1$, the expressions in (67)-(68) convert to

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\Big(2\alpha+2\pi(p+n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{75}$$

and

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\Big(2\alpha-\pi(2p-2n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{76}$$

In the above representations, the sign is selected similarly to those in (61)-(64)

Representations for the tangent function follow directly from (69)-(76). This provide us with the total number of sixteen expansions. Four of those are shown below.

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi(2p-2n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\left|\Big(\alpha+2\pi(p+n)\Big)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi(2p-2n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\left|\Big(\alpha-2\pi(p-n)\Big)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi(2p+2n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\left|\Big(\alpha+2\pi(p+n)\Big)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi(2p+2n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\left|\Big(\alpha-2\pi(p-n)\Big)^2-\pi^2(2k+1)^2\right|}$$

while the others can be derived in a similar manner.

Notes

© 2013 Global Journals Inc. (US)

Acknowledgement

It is with gratitude that the author acknowledges the advises of Prof. Y.Melnikov in the process of preparation of this work.

References Références Referencias

1. C. Caratheodory, Conformal representation, Cambridge university press, 1969.
2. Z. Nehari, Conformal mapping, McGraw-Hill, New York, 1952.
3. I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press Inc., 1980
4. Dean G. Duffy, Green's Functions with Applications, Chapman and Hall/CRC, 2001.
5. R. Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, Pearson, 2012.
6. Yu. A. Melnikov, A new approach to the representation of some trigonometric and hyperbolic functions by infinite products, J. Math. Anal. Appl., 344, 1, 2008, 521-534.
7. Yuri A. Melnikov, Green's functions and Infinite Products: Bridging the Divide, Birkhauser-Springer, New York - London, 2011.
8. P.M. Morse, H. Feshbach, Methods of Theoretical Physics, Vol.2, McGraw-Hill, New York - Toronto- London 1953.

Notes

© 2013 Global Journals Inc. (US)

This page is intentionally left blank

© 2013 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 4 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Comparison of Numerical Schemes for Shallow Water Equation

By Md.Saiduzzaman & Sobuj Kumar Ray

IUBAT-International University of Business Agriculture and Technology Dhaka, Bangladesh

Abstract - Many problems of river management and civil protection consist of the evaluation of the maximum water levels and discharges that may be attained at particular locations during the development of an exceptional meteorological event. Numerical methods have become a useful tool to predict discharges and water levels in hydraulic systems. The shallow water or St. Venant equations, being a hyperbolic quasi- linear partial differential system represents a good candidate for the application of many of the techniques developed originally for Fluid Dynamics.

In this paper we will present different numerical schemes such as Lax-Friedrich scheme, Lax-Wendroff scheme, Leap-Frog scheme for the shallow water equation and implement the numerical schemes by computer programming. Next we will compare these different schemes with respect to their efficiency and the quality of the solution indeed. Here we apply linear advection equation in order to test the accuracy of these schemes for shallow water equation.

Keywords : *equation of continuity, linear advection equation, lax-friedrich scheme, lax-wendroff scheme, leap-frog scheme, shallow water equations. stencil.*

GJSFR-F Classification : *MSC 2010: 32W50*



Strictly as per the compliance and regulations of :

© 2013. Md.Saiduzzaman & Sobuj Kumar Ray. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

eJournal
physical existance Print Journal
epaper
open standard
3d virtual journal

1. Pilar García-Navarro & Pilar Brufau, "One Dimensional Dam Break Flow Modeling: Some results".

Comparison of Numerical Schemes for Shallow Water Equation

Md.Saiduzzaman[α] & Sobuj Kumar Ray[σ]

Abstract - Many problems of river management and civil protection consist of the evaluation of the maximum water levels and discharges that may be attained at particular locations during the development of an exceptional meteorological event. Numerical methods have become a useful tool to predict discharges and water levels in hydraulic systems. The shallow water or St. Venant equations, being a hyperbolic quasi- linear partial differential system represents a good candidate for the application of many of the techniques developed originally for Fluid Dynamics.

In this paper we will present different numerical schemes such as Lax-Friedrich scheme, Lax-Wendroff scheme, Leap-Frog scheme for the shallow water equation and implement the numerical schemes by computer programming. Next we will compare these different schemes with respect to their efficiency and the quality of the solution indeed. Here we apply linear advection equation in order to test the accuracy of these schemes for shallow water equation.

Keywords : equation of continuity, linear advection equation, lax-friedrich scheme, lax-wendroff scheme, leap-frog scheme, shallow water equations. stencil.

I. Introduction

The shallow water equations (SWEs) describe the evolution of a hydrostatic homogeneous (constant density), incompressible fluid in response to gravitational and rotational accelerations and they are derived from the principles of conservation of mass and conservation of momentum. The SWEs (also called Saint–Venant equations) are one of the simplest form of the equations of motion that can be used to describe the horizontal structure of an atmosphere and ocean that model the propagation of disturbances in fluids. They are widely used to model the free surface water flows such as periodic (tidal) flows, transient wave phenomena [1] (tsunamis, flood waves, and dam-break waves) etc. The shallow water equations are only relevant when the horizontal scale of the flow is much smaller than the depth of the fluid.

In fluid dynamics the flow of the fluid is known as the Navier – Stocks equation. The shallow water equations are good approximation to the fluid motion equation when fluid density is homogeneous and depth is small in comparison to characteristic horizontal distance. So the most attention will be given to the flow of shallow water equation [2].

Shallow water equation is a system of first order partial differential equation. It is linear in derivative but non-linear in unknowns and so is called quasi-linear partial differential equation [3], [4].

Authors α : Lecturer, Dept. of Mathematics, IUBAT-International University of Business Agriculture and Technology Dhaka, Bangladesh. E-mail : szaman@iubat.edu

Authors σ : Assistant Manager, DESCO-Dhaka Electric Supply Company Limited, M.Sc Student of DUET-Dhaka University of Engineering and Technology. E-mail : sobuj_kumar_ray@yahoo.com

© 2013 Global Journals Inc. (US)

II. Mathematical Models for Fluid Motions

a) Symbols and Notations

Let $\Omega \subset R^d$, $d \in$ –1,2″ be a region occupied by a fluid flow and let $[t_0, T)$ be time interval with $0 \leq t_0 \leq T$. An arbitrary point in Ω is denoted by $\mathbf{x}=(x_1, \ldots, x_d)^T$. For the description of a general unsteady compressible fluid flow, we introduce the quantities: The density $\rho = \rho(\mathbf{x}, t)$, the velocity vector $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (v_1(\mathbf{x}, t), .., v_d(\mathbf{x}, t))^T$,the pressure $p = p(x, t)$, the energy density $E = E(\mathbf{x}, t)$.We denote the external forces by $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) = (f_1(x, t), \ldots, f_d(x, t))^T$, the mass flux by $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$. For the description of the viscous flow, let λ and μ denote the coefficient of viscosity and coefficient of kinematic viscosity respectively.

All the above quantities describing fluid flows are functions of space and time. The density of such a quantity is generally written as a function of $\Phi = \Phi(\mathbf{x}, t)$, where for given any time instant $t \in [t_0, T)$, $\mathbf{x} \in G(t)$ denote the points of set $G(t) \subset R^d$ occupied by the fluid at time t. So the domain of definition of the function ϕ is the set

$$G = -(\mathbf{x},t); \mathbf{x} \in G(t), t \in [t_0,T)'' \in R^{d+1} \tag{1}$$

In particular $G(t_0)$ represents the domain occupied by the fluid at the initial time t_0. For the motion of a particular fluid particle, the trajectory can be described by an application $\mathbf{x} = \mathbf{X}(x_0, t_0; t)$ where $\mathbf{x}_0 \in G(t_0)$ represents the initial position of the particle at time t_0. Any bounded domain

$$V(t) = -\mathbf{x} = \mathbf{X}(x_0, t_0; t) — \mathbf{x}_0 \in V(t_0) \subset G(t_0)'' \subset G(t) \tag{2}$$

occupied by the fluid at any instant t is called a control volume. The total amount of a quantity with density $\phi(\mathbf{x}, t)$ contained in the volume $V(t)$ at time t is given by the integral

$$\phi(t) = \int_{V(t)} \phi(\mathbf{x}, t)\, d\mathbf{x} \tag{3}$$

In what follows, we are interested in the rate of change of $\phi(t)$.for the analysis of derivative

$$\frac{d}{dt}\phi(t) = \frac{d}{dt}\int_{V(t)} \phi(x, t)\, dx \tag{4}$$

The so-called Reynolds transport theorem plays a crucial rule:

b) The Equation of Continuity

The mass m(V(t)) of the fluid contained in an arbitrary control volume V(t) at some given time t is given by:

$$m(V(t) = \int_{V(t)} \rho(\mathbf{x}, t) d\mathbf{x}, \tag{5}$$

where $\rho(\mathbf{x}, t)$ is the mass density. As the domain V(t) contains the same particles at each time instant t (i.e. no mass is generated or destroyed), the law of conservation of mass can be written as:

The mass of a amount of fluid within the control volume $V(t)$ is independent of time, i.e.

© 2013 Global Journals Inc. (US)

$$\frac{dmV(t)}{dt} = 0, \text{ for all } t \in [t_0, T)$$

Assuming $\rho(\mathbf{x}, t)$ is smooth, one obtain from the Reynolds transport theorem the integral of the mass conservation law

$$\int_{V(t)} \left[\frac{\partial \rho}{\partial t}(\mathbf{x}, t) + \nabla.\left(\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)\right)\right] d\mathbf{x} = 0 \tag{6}$$

Notes

For all $t \in [t_0, T)$

As the control volume can be chosen arbitrary, whenever the integrand is continuous we obtain the differential form of the law of conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0 \tag{7}$$

which is known as the continuity equation.

c) The Linear Advection Equation

In fluid dynamics the flow of the fluid is known as the Navier- Stokes equation. Shallow water equations are good approximation to the fluid motion equation when the flow is one dimensional and incompressible unsteady fluid and derived from the water wave by assuming that the water depth is sufficiently small compared to the water wave length. Now we will discuss the mathematical description of the shallow water equation for flood modeling.

We now consider the linear advection equation, which is given by

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \tag{8}$$

The Cauchy problem is defined by this equation on the domain $-\infty < x < \infty$, $t \geq 0$ together with initial conditions $U(x, 0) = u_0(x)$

Here we assume that the velocity c is constant. For simplicity we assume that the velocity c is positive.

$$c > 0$$

If u_0 is differentiable, then the solution is simply

$$u(t, x) = u_0(x + ct) \tag{9}$$

for $t \geq 0$ (which is based on [5]). As time involved, the initial data simply propagates unchanged to the right ($c > 0$), The solution $u(t, x)$ is constant along each ray $x - ct = x_0$, which are known as characteristics of the equation.

The characteristics are curves in the x−t plane satisfying the ordinary differential equations $x'(t) = c$, $x(0) = x_0$.

If we differentiate $u(x, t)$ along one of this curves to find the rate of change of u along the characteristic, we find that

$$\frac{d}{dt}u(t, x(t)) = \frac{\partial}{\partial t}u(t, x(t)) + \frac{\partial}{\partial x}u(t, x(t))\,x' = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

Confirming that u is constant along these characteristics.

More generally, we might consider a variable coefficient advection equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial (c(x))u}{\partial x} = 0 \tag{10}$$

© 2013 Global Journals Inc. (US)

Where $c(x)$ is a function, then we have

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = -c'(x)$$

$$\text{or, } \left(\frac{\partial}{\partial t} + c(x)\frac{\partial}{\partial x}\right) u(x, t) = -c'(x)\, u(x, t)$$

It follows that the evaluation of u along any curve x(t) satisfying

$$x'(t) = c(x(t))$$

$$x(0) = x_0$$

Satisfies a simple ordinary differential equation:

$$\frac{d}{dt} u(t, x(t)) = -c'(x(t))\, u(x(t), t)$$

The curves determined by (11) are again called characteristics.

In this case the solution is not constant along these curves, but can be easily determined by solving two sets of ordinary differential equations.

It can be shown that if $u_0(x)$ is a smooth function, say $u_0 \in c^k(-\infty, \infty)$ then the solution $u(x, t)$ is equally smooth in space and time, $u \in c^k((-\infty, \infty).(0, \infty))$

In the above discussion we also mention the model of Linear advection equation as well as its solution. [6]

d) Shallow Water Equation

Shallow water waves which arise if the water height is much smaller than the wave length of the water. The flow of water is distributed process because the flow rate, velocity and depth in space and time. To derive the one dimensional equation, we consider,

1. The velocity parallel to x- direction and almost independent of y: $v(t, x) \underline{e}_x$
2. The fluid is incompressible, so the density ρ is constant.

The shallow water equations are based on the laws of conservation of mass and the laws of conservation of momentum [6]. Now we will show the mathematical description of the shallow water equation which shows the direction of the propagation of the wave length as well as the flow of the fluid.

e) Mathematical Description of the Model

Conservation of mass, simply state that the mass of the volume at a given time t equal to the mass of source inside it.

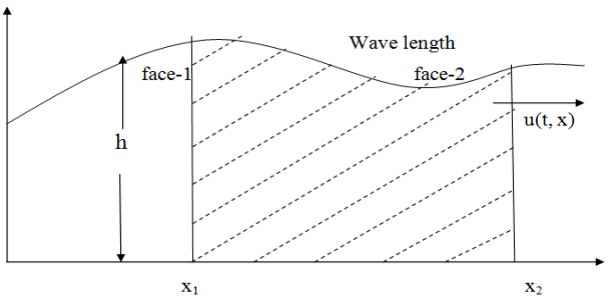


Figure 1: Shallow Water

Ref.
6. Randall J. Le Veque, "Numerical Methods for Conservation Laws".

© 2013 Global Journals Inc. (US)

Now the change of mass inside the volume $((\Delta x\, h(t,x)))$ is

$$\frac{\partial}{\partial t}(\rho \Delta x h(t,x) \tag{12}$$

Mass flow in/out the volume:
Mass flux vector $\rho v\ \underline{e}_x$ then integrate $\rho v\ \underline{e}_x$ at face-1 we get influx: $= q_1$

$$\text{So } q_1 = \rho\, v\, h(t, x_1)$$

Integrate $\rho v\ \underline{e}_x$ at face-2 we get influx: $= q_2$

$$\text{So } q_2 = -\rho\, v\, h(t, x_2)$$

So the change of mass flux $= \rho v h(t,x_1) - \rho v h(t,x_2)$ (2.2)
Hence we obtain from (2.1) and (2.2),

$$\frac{\partial}{\partial t}(\rho \Delta x h(t,\ x)) = \rho v h(t,x_1) - \rho v h(t,x_2)$$

$$\rho\ \frac{\partial}{\partial t}\ (h(t,\ x))\ = \rho\ -v h(t,x_1) - v h(t,x_2)/\Delta x''$$

Taking $\Delta x \to 0$ we have

$$\frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} \tag{13}$$

Conservation of momentum state that the rate of change of the total momentum of a volume of fluid formed by the same particles at any time and occupying the control volume $V(t)$ at the time instant t is equal to the resultant of the forces acting on the fluid in $V(t)$.
Now the change of momentum inside the volume $((\Delta x h(t,\ x)))$ is

$$\frac{\partial}{\partial t}\ (\rho v\ \Delta x\ h(t,\ x)) \tag{14}$$

Net flux of momentum:
Momentum flux vector $(\rho v) v\ \underline{e}_x$, then the net flux momentum at$[x_1, x_2]$ is

$$\rho v^2\ h\ (t,\ x_1) - \rho\ v^2 h(t\ ,\ x_2) \tag{15}$$

The pressure P is determined from a hydrostatic law, stating that the pressure at depth y is $\rho g h$, where g is gravitational force. Then pressure force for dy is,

$$P(t,\ x,\ y) = \rho\ g\ (h\ (t\ ,\ x) - y) + p_0$$

Where p_0 is atmospheric pressure. Consider $P_1 = P(t,\ x_1\ ,\ y)$ and $P_2 = P(t,\ x_2\ ,\ y)$.
Now force acting at $[x_1, x_2]$ is

$$P_1\ dy - p_2\ dy = \rho g(h(t,\ x_1) - h(t,\ x_1))\ dy$$

Integrating from $y = 0$ to $y = h(t,\ x)$ we have total forces

$$\rho g\ (h(t,\ x_1) - h(t\ ,\ x_1))\ h(t\ ,\ x) \tag{16}$$

© 2013 Global Journals Inc. (US)

using (14) , (15) and (16) we have,

$$\rho \frac{\partial}{\partial t}(\rho \Delta x h(t, x)) = \rho v^2 h(t, x_1) - \rho v^2 h(t, x_2) + \rho g (h(t, x_1) - h(t, x_1)) h(t, x)$$

$$\frac{\partial}{\partial t}(hv) = \rho - vh(t, x_1) - vh(t, x_2)/\Delta x'' + \rho g\, h(t, x) - h(t, x_1) - h(t, x_2)/\Delta x''$$

Taking $\Delta x \to 0$ and dividing by ρ then we have,

$$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial t}(hv^2) + gh\left(\frac{\partial h}{\partial x}\right) = 0$$

$$h\frac{\partial v}{\partial t} + v\frac{\partial h}{\partial t} + hv\frac{\partial v}{\partial x} + v\frac{\partial (hv)}{\partial x} + gh\left(\frac{\partial h}{\partial x}\right) = 0$$

$$v\left(\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x}\right) + h\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) + gh\left(\frac{\partial h}{\partial x}\right) = 0$$

$$h\left(\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} + v\frac{\partial v}{\partial x}\right) = 0 \qquad \text{[using (13)]}$$

Hence Shallow water equation become,

$$\frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + g\frac{\partial h}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad (17)$$

If we denote $u := \begin{pmatrix} h \\ v \end{pmatrix}$ and A $(u) := \begin{pmatrix} v & h \\ g & v \end{pmatrix}$

Then we have

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0$$

This is a quasi-linear system of partial differential equations and this is the mathematical description of shallow water equation.

Now we construct two dimensional shallow water model and also add with source or rainfall term (based on [7]). Next we write vector form of shallow water equation. We are considering water flow into the system. Flood is caused by overflow of water. The heavy rainfall is one of the cause of it. So we have

$$\frac{\partial h}{\partial t} + h\frac{\partial u}{\partial x} + u\frac{\partial h}{\partial x} + h\frac{\partial v}{\partial y} + v\frac{\partial h}{\partial y} = f_s$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial h}{\partial x} + g\frac{\partial z}{\partial x} = a_{fx} \qquad (18)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial h}{\partial y} + g\frac{\partial z}{\partial y} = a_{fy}$$

And in vector form

$$\frac{dh}{dt} + \nabla.(Vh) = f_s$$

Ref.
7. C. B. Vreugdenhil, "Numerical Method for Shallow Water Flow" Kluwer Academic Publisher, Dordsecht 1994.

© 2013 Global Journals Inc. (US)

$$\frac{dh}{dt} + V.\nabla V + g\nabla h + g\nabla z - a_f = 0$$

Where

V(x, t)[m/s]: velocity vector, h(x, t) [m]: water level above ground, f_s: source term (Rainfall), z[m]: topographical height, g[m/s^2]: acceleration due to gravity, c_f [m]: Manning coefficient, a_f [m]: friction slop.

Where $a_f = V|V| \,/\, c_f$

III. Numerical Methods

a) Finite Difference Methods

We will develop approximations to solutions of general nonlinear scalar conservation laws on a bounded domain,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \text{ for a} < x < b, 0 < t \qquad (19)$$

Initially we will assume that the characteristic speeds satisfy $\lambda = f'(u) > 0$ for all states that occur in the solution of (19). Thus we also provide initial data

$$u(x, 0) = u_0(x)$$

and data at the left hand boundary

$$u(t, a) = u_a(t) \qquad (20)$$

The numerical techniques have been applied to find their approximate solution of (19), i.e density $u(x, t)$ and flux $f(u(x, t))$ at a certain number of points in the time space domain , in such way as to satisfy the basic equations as far as possible. This procedure is called the discretization(based on [8]). Each point is called a grid point and the set of grid points is called a grid. We will discretize space

$$a = x_{-1/2} < x_{1/2} < x_{3/2} < \ldots\ldots\ldots < x_{l-1/2} = b$$

and time

$$0 = t^0 < t^1 < t^2 < \ldots\ldots\ldots < t^N = T$$

We will define the componential grid cells to be the intervals $(x_{i+1/2}, x_{i-1/2})$ with cell weights

$$\Delta x_i \equiv (x_{i+1/2} - x_{i-1/2})$$

We will also define the time steps to be

$$\Delta t^{n+1/2} \equiv t^{n+1} - t^n$$

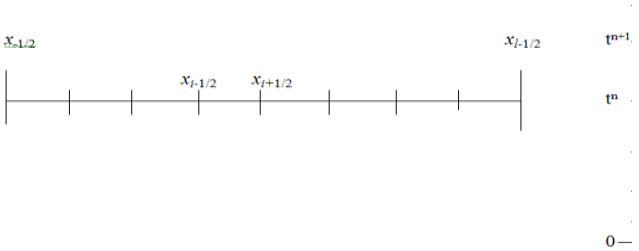


Figure 2 : Spatial and temporal discretization

8. L.S. Andallah,"Finite Difference Method-Explicit Upwind Difference scheme", Lecture note, Department of Mathematics, Jahangirnagar University, 2007.

© 2013 Global Journals Inc. (US)

The main numerical technique applied to the hyperbolic system of partial differential equations is the finite difference method (FDM). Its foundation is the following functions of continuous arguments which describe the state of the flow are replaced by functions defined on a finite number of the grid points within the considered domain. The derivates are then replaced by divided differences. Thus the differential equations are replaced by algebraic finite difference relationships. The different ways in which derivatives and integrals are expressed by discrete functions are called finite difference schemes or methods [9].

b) The CFL Condition

One of the first papers on finite difference methods for PDEs was written in 1928 by Courant, Friedrich and Levy [6]. They use finite difference methods as an analytic tool for proving existence of solutions of certain PDEs. The idea is to define a sequence of approximate solutions (via finite difference equation), prove that they converge as the grid is refined, and then show that the limit function must satisfy the PDE, giving existence of a solution.

In the course of proving convergence of this sequence, they recognized that a necessary stability condition for any numerical method of is that the domain of dependence of the finite difference method should be the domain of dependence of the PDEs, at least in the limit as k, $h \to 0$. This condition is known as the CFL, condition after Courant, Friedrich and Lewy.

The domain of dependence D $(\tilde{x}, \tilde{t})$ for the PDE has already been described in the previous section. The set $D(\tilde{x}, \tilde{t})$ consists of the points $\tilde{x} - c\tilde{t}$, since only initial data at these points can affect the solution at $(\tilde{x}, \tilde{t})$.

c) Lax-Friedrich Scheme

We now describe the Lax-Friedrich scheme for example we take linear advection equation [4].For discretization we use

1. Forward difference for time derivative

$$\text{i.e. } \frac{\partial u}{\partial t} \approx (u_i^n - u_i^n) / \Delta t^{n+1/2} \tag{21}$$

2. Central difference for spatial derivative

$$\text{i.e. } \frac{\partial u}{\partial x} \approx (u_{i+1}^n - u_{i-1}^n)/2\Delta x^{n+1/2} \tag{22}$$

Substituting these values in (8) we get

$$(u_i^n - u_i^n) / \Delta t^{n+1/2} + c(u_{i+1}^n - u_{i-1}^n)/2\Delta x^{n+1/2} = 0 \tag{23}$$

$$u_i^{n+1} = u_i^n - c\Delta t^{n+1/2}/2\Delta x_i (u_{i+1}^n - u_{i-1}^n) \tag{24}$$

Unfortunately, despite the quite natural derivation of this method, it suffers from severe stability problem and is useless in practice. Now we replace u_i^n by $1/2\ (u_{i+1}^n + u_{i-1}^n)$ and is stable provided $\Delta t^{n+1/2}/\Delta x$ is sufficiently small. Hence we have

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - c\Delta t^{n+1/2}/2\Delta x_i (u_{i+1}^n - u_{i-1}^n) \tag{25}$$

Ref.

9. Hanh ngwyen Van, "Numerical Methods for Water Flow Models in Water Resources"1996.

© 2013 Global Journals Inc. (US)

Stencil:

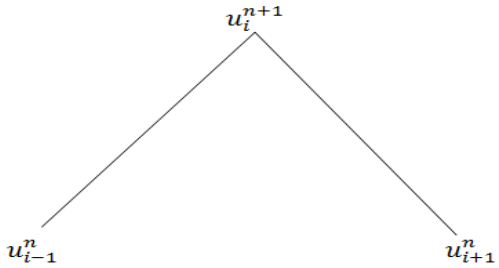


Figure 3 : Stencil of Lax-Friedrich Scheme

Stability condition: The method is stable provided that Δt and Δx are related by the following CFL (The Courant - Friedrichs - Lewy) condition

$$\gamma_i^{n+1} \equiv \text{—c } \Delta t^{n+1/2}/\Delta x_i\text{—} \leq 1$$

If we take equidistance grid and denote $\Delta x = h$ and $\Delta t = k$ then we have,

$$|\frac{ck}{h}| \leq 1 \tag{26}$$

This is the stability restriction for this method.

d) Lax – wendroff scheme for linear system

In a search for stable and more accurate shock capturing numerical schemes, P. Lax and B. Wendroff proposed the idea of combining the spatial and temporal discretization in order to globally achieve second order. Lax-Wendroff's scheme is an explicit second order method [10].

We consider the time dependent Cauchy problem in one space dimension [6],

$$u_t + A u_x = 0, -\infty < x < \infty, \ t \geq 0 \tag{27}$$

$$u\,(x, 0) = u_0\,(x). \tag{28}$$

We discretize that *x-t* plane by choosing a mesh width h ≡ Δx and a time step k ≡ Δt.

A wide variety of methods can be devised for the linear system (27) by using different finite difference approximations. Most of these are based directly on finite difference approximations to the PDE. An exception is the Lax-Wendoff scheme [10], which is based on the Taylor series expansion

$$u(x,\ t+k) = u\,(x,\ t) + k\ u_t\,(x,\ t) + \frac{1}{2}\,k^2\,u_{tt}\,(x,\ t) + \tag{29}$$

The Lax-Wendroff scheme is just the first three terms of (29).

Hence the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{k}{2h}\,A(U_{j+1}^n - U_{j-1}^n) + \frac{k^2}{2h^2}\,A^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n) \tag{30}$$

Ref

10. John A. Trangenstein "Numerical Solution for Partial Differential Equations".

© 2013 Global Journals Inc. (US)

Stencil:

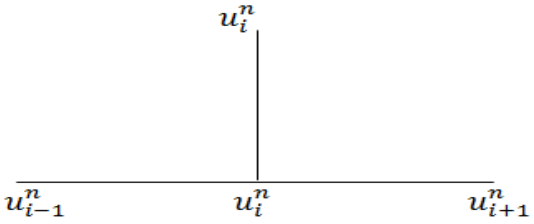


Figure 4 : Stencil of Lax-Wendroff Scheme

Stability condition: The method is stable provided that Δt and Δx are related by the following relation:
It turn out that the resulting scheme is second-order in both space and time and stable for

$$— \frac{\partial f}{\partial u} — \Delta t \leq \Delta x$$

e) Lax – wendroff scheme for non-linear system

Our intension is to apply the Lax-Wendoff scheme for shallow water equation which is a quasi-linear equation i.e. non-linear [10]. It was further simplified by some authors in order to avoid evaluation of the celerity (Jacobian) at an intermediate position. In the case of homogeneous systems of equations, the same guidelines lead to the following scheme [11]

$$U_j^{n+1} = U_j^n - \Delta t \,/2\, \Delta x\, (F_{i+1}^n - F_{i-1}^n) + \frac{1}{2}\,(\,\Delta t/\Delta x)^2$$

$$[J_{i+1/2}^n\,(F_{i+1}^n - F_i^n - J_{i+1/2}^n\,(F_i^n - F_{i-1}^n)] \qquad (31)$$

where now $J = \frac{\partial F}{\partial U}$ is the Jacobian matrix of the system. If the system is linear, the matrix is constant F = JU with J =constant and in the non – linear case JU must be evaluated at an intermediate position $J_{i+1/2} = J(U_{i+1/2})$ .The numerical flux can be written as

$$F_{i+1/2}^n = \frac{1}{2}\,(F_{i+1}^n - F_i^n) + \frac{1}{2}\,\frac{\Delta t}{\Delta x}\,(J_{i+1/2}^n)^2\,\,(F_{i+1}^n - F_{i+1}^n) \qquad (32)$$

The scheme is non-dissipative for J = constant, and displays oscillations near strong gradients (shocks). It can also lead to numerical difficulties near critical or sonic points. Several authors have recommended the addition of extra dissipative terms (pseudo viscosity) in these cases [12]. Lax-Wendroff's scheme is one of the most frequently encountered in the literature related to classical shock-capturing schemes. Difficulties have been reported when trying to include source terms in the discretization and to keep second order of accuracy at the same time.
Hence finally for hyperbolic system the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{1}{2}\,A(U_{j+1}^n - U_{j-1}^n)(\frac{\Delta t}{\Delta x}) + \frac{1}{2}A^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n)(\frac{\Delta t}{\Delta x}) \qquad (33)$$

Ref.
10. John A. Trangenstein "Numerical Solution for Partial Differential Equations".

© 2013 Global Journals Inc. (US)

f) Leap-Frog scheme for linear system

Leap-Frog scheme is a special form of Lax-Wendroff scheme. It is obtained from Lax-Wendroff scheme. It is exceptional to the above schemes [10]. It is also based on the Taylor series expansion as mentioned above. The Leap-Frog scheme is just the first two terms of (31).

Hence the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{k}{2h} A(U_{j+1}^n - U_{j-1}^n) \tag{34}$$

Stencil:

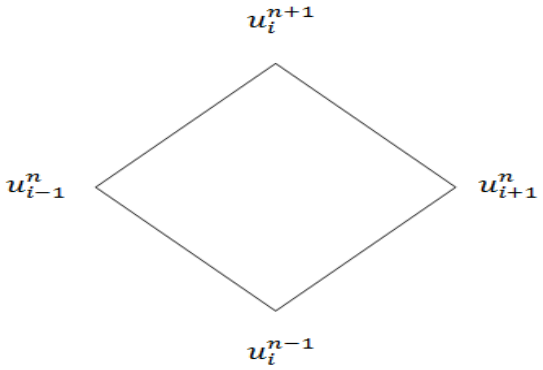


Figure 5 : Stencil of Leap-Frog scheme

Stability condition: The method is stable provided that Δt and Δx are related by the following relation:

It turn out that the resulting scheme is second-order in both space and time and stable for

$$— \frac{\partial f}{\partial u} — \Delta t \leq \Delta$$

g) Leap-Frog scheme for non-linear system

For hyperbolic system the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{1}{2} A(U_{j+1}^n - U_{j-1}^n)\left(\frac{\Delta t}{\Delta x}\right) \tag{35}$$

h) Finite Difference schemes for shallow water equation

In the following, we specify the finite difference methods for solving the shallow water equation [15].

Here $h(t^n, x_i)$ is abbreviated by h_i^n and $v(t^n, x_i)$ is abbreviated by v_i^n. We denote flux $f(t^n, x_i)$ by f_i^n for first equation and flux $f1(t^n, x_i)$ by $f1_i^n$ for second equation.

In Lax-Friedrich scheme, the shallow water equation can be written as

$$h_i^n = \frac{1}{2}\ (h_{i+1}^{n-1} + h_{i-1}^{n-1}) - \frac{1}{2}(f_{i+1}^{n-1} - f_{i-1}^{n-1})\ \frac{\Delta t}{\Delta x}$$

$$v_i^n = \frac{1}{2}\ (v_{i+1}^{n-1} + v_{i-1}^{n-1}) - \frac{1}{2}(f1_{i+1}^{n-1} - f1_{i-1}^{n-1})\ \frac{\Delta t}{\Delta x}$$

Where $f = h * v$

$$f1 = g * h + \frac{1}{2} v^2$$

In Lax-Wendroff scheme, the shallow water equation can be written as

10. John A. Trangenstein "Numerical Solution for Partial Differential Equations".

© 2013 Global Journals Inc. (US)

$$h_i^n = h_i^{n-1} - \frac{1}{2}(f_{i+1}^{n-1} - f_{i-1}^{n-1})\frac{\Delta t}{\Delta x} + \frac{1}{2}[(\frac{\Delta t}{\Delta x})^2\{\frac{1}{2}(v_{i+1}^{n-1} + v_i^{n-1})(f_{i+1}^{n-1} - f_i^{n-1})\} - \{\frac{1}{2}(v_{i-1}^{n-1} + v_i^{n-1})(f_i^{n-1} - f_{i-1}^{n-1})\}]$$

$$v_i^n = v_i^{n-1} - \frac{1}{2}(f1_{i+1}^{n-1} - f1_{i-1}^{n-1})\frac{\Delta t}{\Delta x} + \frac{1}{2}[(\frac{\Delta t}{\Delta x})^2\{\frac{1}{2}(v_{i+1}^{n-1} + v_i^{n-1})(f1_{i+1}^{n-1} - f1_i^{n-1})\} - \{\frac{1}{2}(v_{i-1}^{n-1} + v_i^{n-1})(f1_i^{n-1} - f1_{i-1}^{n-1})\}]$$

Where

$$f = h * v$$

$$f1 = g * h + \frac{1}{2}v^2$$

In Leap-Frog scheme, the shallow water equation can be written as

$$h_i^n = h_i^{n-1} - \frac{1}{2}(f_{i+1}^{n-1} - f_{i-1}^{n-1})\frac{\Delta t}{\Delta x}$$

$$v_i^n = v_i^{n-1} - \frac{1}{2}(f1_{i+1}^{n-1} - f1_{i-1}^{n-1})\frac{\Delta t}{\Delta x}$$

Where

$$f = h * v$$

$$f1 = g * h + \frac{1}{2}v^2$$

Here we consider $t_n = n.\Delta t$, $x_i = i.\Delta x$, where $n = 0, \ldots\ldots\ldots, \Delta t + 1$, $i = 0, \ldots\ldots\ldots, \Delta x + 1$.

IV. Result

Shallow water equation solved by analytical method is too complex that's why in this paper we apply two numerical schemes known as Lax-Friedrich scheme and Lax-Wendroff scheme. To Test the accuracy of the implementation of the numerical schemes we consider the linear advection equation whose analytical solution is known to us.

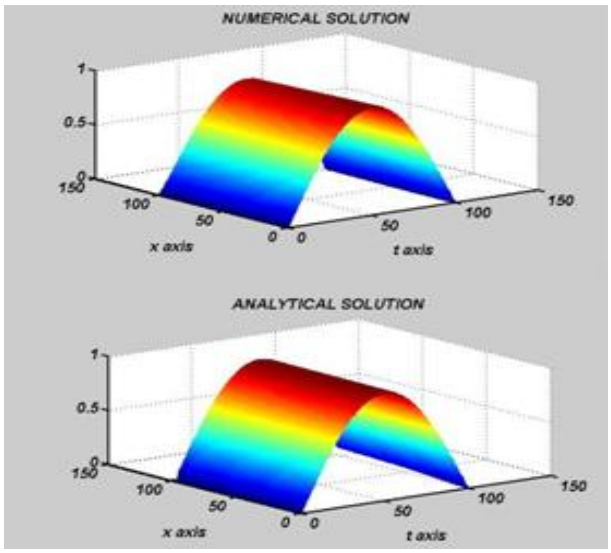


Figure 6 : Numerical and Analytical solution of linear advection equation using Lax-Friedrich scheme

© 2013 Global Journals Inc. (US)

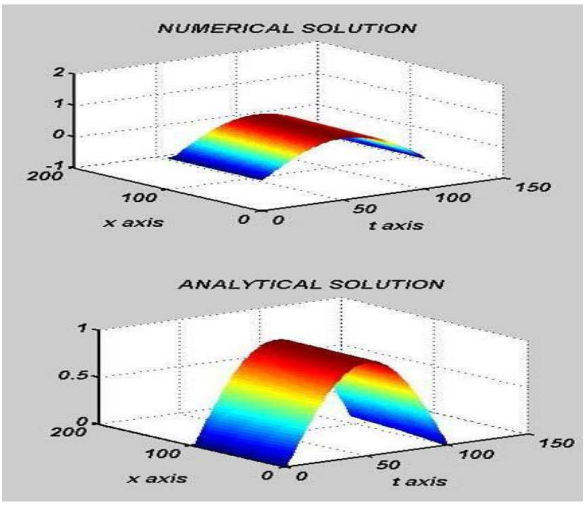


Figure 7 : Numerical and Analytical solution of linear advection equation using Lax-Wendroff scheme

Fig. 6 and fig.7 shows the analytical and numerical solution of linear advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme respectively. From these figure we observe that there is no difference between the analytical and numerical solution. Now we define error term as follows:

$$\text{Error} = \left(\frac{\text{Analytical solution} - \text{numerical solution}}{\text{Analytical solution}}\right) \times 100.$$

From the program we obtain, the error of the numerical solution of linear advection equation is 0.05, 0.048 and .99 using Lax-Friedrich scheme and Lax-Wendroff scheme and Leap-Frog scheme respectively. So we can say the error between them is very negligible and if we take $\Delta x \to 0$ then the error $\to 0$.

Figure 6. and 7 shows numerical and analytical solution of advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme, respectively.

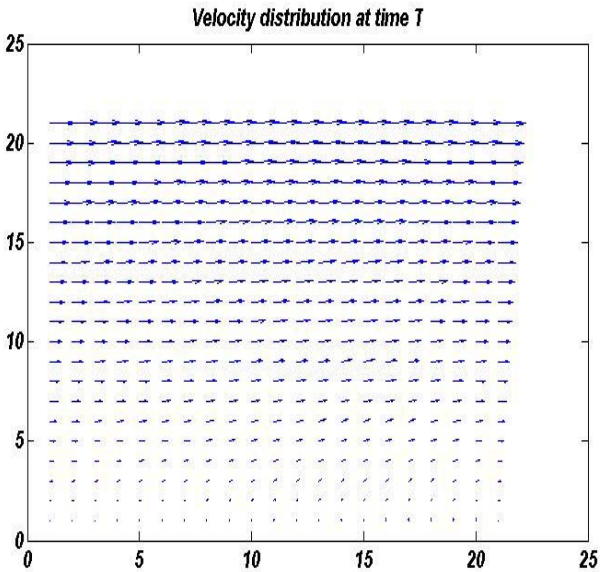


Figure 8 : Velocity distribution for numerical solution of linear advection equation using Lax-Friedrich scheme

© 2013 Global Journals Inc. (US)

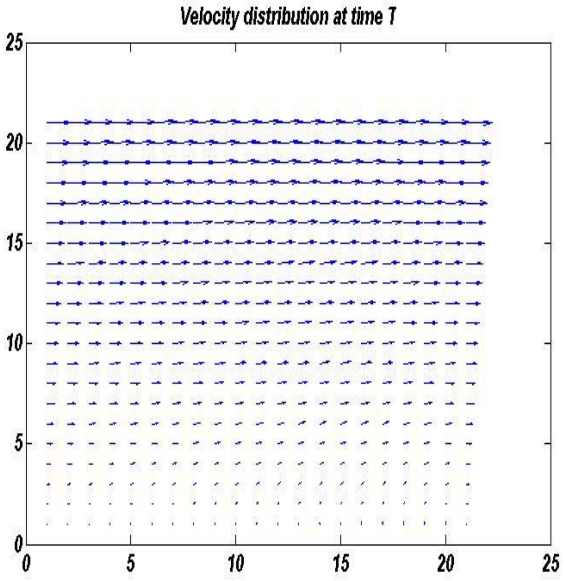


Figure 9 : Velocity distribution for numerical solution of linearadvection equation using Lax-Wendroff scheme

Fig. 8. and Fig. 9. shows water velocity distribution for solution of advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme, respectively.

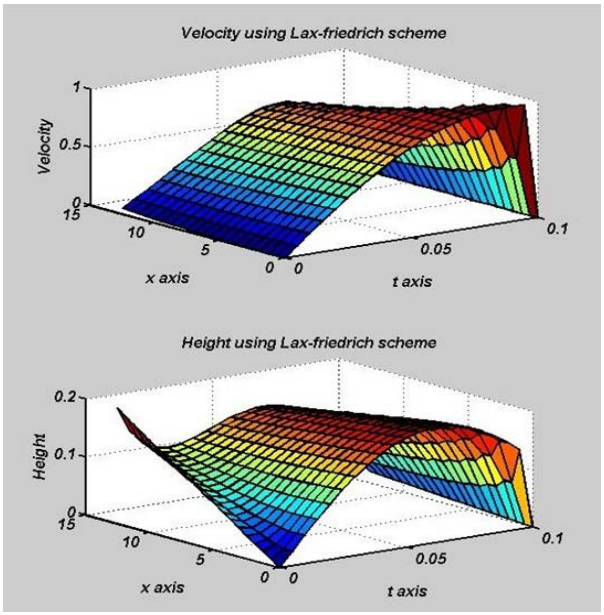


Figure 10 : Water velocity and height of shallow water equation using Lax-Friedrich scheme

© 2013 Global Journals Inc. (US)

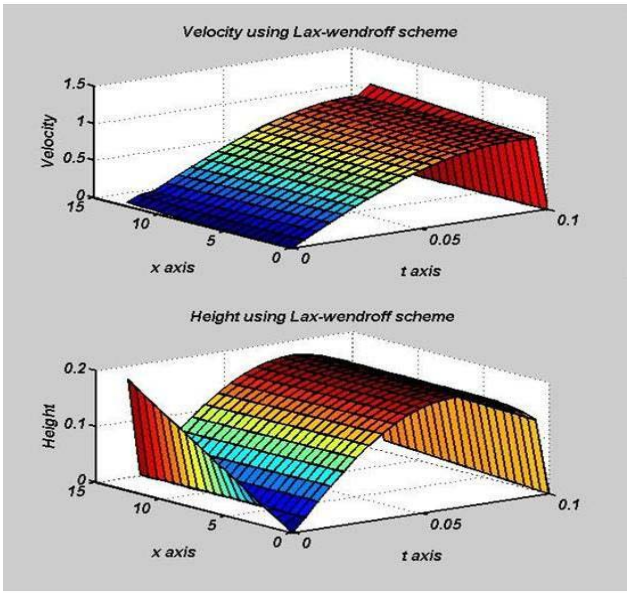


Figure 11 : Water velocity and height of shallow water equation using Lax-Wendroff scheme

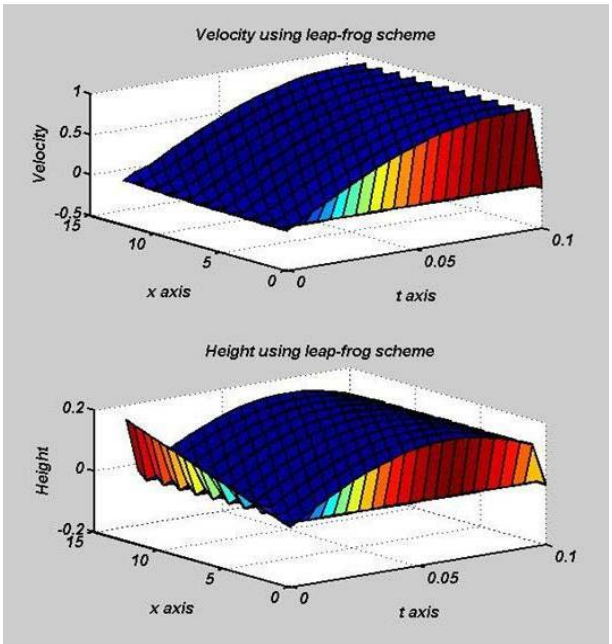


Figure 12 : Water velocity and height of shallow water equation using Leap-Frog scheme

Fig. 10 and 11 and 12 shows water velocity and height of shallow water equation using Lax-Friedrich scheme, Lax-Wendroff scheme and Leap-Frog scheme, respectively.

© 2013 Global Journals Inc. (US)

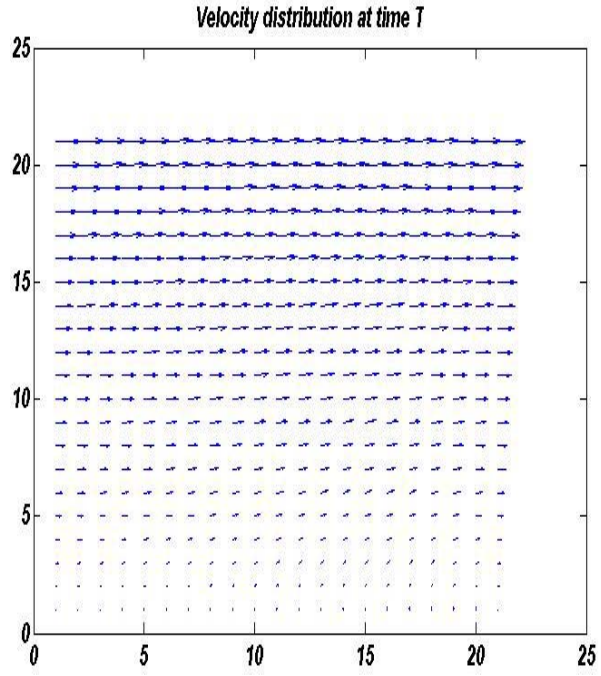


Figure 13 : Velocity distribution for numerical solution of shallow water equation using Lax-Friedrich scheme

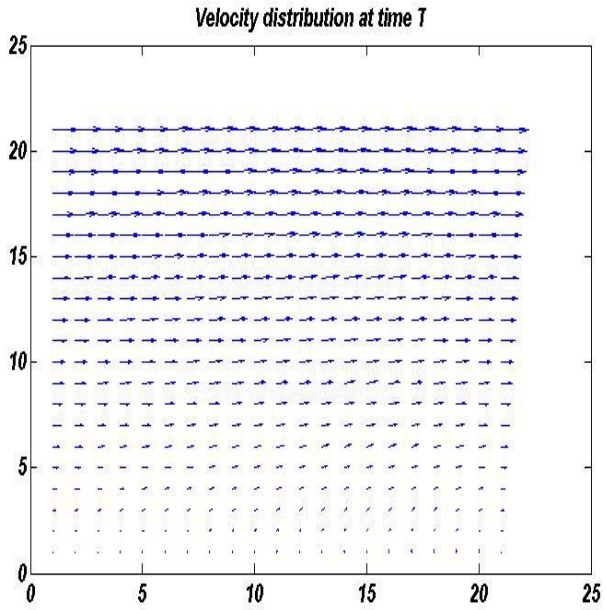


Figure 14 : Velocity distribution for numerical solution of shallow water equation using Lax-Wendroff scheme

Notes

© 2013 Global Journals Inc. (US)

Notes

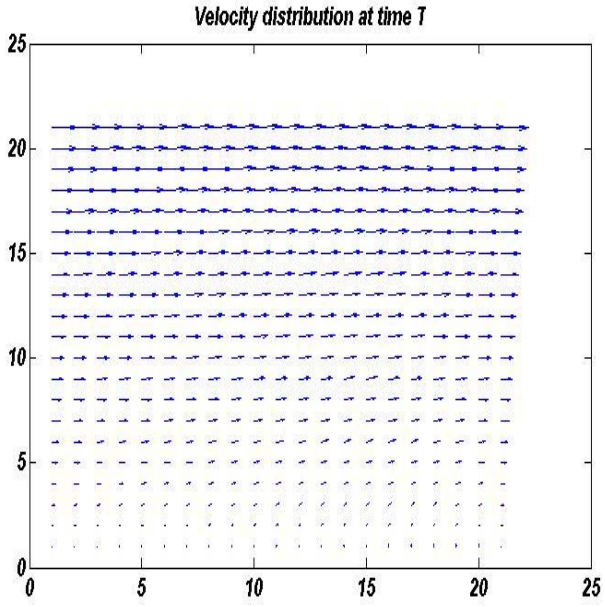


Figure 15 : Velocity distribution for numerical solution of shallow water equation using Leap-Frog scheme

Fig. 13, 14 and 15 shows velocity distribution for numerical solution of shallow water equation using Lax-Friedrich scheme and Lax-Wendroff scheme and Leap-Frog scheme respectively in forms of mesh, which we have simulated from our numerical model.

Fig. 8, and 9 implies that the velocity distribution for numerical solution of linear advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme is identical.

We also observe from the scheme of Leap-Frog scheme that it is obtain from Lax-Wendroff scheme.

V. Conclusion

From the above experiment it has revealed clearly that Lax-Wendroff scheme is more efficient than Lax-Friedrich scheme and Leap-Frog scheme because the error term in Lax-Wendroff scheme is less than the Lax-Friedrich scheme and Leap-Frog scheme.

In this paper we have consider only the 1D shallow water equation. First we have shown the numerical and analytical solution of linear advection equation using Lax-Friedrich scheme, Lax-Friedrich scheme and their qualitative graphs where we have obtained the error terms of these schemes which is essential to find the acceptability of these schemes. Finally we have shown the Lax-Friedrich scheme, Lax-Wendroff scheme and Leap-Frog scheme and their respective graphs for shallow water equation. It may be more effective experiment for flood modeling using higher order shallow water equation and we left this as our future work.

References Références Referencias

1. Pilar García-Navarro & Pilar Brufau, "One Dimensional Dam Break Flow Modeling: Some results".
2. Duane Harselman, Bruce Littlefield "Mastering Matlab".

© 2013 Global Journals Inc. (US)

3. Rokhsan-Ara-Hemel & L.S. Andallah, "Shallow Water Equation for Flood Modeling (study-1)" 2009.
4. Rokhsan -Ara- Hemel "Numerical Simulation of Shallow Water Equation for Flood Modeling (study-1)" 2008.
5. Goel & Mittal, "Numerical Analysis".
6. Randall J. Le Veque, "Numerical Methods for Conservation Laws".
7. C. B. Vreugdenhil, "Numerical Method for Shallow Water Flow" Kluwer Academic Publisher, Dordsecht 1994.
8. L.S. Andallah,"Finite Difference Method-Explicit Upwind Difference scheme", Lecture note, Department of Mathematics, Jahangirnagar University, 2007.
9. Hanh ngwyen Van, "Numerical Methods for Water Flow Models in Water Resources"1996.
10. John A. Trangenstein "Numerical Solution for Partial Differential Equations".
11. P. Garcɹa-Navarro, P. Brufau, J. Burguete and J. Murillo, "The shallow water equations: An example of hyperbolic system".
12. Mike Baines, "A Survey for Numerical Scheme for Advection, The Shallow Water Equation and Dam break problems".

Notes

© 2013 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 4 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Helical-One, Two, Three-Revolutional Cyclical Surfaces

By T. Olejníková
Technical University of Košice

Abstract - This paper describes method for modelling of helical-*n*-revolutional cyclical surfaces. The axis of the cyclical surface Φ_1 is the helix s_1 created by revolving the point about *n* each other revolving axes o_n (n = 1,2,3), that move together with Frenet-Serret moving trihedron along the cylindrical helix *s*. Particular evolutions are determined by its angular velocity and orientation. The moving circle along the helix *s* or s_1, where its center lies on the helix and circle lies in the normal plane of the helix creates the cyclical surface.

Keywords : *cyclical surface, helix, frenet-serret moving trihedron, transformation matrices.*

GJSFR-F Classification : *MSC 2010: 05B20*



Strictly as per the compliance and regulations of :

© 2013. T. Olejníková. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Helical-One, Two, Three-Revolutional Cyclical Surfaces

T. Olejníková

Abstract - This paper describes method for modelling of helical-*n*-revolutional cyclical surfaces. The axis of the cyclical surface Φ_1 is the helix s_1 created by revolving the point about *n* each other revolving axes o_n (n = 1,2,3), that move together with Frenet-Serret moving trihedron along the cylindrical helix *s*. Particular evolutions are determined by its angular velocity and orientation. The moving circle along the helix *s* or s_1, where its center lies on the helix and circle lies in the normal plane of the helix creates the cyclical surface.

Keywords : *cyclical surface, helix, frenet-serret moving trihedron, transformation matrices.*

I. Introduction

Let thre-dimensional Euclidean space E^3 is determined by Cartesian coordinate system $(0, x, y, z)$. In this space is given cylindrical helix *s* with axis identical with coordinate axis *z* determined by vector function (Fig.1)

$$\mathbf{r}(v) = (x_s, y_s, z_s, 1) = (a\cos mv, sg\, a \sin mv, bv, 1),\ v \in \langle 0, 2\pi \rangle, \tag{1}$$

where parameter *a* is radius of the helix, *b* is the reduced pitch, *sg* determined orientation of the helix, ($sg = +1$ for right-handed and $sg = -1$ for left-handed revolution), *m* is number of pitches. Let $(0', n, b, t)$ be Frenet-Serret moving trihedron of the cylindrical helix *s* represented by regular square matrix

$$\mathbf{M}(v) = \begin{pmatrix} n_x(v) & n_y(v) & n_z(v) & 0 \\ b_x(v) & b_y(v) & b_z(v) & 0 \\ t_x(v) & t_y(v) & t_z(v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where the matrix elements are the coordinates of unit vectors of the principle normal *n*, binormal *b* and tangent *t* of the helix *s* in the point $0' \in s$ in the coordinate system $(0, x, y, z)$

$$\mathbf{t}(v) = (t_x(v), t_y(v), t_z(v)) = \frac{\mathbf{r}'(v)}{|\mathbf{r}'(v)|}, \tag{3}$$

$$\mathbf{b}(v) = (b_x(v), b_y(v), b_z(v)) = \frac{\mathbf{r}'(v) \times \mathbf{r}''(v)}{|\mathbf{r}'(v) \times \mathbf{r}''(v)|}, \tag{4}$$

$$\mathbf{n}(v) = (n_x(v), n_y(v), n_z(v)) = \mathbf{b}(v) \times \mathbf{t}(v). \tag{5}$$

Author : *Department of Applied Mathematics Institute of Construction, Technology and Management Faculty of Civil Engineering, Technical University of Košice, Slovakia. E-mail : tatiana.olejnikova@tuke.sk*

© 2013 Global Journals Inc. (US)

Transformations of revolutions about coordinate axes *x*, *y*, *z* are represented by matices $\mathbf{T}_x(\varphi, \psi)$, $\mathbf{T}_y(\varphi, \psi)$, $\mathbf{T}_z(\varphi, \psi)$, where φ is angle and ψ is orientation of the revolution, transformation of translation is represented by matrix $\mathbf{T}(\pm d_x, \pm d_y, \pm d_z)$, where $(\pm d_x, \pm d_y, \pm d_z)$ is translation vector determined by its coordinates (6), (7):

$$\mathbf{T}_x(\varphi, \psi)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \psi\sin\varphi & 0 \\ 0 & -\psi\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{T}_y(\varphi, \psi)=\begin{pmatrix} \cos\varphi & 0 & \psi\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\psi\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

$$\mathbf{T}_z(\varphi, \psi)=\begin{pmatrix} \cos\varphi & \psi\sin\varphi & 0 & 0 \\ -\psi\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{T}(\pm d_x, \pm d_y, \pm d_z)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm d_x & \pm d_y & \pm d_z & 1 \end{pmatrix}. \tag{7}$$

The moving circle $c=(0', r)$ along the helix *s*, where its center $0'$ lies in the normal plane determined by principal normal *n* and binormal *b* of the helix in the point $0' \in s$ creates the cyclical surface Φ. The vector function of this surface is

$$\mathbf{P}(u,v)=\mathbf{r}(v)+\mathbf{c}(u).\mathbf{M}(v),\ u \in \langle 0, 2\pi \rangle,\ v \in \langle 0, 2\pi \rangle, \tag{8}$$

where $\mathbf{r}(v)$ is vector function of the helix *s* expressed in equation (1), $\mathbf{M}(v)$ is transformation matrix of the coordinate system $(0', n, b, t)$ into coordinate system $(0, x, y, z)$ (2) and $\mathbf{c}(u)=(r\cos u, r\sin u, 0, 1), u \in \langle 0, 2\pi \rangle$ is vector function of the circle *c* determined by its center $0'$ and radius *r* (Fig.2). In Fig.3 there are displayed two screws of the right-handed cyclical surface Φ together with the cylindrical surface on which helix *s* is wound.

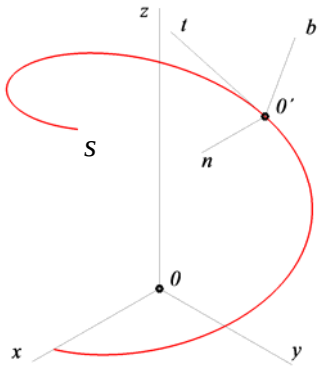


Fig. 1 : Helix *s* with Trihedron

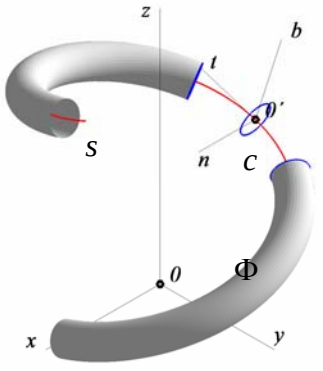


Fig. 2 : Cyclical Surface Φ

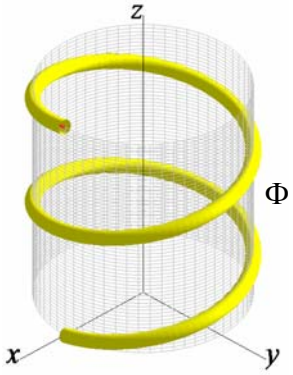


Fig. 3 : Surface Φ and Cylinder

II. Cyclical Helical Surface Created by One Revolution

The helix s_1 created by revolution of the point $P(x_0, y_0, z_0, 1)$ about the axis o_1 connected to the moving trihedron of the helix *s*, is represented by vector function

$$\mathbf{r}_1(v)=\mathbf{r}(v)+(x_0, y_0, z_0, 1).\mathbf{T}_1(m_1 v, sg_1).\mathbf{M}(v) \tag{9}$$

© 2013 Global Journals Inc. (US)

and cyclical surface Φ_1 created in a similar way as surface Φ by vector function

$$\mathbf{P}_1(u,v)=\mathbf{r}_1(v)+\mathbf{c}_1(u).\mathbf{M}_1(v),\ u\in\langle 0,2\pi\rangle,\ v\in\langle 0,2\pi\rangle, \tag{10}$$

where $\mathbf{r}_1(v)$ is vector function of the helix s_1 expressed in equation (9), $\mathbf{M}_1(v)$ is transformation matrix of the coordinate system $(0'',n',b',t')$ into coordinate system $(0,x,y,z)$ (11), $\mathbf{c}_1(u)=(r_1\cos u,\ r_1\sin u,0,1), u\in\langle 0,2\pi\rangle$ is vector function of the circle c_1 determined by center $0''\in s_1$ and radius r_1

$$\mathbf{M}_1(v)=\begin{pmatrix} n'_x(v) & n'_y(v) & n'_z(v) & 0\\ b'_x(v) & b'_y(v) & b'_z(v) & 0\\ t'_x(v) & t'_y(v) & t'_z(v) & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{11}$$

Elements of this matrix are coordinates of unit vectors of the principle normal n', binormal b' and tangent t' of the helix s_1 in the point $0''\in s_1$ in the coordinate system $(0',n,b,t)$

$$\mathbf{t}'(v)=(t'_x(v),t'_y(v),t'_z(v))=\frac{\mathbf{r}'_1(v)}{|\mathbf{r}'_1(v)|} \tag{12}$$

$$\mathbf{b}'(v)=(b'_x(v),b'_y(v),b'_z(v))=\frac{\mathbf{r}'_1(v)\times\mathbf{r}''_1(v)}{|\mathbf{r}'_1(v)\times\mathbf{r}''_1(v)|}, \tag{13}$$

$$\mathbf{n}'(v)=(n'_x(v),n'_y(v),n'_z(v))=\mathbf{b}'(v)\times\mathbf{t}'(v). \tag{14}$$

a) Revolution about tangent t of the helix s

The helix created by the revolution of the point P about the axis $o_1=t$ is expressed by vector function (9), in which matrix $\mathbf{T}_1(m_1v,sg_1)=\mathbf{T}_z(m_1v,sg_1)$. In Fig.4 is displayed cyclical surface Φ, whose axis is helix s with parameters $m=2, sg=+1$ and surface Φ_1, whose axis is helix s_1 created by revolution of the point $P=(d,0,0,1)$ about tangent t of the helix s with parameters $m_1=8m, sg_1=+1$.

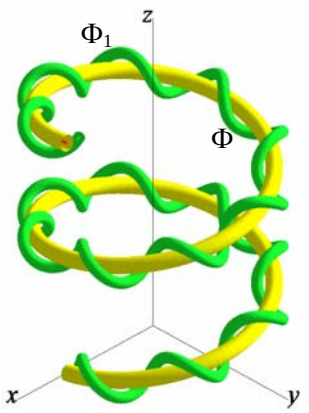


Fig. 4 : Cyclical Surfaces Φ, Φ_1

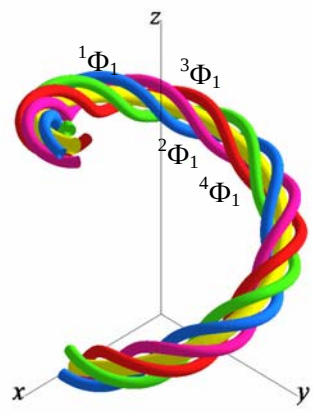


Fig. 5 : 4 Surfaces ${}^{i}\Phi_1$

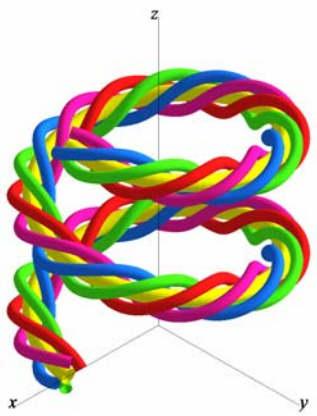


Fig. 6 : Left-handed Surfaces Φ,${}^{i}\Phi_1$

© 2013 Global Journals Inc. (US)

In Fig.5 are displayed $k=4$ surfaces ${}^{i}\Phi_1$, whose axes are helix ${}^{i}s_1$, $i=1,...,k$ created by revolution of the points ${}^{i}P=(d\cos i\alpha, d\sin i\alpha, 0,1)$, $\alpha=2\pi/k$ about tangent t of the helix s with parameters $m_1=4m, sg_1=+1$, in Fig.6 are displayed the same surfaces with altered orientation of the revolution $m=2, sg=-1$, $m_1=4m, sg_1=-1$.

b) Revolution about principal normal n of the helix s

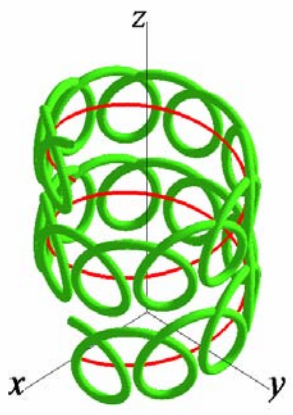


Fig. 7 : Normal Surface Φ_1

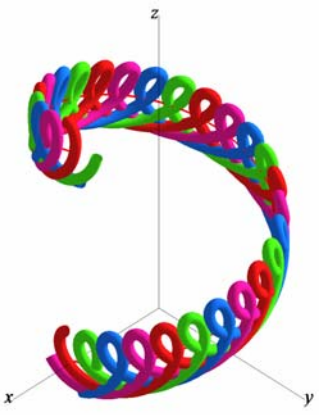


Fig. 8 : 4 Surfaces ${}^{i}\Phi_1$

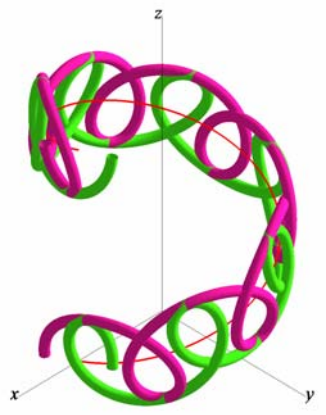


Fig. 9 : Normal Surfaces ${}^{1}\Phi_1$, ${}^{3}\Phi_1$

The helix s_1 created by the revolution of the point P about the axis $o_1=n$ is expressed by vector function (9), in which matrix $\mathbf{T}_1(m_1v, sg_1)=\mathbf{T}_x(m_1v, sg_1)$. In Fig.7 is displayed helix s with parameters $m=2, sg=+1$ and normal surface Φ_1, whose axis is helix s_1 created by revolution of the point $P=(0,d,0,1)$ about normal n of the helix s with parameters $m_1=10m, sg_1=+1$, in Fig.8 are displayed $k=4$ normal surfaces ${}^{i}\Phi_1$, whose axes are helix ${}^{i}s_1$, $i=1,...,k$ created by revolution of the points ${}^{i}P=(0, d\cos i\alpha, d\sin i\alpha, 1)$, $\alpha=2\pi/k$ about normal n of the helix s with parameters $m_1=7m, sg_1=-1$, in Fig.9 are displayed surfaces ${}^{1}\Phi_1$, ${}^{3}\Phi_1$ with altered orientation of the revolution $sg_1=\pm 1$.

c) Revolution about binormal b of the helix s

The helix created by the revolution of the point P about axis $o_1=b$ is expressed by vector function (9), in which matrix $\mathbf{T}_1(m_1v, sg_1)=\mathbf{T}_y(m_1v, sg_1)$. In Fig.10 is displayed helix s with parameters $m=2, sg=+1$ and binormal surface Φ_1, whose axis is helix s_1 created by revolution of the point $P=(d,0,0,1)$ about binormal b of the helix s with parameters $m_1=10m, sg_1=+1$, in Fig.11 is displayed binormal surface with parameters $m_1=8m, sg_1=-1$, in Fig.12 are surfaces ${}^{1}\Phi_1$, ${}^{3}\Phi_1$ created by revolution of the points ${}^{i}P=(d\cos i\alpha, d\sin i\alpha, 0, 1)$, $\alpha=2\pi/k$ about binormal b of the helix with altered orientation of the revolution $sg_1=\pm 1$.

Notes

© 2013 Global Journals Inc. (US)

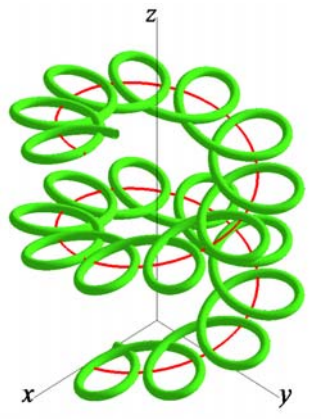


Fig. 10 : Binormal Surface Φ_1

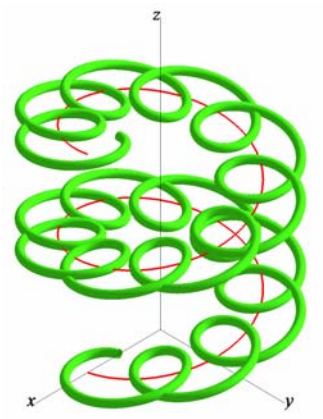


Fig. 11 : 4 Surfaces ${}^{i}\Phi_1$

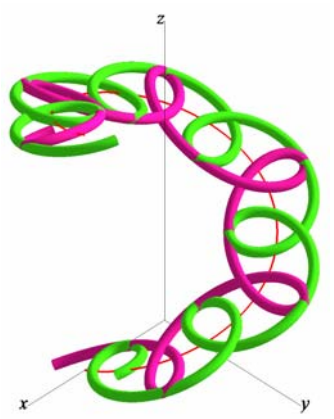


Fig. 12 : Binormal Surfaces ${}^{1}\Phi_1, {}^{3}\Phi_1$

III. Cyclical Helical Surface Created by Two Revolutions

The helix s_1 created by revolution of the point $P(x_0, y_0, z_0, 1)$ about axis o_2, which revolves about the axis o_1 identical with one edge of the moving trihedron of the helix s is represented by vector function

$$\mathbf{r}_1(v) = \mathbf{r}(v) + (x_0, y_0, z_0, 1).\mathbf{T}_2(m_2 v, sg_2).\mathbf{T}_1(m_1 v, sg_1).\mathbf{M}(v), \tag{15}$$

where matrix $\mathbf{T}_2(m_2 v, sg_2)$ represents revolution of the point P about the axis o_2 and matrix $\mathbf{T}_1(m_1 v, sg_1)$ represents revolution of the axis o_2 about the axis o_1.

a) Revolution about two parallel axes

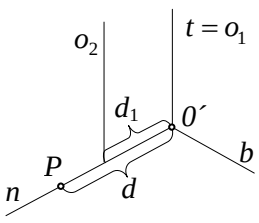


Fig. 13

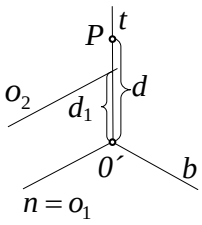


Fig. 14

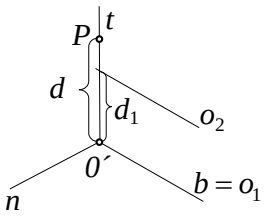


Fig. 15

If the helix s_1 is created by revolution of the point P about two parallel axes $o_2 \parallel o_1$ and $o_1 = t$, where $d_1 = |o_1 o_2|$ is the distance between them, then (Fig.13)

$$\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}(-d_1, 0, 0).\mathbf{T}_z(m_2 v, sg_2).\mathbf{T}(+d_1, 0, 0),\ \mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_z(m_1 v, sg_1). \tag{16}$$

In Fig.16 is displayed this surface Φ_1 with parameters $m_1 = 8m, sg_1 = +1, m_2 = 4m_1, sg_2 = +1$.

If the helix s_1 is created by revolution of the point P about parallel axes $o_2 \parallel o_1$ and $o_1 = n$, where $d_1 = |o_1 o_2|$, then (Fig.14)

$$\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}(0, 0, -d_1).\mathbf{T}_x(m_2 v, sg_2).\mathbf{T}(0, 0, +d_1),\ \mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_x(m_1 v, sg_1). \tag{17}$$

© 2013 Global Journals Inc. (US)

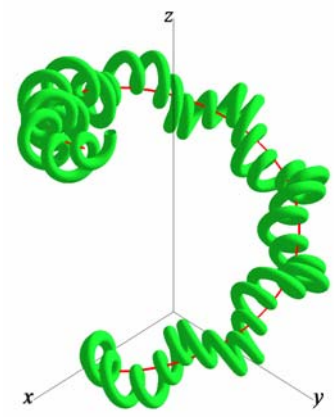


Fig. 16 : $o_2 \parallel o_1,\ o_1 = t$

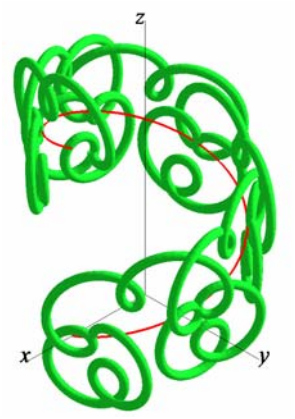


Fig. 17 : $o_2 \parallel o_1,\ o_1 = n$

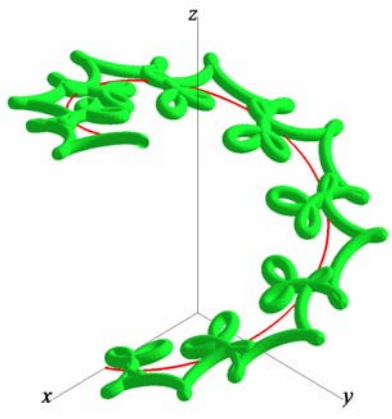


Fig. 18 : $o_2 \parallel o_1,\ o_1 = b$

In Fig.17 is displayed this surface Φ_1 with parameters $m_1 = 6m, sg_1 = +1, m_2 = 4m_1, sg_2 = +1$.

If the helix s_1 is created by revolution of the point $P = (0,0,d,1)$ about parallel axes $o_2 \parallel o_1$ and $o_1 = b$, where $d_1 = |o_1 o_2|$, then (Fig.15)

$$\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}(0,0,-d_1).\mathbf{T}_y(m_2 v, sg_2).\mathbf{T}(0,0,+d_1),\ \mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_y(m_1 v, sg_1). \qquad (18)$$

In Fig.18 is displayed this surface Φ_1 with parameters $m_1 = 8m, sg_1 = -1, m_2 = 5m_1, sg_2 = +1$.

b) Revolution about two intersecting axes

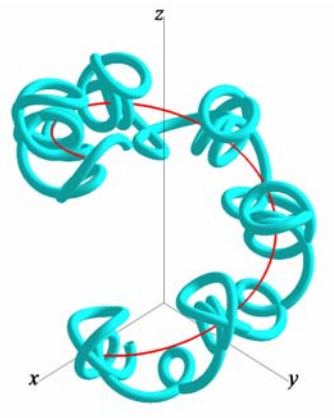


Fig. 19 : $(o_2 = t) \perp (o_1 = n)$

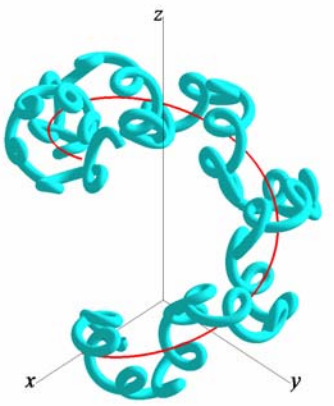


Fig. 20 : $(o_2 = n) \perp (o_1 = t)$

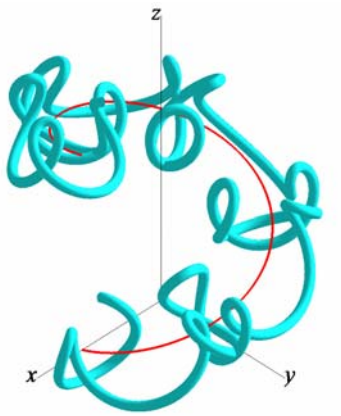


Fig. 21 : $(o_2 = n) \perp (o_1 = b)$

In Fig.19 is displayed surface created by revolution of the point $P = (2,2,0,1)$ about mutually perpendicular axes $(o_2 = t) \perp (o_1 = n)$ determined by the parameters $m_1 = 6m, sg_1 = -1$, $m_2 = 4m_1, sg_2 = -1$, where matrices $\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}_z(m_2 v, sg_2)$, $\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_x(m_1 v, sg_1)$, in Fig.20 is displayed surface created by revolution of the point $P = (2.2,1.2,0,1)$ about mutually perpendicular axes $(o_2 = n) \perp (o_1 = t)$ determined by parameters $m_1 = 6m, sg_1 = +1$, $m_2 = 6m_1, sg_2 = +1$, and matrices $\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}_x(m_2 v, sg_2)$, $\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_z(m_1 v, sg_1)$, here we see action of changing the order of the revolutions to form of the surfaces. In Fig.21 is displayed surface created by revolution of the point $P = (2.5,2.5,0,1)$ about mutually perpendicular axes $(o_2 = n) \perp (o_1 = b)$ determined by parameters

© 2013 Global Journals Inc. (US)

$m_1 = 5m, sg_1 = -1$, $m_2 = 3m_1, sg_2 = +1$, $\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}_x(m_2 v, sg_2)$, $\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_y(m_1 v, sg_1)$.In Figs.22,23 is displayed surface created by revolution of the point $P = (d,0,0,1)$ about intersecting axes $o_2 \times (o_1 = t)$ determined by the parameters $m_1 = 6m, sg_1 = +1$, $m_2 = 6m_1, sg_2 = -1$, $\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_z(m_1 v, sg_1)$, $\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}_y(\alpha,+1).\mathbf{T}_x(m_2 v, sg_2).\mathbf{T}_y(\alpha,-1)$.

Notes

c) Revolution about two skew axes

In Figs.24,25 is displayed surface created by revolution of the point $P = (d,0,0,1)$ about mutually skew axes $(o_2 \,||\, n) / (o_1 = t)$, determined by parameters $m_1 = 4m, sg_1 = +1$, $m_2 = 8m_1, sg_2 = +1$, where transformation matrices of two revolutions are

$$\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_z(m_1 v, sg_1),\ \mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}(0,0,-d_1).\mathbf{T}_x(m_2 v, sg_2).\mathbf{T}(0,0,+d_1).$$

In Figs.26,27 is displayed surface created by revolution of the point $P = (d,0,0,1)$ about mutually skew axes $(o_2 \times t, o_2 \times n) / (o_1 = t)$ determined by parameters $m_1 = 6m, sg_1 = +1$, $m_2 = 4m_1, sg_2 = -1$, and transformation matrices $\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}_y(\alpha,+1).\mathbf{T}_x(m_2 v, sg_2).\mathbf{T}_y(\alpha,-1)$, $\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_y(m_1 v, sg_1)$.

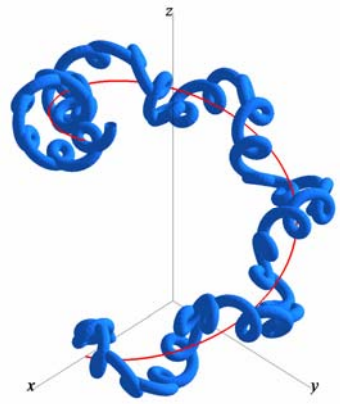


Fig. 22: $o_2 \times (o_1 = t)$

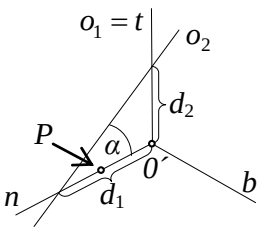


Fig. 23

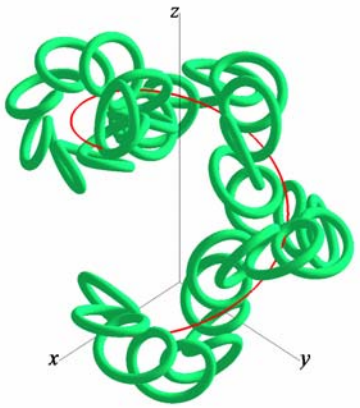


Fig. 24: $(o_2 \,||\, n) / (o_1 = t)$

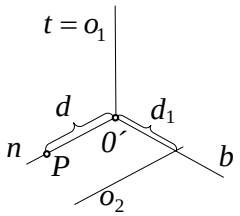


Fig. 25

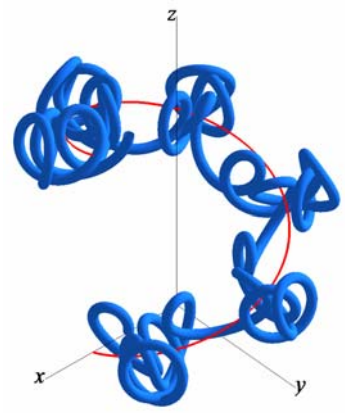


Fig. 26: $(o_2 \times t, o_2 \times n) / (o_1 = t)$

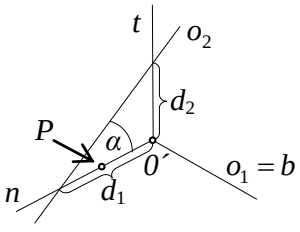


Fig. 27

© 2013 Global Journals Inc. (US)

IV. Cyclical Helical Surface Created by Three Revolutions

The helix s_1 created by the revolution of the point $P=(x_0,y_0,z_0,1)$ about the axis o_3, which revolves about the axis o_2 and this revolves about the axis o_1 identical with any edge of the moving trihedron of the helix *s* is represented by vector function

$$\mathbf{r}_1(v)=\mathbf{r}(v)+(x_0,y_0,z_0,1).\mathbf{T}_3(m_3v,sg_3).\mathbf{T}_2(m_2v,sg_2).\mathbf{T}_1(m_1v,sg_1).\mathbf{M}(v), \tag{19}$$

where matrix $\mathbf{T}_3(m_3v,sg_3)$ represents revolution of the point *P* about the axis o_3, matrix $\mathbf{T}_2(m_2v,sg_2)$ represents revolution of the axis o_3 about the axis o_2 and matrix $\mathbf{T}_1(m_1v,sg_1)$ represents revolution of the axis o_2 about the axis o_1.

a) Revolution about three parallel axes

In Fig.28 is displayed surface created by revolution about three parallel axes $o_3 \parallel o_2 \parallel o_1 = t$ determined by parameters $m_1=4m, sg_1=+1$, $m_2=4m_1, sg_2=+1$, $m_3=3m_2, sg_1=+1$, matrices $\mathbf{T}_3(m_3v,sg_3)=\mathbf{T}(-d_2,0,0).\mathbf{T}_z(m_3v,sg_3).\mathbf{T}(+d_2,0,0)$, $\mathbf{T}_2(m_2v,sg_2)=\mathbf{T}(-d_1,0,0).\mathbf{T}_z(m_2v,sg_2).\mathbf{T}(+d_1,0,0)$, $\mathbf{T}_1(m_1v,sg_1)=\mathbf{T}_z(m_1v,sg_1)$. In Fig.29 is displayed surface created by revolution about three parallel axes $o_3 \parallel o_2 \parallel o_1 = n$ determined by parameters $m_1=4m, sg_1=+1$, $m_2=4m_1, sg_2=+1$, $m_3=4m_2, sg_1=+1$ and by transformation matrices $\mathbf{T}_1(m_1v,sg_1)=\mathbf{T}_x(m_1v,sg_1)$
$\mathbf{T}_2(m_2v,sg_2)=\mathbf{T}(0,0,-d_1).\mathbf{T}_z(m_2v,sg_2).\mathbf{T}(0,0,+d_1)$, $\mathbf{T}_3(m_3v,sg_3)=\mathbf{T}(0,0,-d_2).\mathbf{T}_x(m_3v,sg_3).\mathbf{T}(0,0,+d_2)$.

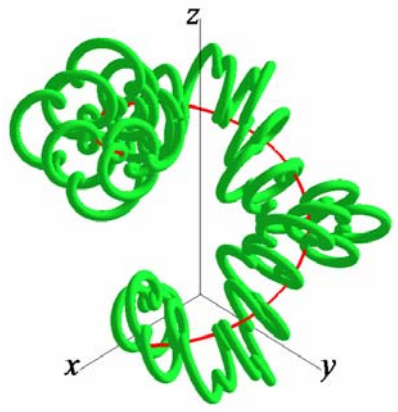


Fig. 28 : $o_3 \parallel o_2 \parallel o_1 = t$

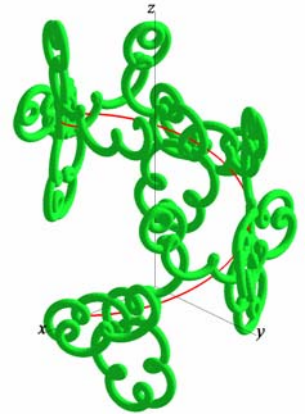


Fig. 29 : $o_3 \parallel o_2 \parallel o_1 = n$

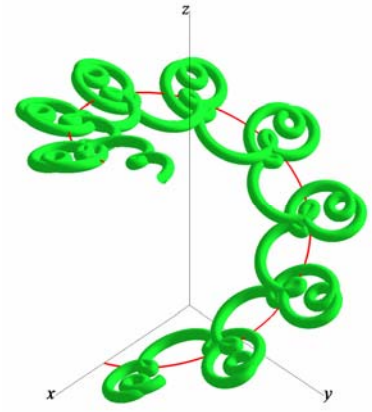


Fig. 30 : $o_3 \parallel o_2 \parallel o_1 = b$

In Fig.30 is displayed surface created by revolution about three parallel axes $o_3 \parallel o_2 \parallel o_1 = b$ determined by parameters $m_1=3m, sg_1=+1$, $m_2=3m_1, sg_2=+1$, $m_3=3m_2, sg_1=+1$ and transformation matrices $\mathbf{T}_3(m_3v,sg_3)=\mathbf{T}(0,0,-d_2).\mathbf{T}_y(m_3v,sg_3).\mathbf{T}(0,0,+d_2)$,
$\mathbf{T}_2(m_2v,sg_2)=\mathbf{T}(0,0,-d_1).\mathbf{T}_y(m_2v,sg_2).\mathbf{T}(0,0,+d_1)$, $\mathbf{T}_1(m_1v,sg_1)=\mathbf{T}_y(m_1v,sg_1)$.

b) Revolution about three perpendicular axes

In Figs.31,32,33 are displayed surfaces created by revolution of the point $P=(d,d,0,1)$ about three perpendicular axes with common point $o_3 \perp o_2 \perp o_1$, which are identical with edges of the trihedron of the helix *s*, where parameters are the same $m_1=3m, sg_1=+1$, $m_2=3m_1, sg_2=+1$, $m_3=3m_2, sg_1=+1$, but the order of the revolutions changes.

Notes

© 2013 Global Journals Inc. (US)

Notes

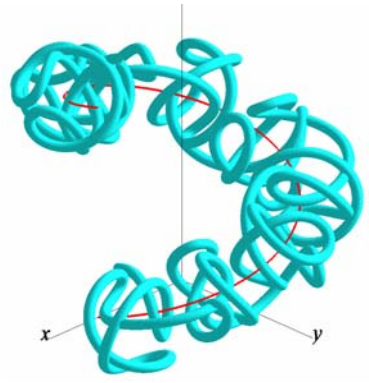


Fig. 31 : $o_3 = n, o_2 = b, o_1 = t$

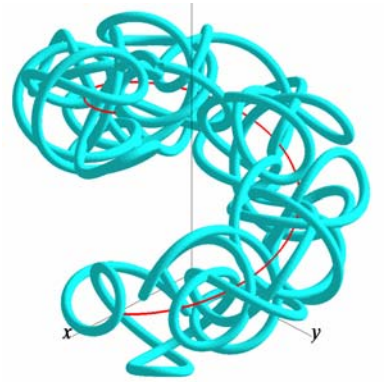


Fig. 32 : $o_3 = n, o_2 = t, o_1 = b$

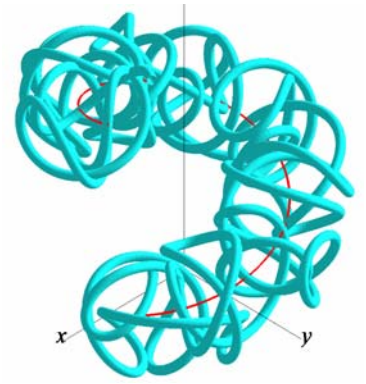


Fig. 33 : $o_3 = b, o_2 = n, o_1 = t$

c) Revolution about three skew axes

In Fig.34 are displayed surfaces created by revolution of the point $P = (0,0,0,1)$ about three skew axes $o_3 / o_2 / o_1$, which are parallel with edges of the trihedron of the helix *s*, $o_3 \| n$, $o_2 \| b$, $o_3 \| t$, where parameters are $m_1 = 4m, sg_1 = +1$, $m_2 = 2m_1, sg_2 = +1$, $m_3 = 6m_2, sg_1 = +1$, transformation matrices of three revolutions are

$\mathbf{T}_3(m_3 v, sg_3) = \mathbf{T}(0,0,-d_3).\mathbf{T}_x(m_3 v, sg_3).\mathbf{T}(0,0,+d_3)$, $\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}(-d_2,0,0).\mathbf{T}_y(m_2 v, sg_2).\mathbf{T}(+d_2,0,0)$,

$\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}(0,-d_1,0).\mathbf{T}_z(m_1 v, sg_1).\mathbf{T}(0,+d_2,0)$.

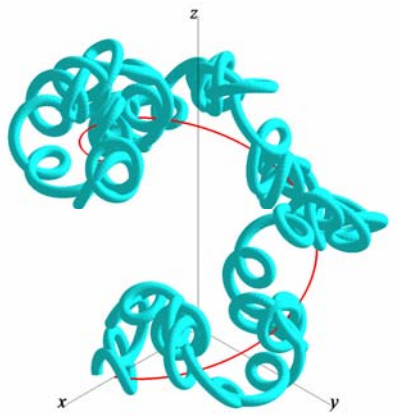


Fig. 34 : $o_3 \| n,\ o_2 \| b,\ o_3 \| t$

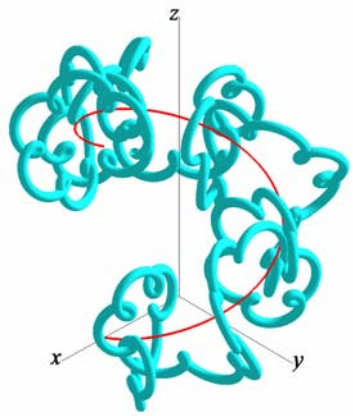


Fig. 35 : $o_3 \| n, o_2 = n, o_1 = b$

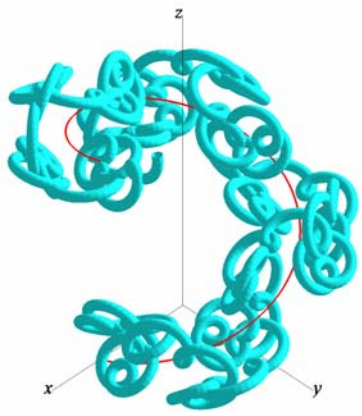


Fig. 36 : $o_3 \| n, o_2 = n, o_1 = t$

In Figs.35,36 are displayed surfaces created by revolution about the axes $o_3 \| n, o_2 = n, o_1 = b$ or axes $o_3 \| n, o_2 = n, o_1 = t$.

V. Conclusion

The described method of modeling of the helical-*n*-revolutional cyclical surfaces makes it possible to model different interest surfaces simply by changing the parameters.

References Références Referencias

[1] BUDINSKÝ, B., KEPR, B.: Introduction to differential geometry with technical applications. SNTL–Publishers of technical literature, Praha, 1970

[2] GRANÁT, L., SECHOVSKÝ, H.: Computer graphics. SNTL– Publishers of technical literature, Praha, 1980

© 2013 Global Journals Inc. (US)

[3] OLEJNÍKOVÁ, T.: Two helical surfaces. In: Journal of civil engineering, Selected scientific papers, Košice, 2010, Vol. 5, Issue 1, ISSN 1336-9024, pp.7-16

[4] OLEJNÍKOVÁ, T.: Composed Cyclical Surfaces. Transactions of the universities of Košice, 2007, Issue.3, ISSN 1335-2334, pp.54-60

[5] OLEJNÍKOVÁ, T.: Rope of Cyclical Helical surfaces. In: Journal of civil engineering, Selected scientific papers, Košice, 2012, Vol. 7, Issue 2, ISSN 1336-9024, pp.23-32, DOI: 10.2478/v10299-012-0003-4, http://www.degruyter.com/view/j/sspjce.2012.7.issue-2/issue-files/sspjce.2012.7.issue-2.xml

[6] STANOVÁ, E.: Geometry and Modeling of the Oval Strand of $n_0 + (2n_0 + 4) + n_2$ Type. In: Journal of civil engineering, Selected scientific papers, Košice, 2012, Vol. 7, Issue 2, ISSN 1336-9024,pp.33-40, DOI: 10.2478/v10299-012-0004-3, http://www.degruyter.com/view/j/sspjce.2012.7.issue-2/issue-files/sspjce.2012.7.issue-2.xml

Notes

© 2013 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 4 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Estimation of Population Ratio in Simple Random Sampling using Variable Transformation

By Onyeka, A.C., Nlebedim, V.U. & Izunobi, C.H.

Federal University of Technology

Abstract - This paper proposes six new estimators of the population ratio (R) of the population means of two variables (y and x) in Simple Random Sampling (SRS) scheme, using a variable transformation of the auxiliary variable, x. Properties of the proposed estimators, including optimality conditions, are derived up to first order approximation, and conditions under which the proposed estimators perform better than the customary ratio estimator ($\hat{R} = \bar{y}/\bar{x}$) are also obtained. The results are supported with empirical illustrations, which show that some of the proposed estimators have relatively large gains in efficiency over the customary ratio estimator, $\hat{R}$ for the data set considered.

Keywords : *variable transformation, ratio, product and regression-type estimators, mean squared error.*

GJSFR-F Classification : *MSC 2010: 62D05*



Strictly as per the compliance and regulations of :

© 2013. Onyeka, A.C., Nlebedim, V.U. & Izunobi, C.H. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Estimation of Population Ratio in Simple Random Sampling using Variable Transformation

Onyeka, A.C. [α], Nlebedim, V.U. [σ] & Izunobi, C.H. [ρ]

Abstract - This paper proposes six new estimators of the population ratio (R) of the population means of two variables (y and x) in Simple Random Sampling (SRS) scheme, using a variable transformation of the auxiliary variable, x. Properties of the proposed estimators, including optimality conditions, are derived up to first order approximation, and conditions under which the proposed estimators perform better than the customary ratio estimator ($\hat{R} = \bar{y}/\bar{x}$) are also obtained. The results are supported with empirical illustrations, which show that some of the proposed estimators have relatively large gains in efficiency over the customary ratio estimator, $\hat{R}$ for the data set considered.

Keywords : *variable transformation, ratio, product and regression-type estimators, mean squared error.*

I. Introduction

It is a common practice, in sample surveys, to use information obtained on an auxiliary variable to improve the efficiency of estimates of the population mean and total of the study variable. In some studies, however, the ratio of the population means (or totals) of the study and auxiliary variables might be of great significance, hence the need to estimate such ratios. For instance, one might be interested in the unemployment rate, which is usually obtained as the ratio of the number of unemployed people to the number of employed individuals in a country's labour force. Other parameters that could be obtained as a ratio of two parameters include income per capita, which is the ratio of the total income of a household to the total number of members of the household; the average salary of workers in a given establishment or company, which is usually obtained as the ratio of the total salary funds of the establishment to the company's total number of employees; and the employment sex ratio of a country, which is the ratio of the number of employed men and women in the country's labour force.

The usual or customary estimator of the population ratio, $R = \bar{Y}/\bar{X}$, of the population means of two variables, y and x, under the simple random sampling scheme, is given as $\hat{R} = \bar{y}/\bar{x}$, which is the ratio of the sample means of the two variables [Cochran(1977)]. However, several authors have contributed to the problem of estimating the population ratio of two means, often utilizing information on single or more auxiliary variables. These include Singh (1965), Srivastava (1967), Srivastava et al. (1988), Upadhyaya et al. (2000), Khare and Sinha (2007), and Khare et al. (2012). In using information on one or more auxiliary characters to

Author α σ ρ : Department of Statistics Federal University of Technology PMB 1526, Owerri, Nigeria. E-mail : aloyonyeka@yahoo.com

© 2013 Global Journals Inc. (US)

estimate the population ratio, $R = \overline{Y}/\overline{X}$, the two variables, y and x, are considered as the study variables, while other variables, say $z_i\,(i = 1, 2, \cdots, k)$, are considered as auxiliary variables, known to have some strong correlation with the variables, y and x. This implies that after observing the variables, y and x, more funds would be required to obtain information on the auxiliary variables, z_i's. If the variable y is the study variable, as it is often the case in most practical surveys, then the variable x, together with the variables, z_i's, would all be considered as auxiliary variables, which require extra funds in order to be observed. In the present study, we restrict observations, and consequently, funding costs, to only two variables, y and x, taking the variable x as an auxiliary variable having strong correlation with the variable, y. The parameter of interest still remains the population ratio, $R = \overline{Y}/\overline{X}$, and the objective of the study is to estimate R using variable transformation of the (auxiliary) variable, x, on the assumption that the population mean ($\overline{X}$) of x is known.

II. The Proposed Estimators

Let $y_i(x_i)$ be observations on two variables, and let a random sample of size n be drawn from a population of N units using simple random sampling without replacement (SRSWOR) scheme. Consider the variable transformation,

$$x_i^* = \frac{N\overline{X} - nx_i}{N - n},\ i = 1, 2, \cdots, N, \tag{2.1}$$

The transformation (2.1) has been used by authors like Srivenkataramana (1980), Singh and Tailor (2005), Tailor and Sharma (2009), and Sharma and Tailor (2010) to improve estimates of the population mean, $\overline{Y}$, under the simple random sampling scheme, and Onyeka (2013) under the poststratified sampling scheme. The associated sample mean is obtained as:

$$\overline{x}^* = (1 + \pi)\overline{X} - \pi\overline{x}, \qquad \pi = \frac{n}{N - n} \tag{2.2}$$

Using the (sample) means, $\overline{y}$, $\overline{x}$, and $\overline{x}^*$, and assuming knowledge of the population mean, $\overline{X}$, of the (auxiliary) variable, x, we propose the following six estimators of the population ratio, $R = \overline{Y}/\overline{X}$, in simple random sampling without replacement (SRSWOR) scheme:

$$\hat{R}_1 = \frac{\overline{y}}{\overline{x} - b(\overline{x}^* - \overline{X})} \quad \text{(regression-type estimator of sample mean, } \overline{x}\text{)} \tag{2.3}$$

$$\hat{R}_2 = \frac{\overline{y}}{\left(\frac{\overline{x}}{\overline{x}^*}\overline{X}\right)} = \frac{\overline{y}\overline{x}^*}{\overline{x}\overline{X}} \quad \text{(ratio-type estimator of sample mean, } \overline{x}\text{)} \tag{2.4}$$

$$\hat{R}_3 = \frac{\overline{y}}{\left(\frac{\overline{x}\overline{x}^*}{\overline{X}}\right)} = \frac{\overline{y}\overline{X}}{\overline{x}\overline{x}^*} \quad \text{(product-type estimator of sample mean, } \overline{x}\text{)} \tag{2.5}$$

$$\hat{R}_4 = \frac{\overline{y}}{\overline{x}^*} \quad \text{(transformed mean estimator, } \overline{x}^*\text{)} \tag{2.6}$$

$$\hat{R}_5 = \frac{\overline{y}}{\overline{x}^* - b(\overline{x} - \overline{X})} \quad \text{(regression-type estimator of transformed mean, } \overline{x}^*\text{)} \tag{2.7}$$

© 2013 Global Journals Inc. (US)

$$\hat{R}_6 = \frac{\bar{y}}{\left(\frac{\bar{x}^*}{\bar{x}}\bar{X}\right)} = \frac{\bar{y}\bar{x}}{\bar{x}^*\bar{X}} \quad \text{(ratio-type estimator of transformed mean, } \bar{x}^*\text{)} \tag{2.8}$$

where b is a suitable constant, often chosen to be very close to the population regression coefficient of y on x.

Notes

It is worth mentioning here, that Adewara et al. (2012) proposed some modified estimators of the population mean, $\bar{Y}$, involving the transformed (sample) means, $\bar{x}^*$ and $\bar{y}^*$, having the relationships:

$$\bar{X} = f\bar{x} + (1-f)\bar{x}^* \text{ and } \bar{Y} = f\bar{y} + (1-f)\bar{y}^* \ , \ f = n/N \tag{2.9}$$

This is quite worrisome in view of the fact that the transformed mean, $\bar{y}^*$, is a function of the population mean, $\bar{Y}$, which is usually unknown. If the population mean ($\bar{Y}$) of the study variable, y, is already known, then there is no need constructing estimators to estimate what is already known. Consequently, there seems to be no justification for the use of the transformed mean, $\bar{y}^*$, in estimating the population mean, $\bar{Y}$, as well as the population ratio, $R = \bar{Y}/\bar{X}$. In estimating the population ratio, in particular, the much one could do is to assume that one of the population means, (say $\bar{X}$), is known, hence the justification for the use of the transformed mean, $\bar{x}^*$. To use both the transformed means, $\bar{x}^*$ and $\bar{y}^*$, in constructing estimators of the population ratio is to suggest or assume that both the population means, $\bar{X}$ and $\bar{Y}$, are already known, which implies that the population ratio is equally known and needs not to be estimated in the first place.

Let $$e_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}} \text{ and } e_1 = \frac{\bar{x}-\bar{X}}{\bar{X}} \ . \tag{2.10}$$

Then,

$$E(e_0) = E(e_1) = 0 \tag{2.11}$$

$$E(e_0^2) = \frac{V(\bar{y})}{\bar{Y}^2} = \left(\frac{1-f}{n}\right)\frac{S_y^2}{\bar{Y}^2} \tag{2.12}$$

$$E(e_1^2) = \frac{V(\bar{x})}{\bar{X}^2} = \left(\frac{1-f}{n}\right)\frac{S_x^2}{\bar{X}^2} \tag{2.13}$$

and

$$E(e_0e_1) = \frac{Cov(\bar{y},\bar{x})}{\bar{Y}\bar{X}} = \left(\frac{1-f}{n}\right)\frac{S_{yx}}{\bar{Y}\bar{X}} \tag{2.14}$$

where $S_y^2(S_x^2)$ is the variance of $y(x)$ and S_{yx} is the covariance of y and x.

To obtain the properties of the proposed estimator, $\hat{R}_1$, we first rewrite (2.3) in terms of e_0 and e_1 and expand up to first order in expected values, to obtain:

$$(\hat{R}_1 - R) = R[e_0 - (1+\pi b)e_1 - (1+\pi b)e_0e_1 + (1+\pi b)^2e_1^2] \tag{2.15}$$

and

$$(\hat{R}_1 - R)^2 = R^2[e_0^2 + (1+\pi b)^2e_1^2 - 2(1+\pi b)e_0e_1] \tag{2.16}$$

© 2013 Global Journals Inc. (US)

Taking the expectations of (2.15) and (2.16), and using (2.11) – (2.14) to make the necessary substitutions, gives the bias and mean squared error of the proposed estimators, $\hat{R}_1$, up to first order approximation, respectively as:

$$B(\hat{R}_1) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1+\pi b)\left[(1+\pi b)RS_x^2 - S_{yx}\right] \tag{2.17}$$

and

$$MSE(\hat{R}_1) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1+\pi b)^2 R^2 S_x^2 - 2(1+\pi b)RS_{yx}\right] \tag{2.18}$$

Following similar procedure, we obtain the biases and mean squared errors of the six proposed estimators, together with those of the usual or customary ratio estimator, $\hat{R} = \bar{y}/\bar{x}$, up to first order approximations as:

$$B(\hat{R}) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[RS_x^2 - S_{yx}\right] \tag{2.19}$$

$$B(\hat{R}_1) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1+\pi b)\left[(1+\pi b)RS_x^2 - S_{yx}\right] \tag{2.20}$$

$$B(\hat{R}_2) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1+\pi)\left[RS_x^2 - S_{yx}\right] \tag{2.21}$$

$$B(\hat{R}_3) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[(1-\pi+\pi^2)RS_x^2 - (1-\pi)S_{yx}\right] \tag{2.22}$$

$$B(\hat{R}_4) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\pi\left[\pi RS_x^2 + S_{yx}\right] \tag{2.23}$$

$$B(\hat{R}_5) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(\pi+b)\left[(\pi+b)RS_x^2 + S_{yx}\right] \tag{2.24}$$

$$B(\hat{R}_6) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1+\pi)\left[\pi RS_x^2 + S_{yx}\right] \tag{2.25}$$

and,

$$MSE(\hat{R}) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + R^2 S_x^2 - 2RS_{yx}\right] \tag{2.26}$$

$$MSE(\hat{R}_1) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1+\pi b)^2 R^2 S_x^2 - 2(1+\pi b)RS_{yx}\right] \tag{2.27}$$

$$MSE(\hat{R}_2) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1+\pi)^2 R^2 S_x^2 - 2(1+\pi)RS_{yx}\right] \tag{2.28}$$

$$MSE(\hat{R}_3) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1-\pi)^2 R^2 S_x^2 - 2(1-\pi)RS_{yx}\right] \tag{2.29}$$

$$MSE(\hat{R}_4) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + \pi^2 R^2 S_x^2 + 2\pi RS_{yx}\right] \tag{2.30}$$

Notes

© 2013 Global Journals Inc. (US)

$$MSE(\hat{R}_5) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (\pi+b)^2 R^2 S_x^2 + 2(\pi+b)RS_{yx}\right] \quad (2.31)$$

$$MSE(\hat{R}_6) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1+\pi)^2 R^2 S_x^2 + 2(1+b)RS_{yx}\right] \quad (2.32)$$

Notes

Generally, the mean squared errors of the estimators could be written as:

$$MSE(\hat{R}_d) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + \theta_d^2 R^2 S_x^2 - 2\theta_d RS_{yx}\right] \quad (2.33)$$

where

$$\theta_1 = (1+\pi b),\ \theta_2 = (1+\pi),\ \theta_3 = (1-\pi),\ \theta_4 = (-\pi),\ \theta_5 = [-(\pi+b)],\ \theta_6 = [-(1+\pi)] \quad (2.34)$$

III. Efficiency Comparison

The efficiencies of the six proposed estimators, $\hat{R}_d$, d = 1, 2, …, 6 are first compared to that of the customary ratio estimator, $\hat{R}$, in estimating the population ratio, R, of two population means in simple random sampling scheme. The performance of the proposed estimators among themselves is also considered. Further consideration is also given to the optimum or best estimators among the proposed set of estimators.

a) Efficiency of Proposed Estimators over the Customary Ratio Estimator, $\hat{R}$

Using (2.26) and (2.33), the proposed estimators, $\hat{R}_d$, d = 1, 2, …, 6 would perform better than the customary ratio estimator, $\hat{R}$, in terms of having a smaller mean squared error if:

$$\text{or} \quad \left.\begin{array}{ll} (1) & \theta_d < 1 \text{ and } B < R \\ (2) & \theta_d > 1 \text{ and } B > R \end{array}\right\} \quad (3.1)$$

where θ_d is as given in (2.34), and $B = S_{yx}/S_x^2$ is the population regression coefficient of y on x. This shows that the proposed estimators are not always more efficient than the customary ratio estimator, $\hat{R}$. For instance, the proposed estimator, $\hat{R}_2$, in (2.4) would only be more efficient than the estimator, $\hat{R}$, if and only if B > R, since, from (2.2) and (2.34), the quantity, $\theta_2 = 1+\pi$ is greater than unity. This means that the customary ratio estimator, $\hat{R}$, would be more efficient than the proposed estimator, $\hat{R}_2$ for data sets in which the value of the population regression coefficient, B, is smaller than the value of the population ratio, R. However, it would be shown later in this study that the proposed estimators, under certain general optimality conditions, always perform better than the customary ratio estimator, $\hat{R}$.

b) Efficiency Comparison among the Proposed Estimators

Let $\hat{R}_j$ and $\hat{R}_k$, $j \neq k$, and $j, k = 1, 2, \cdots, 6$ be any two particular estimators from the six proposed estimators in (2.3) to (2.8). Then using (2.33), the estimator, $\hat{R}_j$ would be more efficient than the estimator, $\hat{R}_k$, in terms of having a smaller mean squared error, if:

© 2013 Global Journals Inc. (US)

or

$$\left.\begin{array}{ll} (1) & \theta_j < \theta_k \text{ and } B < R\theta_k \\ (2) & \theta_j > \theta_k \text{ and } B > R\theta_k \end{array}\right\} \qquad (3.2)$$

where θ_j and θ_k are obtained from θ_d, as given in (2.34). For instance, in comparing the proposed estimators, $\hat{R}_2$ and $\hat{R}_3$, we observe, from (2.34) that $\theta_2 = 1+\pi$ and $\theta_3 = 1-\pi$, indicating that the quantity, θ_2, is greater than θ_3, since $\pi = \frac{n}{N-n}$ is always positive. Consequently, and by using condition (2) of (3.2), the proposed estimator, $\hat{R}_2$ would be more efficient than the proposed estimator, $\hat{R}_3$ if and only if, $B > R(1-\pi)$. Comparison of the efficiencies of the remaining proposed estimators could be carry out in a similar manner, using the efficiency conditions in (3.2).

c) Optimum Estimators

The optimum estimators, among the six proposed estimators, could be obtained by minimizing the mean squared error of the proposed estimators, $\hat{R}_d$, in (2.33) with respect to the quantity, θ_d defined in (2.34). Applying the least square method, this gives the optimum value of θ_d, say θ_d^0, as

$$\theta_d^0 = \frac{B}{R} \qquad (3.4)$$

with the associated minimum mean squared error of $\hat{R}_d$ obtained as:

$$MSE_{opt}(\hat{R}_d) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1-\rho^2)S_y^2 \qquad (3.5)$$

where $\rho = S_{yx}/(S_y S_x)$ is the population correlation coefficient of the variables y and x.

Notice, from (2.3) and (2.7), that the proposed estimators, $\hat{R}_1$ and $\hat{R}_5$, particularly give us the opportunity to choose suitable values of the constant, b. This means that with both estimators, it is possible and a lot easier to meet the optimality condition (3.4) by making appropriate choice of the constant, b. Using (3.4) and (2.34), it follows that the proposed estimator, $\hat{R}_1$ would be an optimum estimator if we choose the value of b, say b_1^0, as:

$$b_1^0 = \frac{B-R}{\pi R} \qquad (3.6)$$

Similarly, the proposed estimator, $\hat{R}_5$ would be an optimum estimator if we choose the value of b, say b_5^0, as:

$$b_5^0 = -\left(\frac{B+\pi R}{R}\right) \qquad (3.7)$$

The associated minimum mean squared errors of the estimators, $\hat{R}_1$ and $\hat{R}_5$ are the same as already given in (3.5). Comparing (3.5) and (2.26), we obtain the difference between the

© 2013 Global Journals Inc. (US)

mean squared error of the optimum estimators and that of the customary ratio estimator, $\hat{R}$, as:

$$\Delta = \mathrm{MSE}(\hat{R}) - \mathrm{MSE}_{opt}(\hat{R}_d) = \frac{1}{\overline{X}^2}\left(\frac{1-f}{n}\right)(\rho S_y - R S_x)^2, \qquad (3.8)$$

which is always greater than zero. Hence the optimum estimators, using the optimality condition (3.4), are always more efficient than the customary ratio estimator, $\hat{R}$, for the purpose of estimating the population ratio, R, of two population means, under the simple random sampling scheme.

IV. Numerical Illustration

The theoretical results obtained in the present study are illustrated here numerically, using the data given on page 171 of Johnston (1982). The data set is summarized as follows:

y = Percentage of hives affected by disease

x = Date of flowering of a particular summer species (number of days from January 1)

$N = 10,\ n = 4,\ \overline{Y} = 52,\ \overline{X} = 200,\ S_y^2 = 65.97338,\ S_x^2 = 84.01556,\ S_{yx} = -69.98292$

We assume, for illustration purposes, that we are interested in the ratio of the percentage of hives affected by the disease to the number of days of flowering. That is, $R = \overline{Y}/\overline{X}$. Then, the computed percentage relative efficiencies (PRE) of the six proposed estimators, $\hat{R}_d$, $d = 1, 2, \cdots, 6$, and the optimum estimators, $\hat{R}_d^0$, over the customary ratio estimator, $\hat{R} = \bar{y}/\bar{x}$ are displayed in Table 1.

Table 1 shows that apart from the estimator, $\hat{R}_2$, the remaining five proposed estimators are more efficient than the customary ratio estimator, $\hat{R}$, for the data under consideration, and the gains in efficiency of some of the estimators, like $\hat{R}_6$ and $\hat{R}_4$, are relatively large. Notice that the values of B and R are respectively obtained as $B = -0.83298$ and $R = 0.26$, showing that B is smaller than R. That is, $B < R$. Consequently, and using the efficiency condition (1) of (3.1), the proposed estimators would be more efficient than $\hat{R}$ only if the value of the associated θ_d is less than unity. Table 1 confirms this efficiency condition, since all the estimators whose values of θ_d are less than unity are found to perform better than the estimator, $\hat{R}$.

Table 1 : PRE of Proposed Estimators over the estimator, R

d	Estimator	π	$b = B$	θ_d	MSE	PRE
-	$\hat{R}$	0.6667		1	0.0004052	100
1	$\hat{R}_1$	0.6667	-0.83298	0.44465	0.0003123	130
2	$\hat{R}_2$	0.6667		1.66670	0.0005340	76
3	$\hat{R}_3$	0.6667		0.33330	0.0002953	137
4	$\hat{R}_4$	0.6667		-0.66670	0.0001659	244
5	$\hat{R}_5$	0.6667	-0.83298	0.16628	0.0002707	150
6	$\hat{R}_6$	0.6667		-1.66670	0.0000791	512
-	$\hat{R}_d^0$	**0.6667**		**-3.20377**	**0.0000288**	**1407**

© 2013 Global Journals Inc. (US)

Also to be observed from Table 1 is the fact that the first three proposed estimators, $\hat{R}_1$, $\hat{R}_2$ and $\hat{R}_3$ have smaller gains in efficiency than the last three proposed estimators, $\hat{R}_4$, $\hat{R}_5$ and $\hat{R}_6$. Notice, from (2.3) to (2.8), that while the first three proposed estimators have the sample mean, $\bar{x}$, as the lead statistic in the denominator, the transformed mean, $\bar{x}^*$, is the lead statistic in the denominator of the last three proposed estimators. Consequently, Table 1 suggests that estimators with the transformed mean, $\bar{x}^*$, as the lead statistic in the denominator are likely to be more efficient than those with the sample mean, $\bar{x}$, as the lead statistic in the denominator, when there is a strong negative correlation between the two variables, like we presently have in the data under consideration. The optimum estimators, as expected, are the most efficient estimators, in terms of having the smallest mean squared error when compared with the customary ratio estimator as well as all the proposed estimators.

V. Concluding Remarks

Here, we have proposed and considered six new estimators of the population ratio (R) of two population means in SRSWOR scheme, using a variable transformation of the auxiliary variable, x. The biases and mean squared errors of the proposed estimators were obtained up to first order approximation. Conditions under which the proposed estimators perform better than the customary ratio estimator ($\hat{R} = \bar{y}/\bar{x}$) were derived. Also obtained were the optimality conditions under which some of the proposed estimators could become the best (optimum) estimators. The results of the study were supported and illustrated numerically. The empirical illustration confirmed, among other things, both the optimality and efficiency conditions, which we had earlier obtained theoretically in the study. The empirical study revealed that relatively large gains in efficiency over the customary ratio estimator could be obtained by using some of the new estimators proposed in the present study. Again, the direction (positive or negative) of the linear relationship between the two variables plays a role in identifying some of the proposed estimators that are likely to be more efficient than the others, for a given set of data. The first three proposed estimators make use of the sample mean, $\bar{x}$ as the lead statistic in the denominator, and are likely to perform better than the last three proposed estimators, when there is a strong positive linear relationship between the two variables. When there is a strong negative correlation between the variables, the last three proposed estimators, which incidentally make use of the transformed sample mean, $\bar{x}^*$, as the lead statistic in the denominator are likely to be more efficient than the first three proposed estimators. However, the best estimators to use for any given set of data could be obtained by using the optimality conditions given in (3.4).

References Références Referencias

1. Adewara, A.A., Singh, R. and Kumar, M. (2012): Efficiency of Some Modified Ratio and Product Estimators Using Known Value Of Some Population Parameters. International Journal of Applied Science and Technology, Vol.2. No.2, 76-79.
2. Cochran, W.G. (1977): Sampling Techniques, John Wiley and Sons, New York.
3. Johnston, J. (1982): Econometric Methods, 2nd Edition, McGraw-Hill Kogakusha, Ltd.
4. Khare, B. B. and Sinha, R. R. (2007): Estimation of the ratio of the two populations means using multi-auxiliary characters in the presence of non-response, Published in "Statistical Techniques in Life Testing, Reliability, Sampling Theory and Quality Control edited by B. N. Pandey, Narosa Publishing House, New House, New Delhi, pp. 163-171.

© 2013 Global Journals Inc. (US)

5. Khare, B.B., Srivastava, U. and Kumar, K. (2012): Chain type estimators for ratio of two population means using auxiliary characters in the presence of non-response. Journal of Scientific Research, B.H.U., Varanasi, Vol.56, pp 183 196.
6. Onyeka, A.C. (2013): Dual to Ratio Estimators of Population Mean in Post-Stratified Sampling using Known Value of Some Population Parameters. Global Journal of Science Frontier Research, Vol. 13, Issue 2 (Ver 1.0), 13-23.
7. Sharma, B. and Tailor, R. (2010). A New Ratio-Cum-Dual to Ratio Estimator of Finite Population Mean in Simple Random Sampling. Global Journal of Science Frontier Research, Vol. 10, Issue 1 (Ver 1.0), 27-31
8. Singh, H. P. and Tailor, R. (2005). Estimation of finite population mean using known correlation coefficient between auxiliary characters. Statistica, Anno LXV, 4, 407-418.
9. Singh, M.P. (1965): On the estimation of ratio and product of the population parameters. Sankhya, B, 27, 321-328.
10. Srivastava, S.K. (1967): An estimator using auxiliary information in sample surveys. Calcutta Statistical Association Bulletin, 16, 121-132.
11. Srivastava, S.R., Khare, B.B. and Srivastaca, S.R. (1988): On generalized chain estimator for ratio and product of two population means using auxiliary characters. Assam Statistical Review, 2, 21-29.
12. Srivenkataramana, T. (1980). A dual of ratio estimator in sample surveys. Biometrika, 67, 1, 199-204.
13. Tailor, R. and Sharma, B. K. (2009). A Modified Ratio-Cum-Product Estimator of Finite Population Mean Using Known Coefficient of Variation and Coefficient of Kurtosis. Statistics in Transition-new series, Jul-09, Vol. 10, No. 1, 15-24.
14. Upadhyaya, L. N., Singh, G. N. and Singh, Housila P. (2000): Use of transformed auxiliary variable in the estimation of population ratio in sample survey. Statistics in Transition 4(6), pp. 1019-1027.

Notes

© 2013 Global Journals Inc. (US)

This page is intentionally left blank

© 2013 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 4 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Certain Summation Formulae Involving Gauss Theorem

By Salahuddin & Intazar Husain

P.D.M College of Engineering

Abstract - The main object of present paper is to obtain some summation formuale involving Contiguous relation, Recurrence relation, Gauss second summation theorem and Legendre duplication formula.

Keywords : *contiguous relation, recurrence relation, gauss second summation theorem, legendre duplication formula.*

GJSFR-F Classification : *MSC 2010: 34M30, 11T24*



Strictly as per the compliance and regulations of :

© 2013. Salahuddin & Intazar Husain. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

On Certain Summation Formulae Involving Gauss Theorem

Salahuddin [α] & Intazar Husain [σ]

Abstract - The main object of present paper is to obtain some summation formuale involving Contiguous relation, Recurrence relation, Gauss second summation theorem and Legendre duplication formula.

Keywords and Phrases : contiguous relation, recurrence relation, gauss second summation theorem, legendre duplication formula.

I. Introduction

The Pochhammer's symbol is defined by

$$(\alpha,\ k) = (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1); & \text{if } k = 1,2,3,\cdots \\ 1 \quad ; & \text{if } k = 0 \\ k! \quad ; & \text{if } \alpha = 1 \end{cases} \tag{1}$$

Generalized Gaussian Hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{array}{ll} a_1, a_2, \cdots, a_A & ; \\ & z \\ b_1, b_2, \cdots, b_B & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

or

$${}_AF_B\left[\begin{array}{ll} (a_A) & ; \\ & z \\ (b_B) & ; \end{array}\right] \equiv {}_AF_B\left[\begin{array}{ll} (a_j)_{j=1}^{A} & ; \\ & z \\ (b_j)_{j=1}^{B} & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{2}$$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation is defined by

[Andrews p.367(8), E. D. p.52(19)]

$$c\,(1-z)\ {}_2F_1\left[\begin{array}{ll} a,\ b & ; \\ & z \\ c & ; \end{array}\right] = c\ {}_2F_1\left[\begin{array}{ll} a-1,\ b & ; \\ & z \\ c & ; \end{array}\right] - (c-b)\ z\ {}_2F_1\left[\begin{array}{ll} a,\ b & ; \\ & z \\ c+1 & ; \end{array}\right] \tag{3}$$

Author α : P.D.M College of Engineering, Bahadurgarh, Haryana, India.
Author σ : Delhi Institute of Technology, Management and Research, Ballabhgarh, Haryana, India. E-mail : vsludn@gmail.com

© 2013 Global Journals Inc. (US)

Recurrence relation

$$\Gamma(z+1) = z\,\Gamma(z) \tag{4}$$

Legendre's duplication formula

$$\sqrt{\pi}\,\Gamma(2z) = 2^{(2z-1)}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) \tag{5}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\,\Gamma(\frac{b}{2})\,\Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{6}$$

$$= \frac{2^{(a-1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{7}$$

Gauss second summation theorem [Prudnikov., 491(7.3.7.8)]

$${}_2F_1\left[\begin{matrix} a,\ b \\ \frac{a+b+1}{2} \end{matrix};\ \frac{1}{2}\right] = \frac{\Gamma(\frac{a+b+1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} \tag{8}$$

$$= \frac{2^{(b-1)}\,\Gamma(\frac{b}{2})\,\Gamma(\frac{a+b+1}{2})}{\Gamma(b)\,\Gamma(\frac{a+1}{2})} \tag{9}$$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov., p.491(7.3.7.8)]

$${}_2F_1\left[\begin{matrix} a,\ b \\ \frac{a+b-1}{2} \end{matrix};\ \frac{1}{2}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} + \frac{2\,\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\,\Gamma(b)}\right] \tag{10}$$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$${}_2F_1\left[\begin{matrix} a,\ b \\ \frac{a+b-1}{2} \end{matrix};\ \frac{1}{2}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{11}$$

II. Main Results of Summation Formulae

For $a < 1$ and $a > 9$

$${}_2F_1\left[\begin{matrix} a,\ b \\ \frac{a+b-9}{2} \end{matrix};\ \frac{1}{2}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\left\{\frac{(128a+384b-1152)}{(a-9)} -\right.\right.$$

$$\left.\left. - \frac{(a-b-9)(256a+640b-1792)}{(a-9)(a-7)} + \frac{(a-b-9)(a-b-7)(160a+320b-800)}{(a-9)(a-7)(a-5)} -\right.\right.$$

© 2013 Global Journals Inc. (US)

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(32a+48b-96)}{(a-9)(a-7)(a-5)(a-3)}+$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-$$

$$-\frac{(a-b-9)(416a+352b-2144)}{(a-8)(a-6)}+\frac{(a-b-9)(a-b-7)(192a+128b-640)}{(a-8)(a-6)(a-4)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(22a+10b-34)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg] \tag{12}$$

For $a<1$ and $a>10$

$${}_2F_1\left[\begin{matrix}a,\ b\\ \frac{a+b-10}{2}\end{matrix};\ \frac{1}{2}\right]=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-10}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-10}{2})}\Bigg\{\frac{(256a+768b-2560)}{(a-10)}-$$

$$-\frac{(a-b-10)(576a+1472b-4608)}{(a-10)(a-8)}+\frac{(a-b-10)(a-b-8)(432a+912b-2592)}{(a-10)(a-8)(a-6)}-$$

$$-\frac{(a-b-10)(a-b-8)(a-b-6)(120a+200b-480)}{(a-10)(a-8)(a-6)(a-4)}+$$

$$+\frac{(a-b-10)(a-b-8)(a-b-6)(a-b-4)(9a+11b-18)}{(a-10)(a-8)(a-6)(a-4)(a-2)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(512a+512b-4096)}{(a-9)}-\frac{(a-b-10)(960a+832b-5888)}{(a-9)(a-7)}+$$

$$+\frac{(a-b-10)(a-b-8)(560a+400b-2400)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-10)(a-b-8)(a-b-6)(104a+56b-256)}{(a-9)(a-7)(a-5)(a-3)}+$$

$$+\frac{(a-b-10)(a-b-8)(a-b-6)(a-b-4)(3a+b-2)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}\Bigg] \tag{13}$$

For $a<1$ and $a>11$

$${}_2F_1\left[\begin{matrix}a,\ b\\ \frac{a+b-11}{2}\end{matrix};\ \frac{1}{2}\right]=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-11}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-11}{2})}\Bigg\{\frac{(512a+1536b-5632)}{(a-11)}-$$

$$-\frac{(a-b-11)(1280a+3328b-11520)}{(a-11)(a-9)}+\frac{(a-b-11)(a-b-9)(1120a+2464b-7840)}{(a-11)(a-9)(a-7)}-$$

$$-\frac{(a-b-11)(a-b-9)(a-b-7)(400a+720b-2000)}{(a-11)(a-9)(a-7)(a-5)}+$$

© 2013 Global Journals Inc. (US)

$$+\frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(50a+70b-150)}{(a-11)(a-9)(a-7)(a-5)(a-3)}-$$

$$-\frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-10}{2})}\Bigg\{\frac{(1024a+1024b-9216)}{(a-10)}-\frac{(a-b-11)(2176a+1920b-15488)}{(a-10)(a-8)}+$$

$$+\frac{(a-b-11)(a-b-9)(1536a+1152b-8064)}{(a-10)(a-8)(a-6)}-$$

$$-\frac{(a-b-11)(a-b-9)(a-b-7)(400a+240b-1360)}{(a-10)(a-8)(a-6)(a-4)}+$$

$$+\frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(28a+12b-44)}{(a-10)(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg] \tag{14}$$

The above summation theorems can be easily verified by using computer algebra system programming languages, like Maple, Matlab, or Mathematica.

III. Derivations of Summation Formulae (12) to (14)

Derivation of (12): Substituting $c=\frac{a+b-9}{2}$ and $z=\frac{1}{2}$ in equation (3), we get

$$\left(\frac{a+b-9}{4}\right)\,{}_2F_1\left[\begin{matrix}a,\ b\\ \frac{a+b-9}{2}\end{matrix};\quad\frac{1}{2}\right]=\left(\frac{a+b-9}{2}\right)\,{}_2F_1\left[\begin{matrix}a-1,\ b\\ \frac{a+b-9}{2}\end{matrix};\quad\frac{1}{2}\right]-$$

$$-\left(\frac{a-b-9}{4}\right)\,{}_2F_1\left[\begin{matrix}a,\ b\\ \frac{a+b-7}{2}\end{matrix};\quad\frac{1}{2}\right]$$

$${}_2F_1\left[\begin{matrix}a,\ b\\ \frac{a+b-9}{2}\end{matrix};\quad\frac{1}{2}\right]=2.\,{}_2F_1\left[\begin{matrix}a-1,\ b\\ \frac{a+b-9}{2}\end{matrix};\quad\frac{1}{2}\right]-\left(\frac{a-b-9}{a+b-9}\right)\,{}_2F_1\left[\begin{matrix}a,\ b\\ \frac{a+b-7}{2}\end{matrix};\quad\frac{1}{2}\right]$$

Now involving the derived from Asish et all , we get

$${}_2F_1\left[\begin{matrix}a,\ b\\ \frac{a+b-9}{2}\end{matrix};\quad\frac{1}{2}\right]=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(128a+384b-1152)}{(a-9)}-$$

$$-\frac{(a-b-9)(224a+544b-1568)}{(a-9)(a-7)}+\frac{(a-b-9)(a-b-7)(112a+208b-560)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(14a+18b-42)}{(a-9)(a-7)(a-5)(a-3)}\Bigg\}+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-$$

$$-\frac{(a-b-9)(352a+288b-1824)}{(a-8)(a-6)}+\frac{(a-b-9)(a-b-7)(120a+72b-408)}{(a-8)(a-6)(a-4)}-$$

Notes

© 2013 Global Journals Inc. (US)

Notes

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(6a+2b-10)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]-$$

$$-\left(\frac{a-b-9}{a+b-9}\right)\frac{2^{(b-1)}\ \Gamma(\frac{a+b-7}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-7}{2})}\Bigg\{\frac{(32a+96b-224)}{(a-7)}-\frac{(a-b-7)(48a+112b-240)}{(a-7)(a-5)}+$$

$$+\frac{(a-b-7)(a-b-5)(18a+30b-54)}{(a-7)(a-5)(a-3)}-\frac{(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-6}{2})}\Bigg\{\frac{(64a+64b-320)}{(a-6)}-\frac{(a-b-7)(72a+56b-232)}{(a-6)(a-4)}+$$

$$+\frac{(a-b-7)(a-b-5)(16a+8b-24)}{(a-6)(a-4)(a-2)}\Bigg\}\Bigg]$$

$$=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(128a+384b-1152)}{(a-9)}-\frac{(a-b-9)(224a+544b-1568)}{(a-9)(a-7)}+$$

$$+\frac{(a-b-9)(a-b-7)(112a+208b-560)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(14a+18b-42)}{(a-9)(a-7)(a-5)(a-3)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-\frac{(a-b-9)(352a+288b-1824)}{(a-8)(a-6)}+$$

$$+\frac{(a-b-9)(a-b-7)(120a+72b-408)}{(a-8)(a-6)(a-4)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(6a+2b-10)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]-$$

$$-\frac{2^{(b-1)}\ \Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(a-b-9)(32a+96b-224)}{(a-9)(a-7)}-$$

$$-\frac{(a-b-9)(a-b-7)(48a+112b-240)}{(a-9)(a-7)(a-5)}+$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(18a+30b-54)}{(a-9)(a-7)(a-5)(a-3)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(a-b-9)(64a+64b-320)}{(a-8)(a-6)}-\frac{(a-b-9)(a-b-7)(72a+56b-232)}{(a-8)(a-6)(a-4)}+$$

© 2013 Global Journals Inc. (US)

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(16a+8b-24)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]$$

$$=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\ \Bigg\{\frac{(128a+384b-1152)}{(a-9)}\ -$$

$$-\ \frac{(a-b-9)(256a+640b-1792)}{(a-9)(a-7)}+\ \frac{(a-b-9)(a-b-7)(160a+320b-800)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(32a+48b-96)}{(a-9)(a-7)(a-5)(a-3)}+$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-\frac{(a-b-9)(416a+352b-2144)}{(a-8)(a-6)}+$$

$$+\frac{(a-b-9)(a-b-7)(192a+128b-640)}{(a-8)(a-6)(a-4)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(22a+10b-34)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]$$

Thus , we prove the result (12)

Similarly, we can prove the other results.

References Références Referencias

1. Andrews, L.C.(1992) ; *Special Function of mathematics for Engineers*,secondEdition , McGraw-Hill Co Inc., New York.
2. Arora, Asish, Singh, Rahul , Salahuddin. ; Development of a family of summation formulae of half argument using Gauss and Bailey theorems , *Journal of Rajasthan Academy of Physical Sciences.*, 7(2008), 335-342.
3. Choi, J., Harsh, H. and Rathie, A. K.; Some summation formulae for the Apple's function F_1, *East Asian Math. Journal*, 17(2001), 233-237.
4. Erdélyi, A., Magnus, W., Okerhettinger, F. and Tricomi, F. G.; *Higher transcendental functions* Vol.1 (Bateman Manuscript Project) McGraw-Hill book P. Inc. New York, Toronto and London, 1953.
5. Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O.I.; *Integrals and Series Vol. 3: More Special Functions.* Nauka, Moscow, 1986. Translated from the Russian by G.G. Gould, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.

© 2013 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 4 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Semi-Invariant Submanifolds of Nearly Hyperbolic Cosymplectic Manifold

By Mobin Ahmad & Kashif Ali

Integral University

Abstract - We consider a nearly hyperbolic cosymplectic manifold and study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold. We also study parallel distributions on nearly hyperbolic cosymplectic manifold and find the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold.

Keywords : *semi-invariant submanifolds, nearly hyperbolic cosymplectic manifold, intrgrability conditions, parallel distribution.*

GJSFR-F Classification : *MSC 2010: 20F67, 90C34*



Strictly as per the compliance and regulations of :

© 2013. Mobin Ahmad & Kashif Ali. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

eJournal
physical existance Print Journal
epaper
open standard
3d virtual journal

1. Ahmad, M., Semi-invariant submanifolds of nearly Kenmotsu manifold with semi-symmetric semi-metric connection, MathematickiVesnik 62 (2010), 189-198.

Semi-Invariant Submanifolds of Nearly Hyperbolic Cosymplectic Manifold

Mobin Ahmad [α] & Kashif Ali [σ]

Abstract - We consider a nearly hyperbolic cosymplectic manifold and study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold. We also study parallel distributions on nearly hyperbolic cosymplectic manifold and find the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold.

Keywords and phrases : *semi-invariant submanifolds, nearly hyperbolic cosymplectic manifold, intrgrability conditions, parallel distribution.*

I. Introduction

The notion of CR-submanifolds of a Kaehler manifold as generalization of invariant and anti-invariant submanifolds was initiated by A. Bejancu in [7]. A semi-invariant submanifold is the extension of the concept of a CR-submanifold of a Kaehler manifold to submanifolds of almost contact manifolds. The study of Semi-invariant submanifolds of Sasakian manifolds was initiated by Bejancu-Papaghuic in [9]. The same concept was studied under the name contact CR-submanifold by Yano-Kon in [16] and K. Matsumoto in [14]. The study of semi-invariant submanifolds in almost contact manifold was enriched by several geometers (see, [1], [2], [3], [4], [5], [6], [11], [14]). On the other hand, almost hyperbolic (f, g, η, ξ)-structure was defined and studied by Upadhyay and Dube in [15]. Joshi and Dube studied semi-invariant submanifolds of an almost r-contact hyperbolic metric manifolds in [12]. In this paper, we study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold.

II. Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with almost hyperbolic contact metric structure-(ϕ, ξ, η, g), where a tensor ϕ of type $(1,1)$, a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the followings

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \eta(\xi) = -1, \tag{2.2}$$

$$\phi(\xi) = 0, \ \ \eta o \phi = 0, \tag{2.3}$$

Author α σ : Department of Mathematics, Integral University, P.O. Bas-ha, Kursi Road, Lucknow-226026, India.
E-mails : mobinahmad@rediffmail.com, kashifmb37@gmail.com

© 2013 Global Journals Inc. (US)

$$g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y) \tag{2.4}$$

for any X, Y tangent to $\bar{M}$ [15]. In this case

$$g(\phi X, Y) = -g(\phi Y, X). \tag{2.5}$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact metric structure-(ϕ, ξ, η, g) is said to be nearly hyperbolic cosymplectic manifold [10] if

$$(\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_Y X) = 0, \tag{2.6}$$

$$\bar{\nabla}_X \xi = 0 \tag{2.7}$$

for all X, Y tangent to $\bar{M}$.

The Nijenhuis tensor $N(X,Y)$ of a nearly hyperbolic cosymplectic manifold $\bar{M}$ is defined as

$$N(X,Y) = (\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X - \phi(\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_Y \phi)X \tag{2.8}$$

$for\ any\ X, Y \in T\bar{M}.$

From (2.6), we have

$$(\bar{\nabla}_{\phi X}\phi)Y = -(\bar{\nabla}_Y \phi)\phi X. \tag{2.9}$$

Also, we have

$$(\bar{\nabla}_Y \phi)\phi X = (\bar{\nabla}_Y \eta)(X)\xi + \phi(\bar{\nabla}_Y \phi)X. \tag{2.10}$$

From (2.9) and (2.10), we get

$$(\bar{\nabla}_{\phi X}\phi)Y = (\bar{\nabla}_Y \eta)(X)\xi + \phi(\bar{\nabla}_Y \phi)X. \tag{2.11}$$

Using (2.11) in (2.8), we get

$$N(X,Y) = 4\phi(\bar{\nabla}_Y \phi)X + 2g(\phi X, Y)\xi \tag{2.12}$$

$for\ X, Y \in TM.$

The paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic cosymplectic manifold. In section 3, we study some properties semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold. In section 4, we discuss the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold. In section 5, we study parallel horizontal distribution on nearly hyperbolic Kenmotsu manifold.

III. Semi-Invariant Submanifolds

Let M be a submanifold immersed in $\bar{M}$. We assume that the vector field ξ is tangent to M. Denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M. Then M is called a semi-invariant submanifold [8] of $\bar{M}$ if there exist two differentiable distributions D and $D^{\perp}$ on M satisfying.

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where D, $D^{\perp}$ and $\{\xi\}$ are mutually orthogonal to each other.

(ii) The distribution D is invariant by ϕ, that is, $\phi D_X = D_X$ for each $X \epsilon M$,

(iii) The distribution $D^{\perp}$ is anti-invariant by ϕ,that is, $\phi D_X^{\perp} \subset T_X M^{\perp}$ for each $X \epsilon M$,

Ref.

10. Blair, D.E., Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin, 1976.

© 2013 Global Journals Inc. (US)

where TM and $T^{\perp}M$ be the Lie algebra of vector fields tangential to M and normal to M respectively. Let the Riemannian metric induced on M is denoted by the same symbol g and ∇ be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{3.1}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{3.2}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X,Y), N) \tag{3.3}$$

for any $x \in M$ and $X \in T_x M$. We write

$$X = PX + QX, \tag{3.4}$$

where $PX \in D$ and $QX \in D^{\perp}$.

Similarly, for N normal to M we have

$$\phi N = BN + CN, \tag{3.5}$$

where $BN(resp. CN)$ is the tangential component $(resp. normal\ component)$ of ϕN.

Lemma 3.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$, then

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y]$$

for all $X, Y \in D$.

Proof. By Gauss formula (3.1), we have

$$\bar{\nabla}_X Y - \bar{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y]. \tag{3.6}$$

Also, by covariant differentiation we get

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X,Y]. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y]. \tag{3.8}$$

Adding (2.6) and (3.8), we get

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y]$$

for all $X, Y \in D$.

Hence Lemma is proved.

Lemma 3.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$, then

© 2013 Global Journals Inc. (US)

$$2(\bar{\nabla}_X\phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X - \phi[X,Y] \tag{3.9}$$

for all $X, Y \in D^{\perp}$.

Proof. By Weingarten formula (3.2), we have

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X$$

for any $X, Y \in D^{\perp}$.

Also, by covariant differentiation, we have

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y]. \tag{3.10}$$

From (3.9) and (3.10), we get

$$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X - \phi[X,Y]. \tag{3.11}$$

Adding (2.6) and (311), we get

$$2(\bar{\nabla}_X\phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X - \phi[X,Y]$$

for all $X, Y \in D^{\perp}$.

Hence Lemma is proved.

Lemma 3.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

for all $X \in D$ *and* $Y \in D^{\perp}$.

Proof. Using Gauss and Weingarten formulas, we have

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X). \tag{3.12}$$

Also, by covariant differentiation, we have

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y]. \tag{3.13}$$

From (3.12) and (3.13), we get

$$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]. \tag{3.14}$$

Adding (2.6) and (3.14), we obtain

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

for all $X \in D$ *and* $Y \in D^{\perp}$.

Hence Lemma is proved.

Notes

© 2013 Global Journals Inc. (US)

Lemma 3.4. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y = \phi P(\nabla_X Y) + \phi P(\nabla_Y X), \tag{3.15}$$

$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - QA_{\phi QY}X - QA_{\phi QX}Y = 2Bh(X,Y), \tag{3.16}$$

$$h(Y,\phi PX) + h(X,\phi PY) + \nabla^{\perp}_X \phi QY + \nabla^{\perp}_Y \phi QX = -2\eta(Y)\phi QX - 2\eta(X)\phi QY + \phi Q(\nabla_X Y)$$

$$+\phi Q(\nabla_Y X) + 2Ch(X,Y), \tag{3.17}$$

$$\eta(\nabla_X \phi PY) + \eta(\nabla_Y \phi PX) - \eta\big(A_{\phi QY}X\big) - \eta\big(A_{\phi QX}Y\big) = 0 \tag{3.18}$$

for any $X, Y \in T\bar{M}$.

Proof. Differentiating (3.4) covariantly and using (3.1) and (3.2), we get

$$\bar{\nabla}_Y \phi X = (\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(X,Y).$$

Also,

$$\bar{\nabla}_Y \phi X = P\nabla_X \phi PY + Q\nabla_Y \phi PX + \eta(\nabla_Y \phi PX)\xi + h(Y,\phi PX) + \nabla^{\perp}_{\phi QX}Y$$

$$-PA_{\phi QX}Y - QA_{\phi QX}Y - \eta\big(A_{\phi QX}Y\big)\xi.$$

Thus, we have

$$(\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(X,Y) = P\nabla_X \phi PY + Q\nabla_Y \phi PX + \eta(\nabla_Y \phi PX)\xi$$

$$+h(Y,\phi PX) + \nabla^{\perp}_{\phi QX}Y - PA_{\phi QX}Y - QA_{\phi QX}Y - \eta\big(A_{\phi QX}Y\big)\xi. \tag{3.19}$$

Interchanging X *and* Y, we have

$$(\bar{\nabla}_X \phi)Y + \phi(\nabla_X Y) + \phi h(Y,X) = P\nabla_Y \phi PX + Q\nabla_X \phi PY + \eta(\nabla_X \phi PY)\xi$$

$$+h(X,\phi PY) + \nabla^{\perp}_{\phi QY}X - PA_{\phi QY}X - QA_{\phi QY}X - \eta\big(A_{\phi QY}X\big)\xi. \tag{3.20}$$

Adding (3.19) and (3.20), we have

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi(\nabla_X Y) + 2\phi h(X,Y) = P\nabla_X \phi PY + P\nabla_Y \phi PX$$

$$+Q\nabla_Y \phi PX + Q\nabla_X \phi PY + \eta(\nabla_Y \phi PX)\xi + \eta(\nabla_X \phi PY)\xi + h(Y,\phi PX) + h(X,\phi PY)$$

$$+\nabla^{\perp}_{\phi QX}Y + \nabla^{\perp}_{\phi QY}X - PA_{\phi QX}Y - PA_{\phi QY}X - QA_{\phi QX}Y - QA_{\phi QY}X - \eta\big(A_{\phi QX}Y\big)\xi$$

$$-\eta\big(A_{\phi QY}X\big)\xi. \tag{3.21}$$

By virtue of (2.6) and (3.21), we obtain

© 2013 Global Journals Inc. (US)

$$\phi P(\nabla_Y X) + \phi P(\nabla_X Y) + \phi Q\nabla_Y X + \phi Q\nabla_X Y + 2Bh(X,Y) + 2Ch(X,Y) = P\nabla_Y \phi PX$$

$$+P\nabla_X \phi PY + Q\nabla_Y \phi PX + Q\nabla_X \phi PY + \eta(\nabla_Y \phi PX)\xi + \eta(\nabla_X \phi PY)\xi + h(Y, \phi PX)$$

$$+h(X, \phi PY) + \nabla^{\perp}_{\phi QX} Y + \nabla^{\perp}_{\phi QY} X - PA_{\phi QX} Y - PA_{\phi QY} X - QA_{\phi QX} Y - QA_{\phi QY} X$$

$$-\eta(A_{\phi QX} Y)\xi - \eta(A_{\phi QY} X)\xi.$$

Comparing horizontal, vertical and normal components we get the desired result.
Hence the Lemma is proved.

Definition 3.5. The horizontal distribution D is said to be parallel [10] on $M if \nabla_X Y \in D$ for all vector field $X, Y \in D$.

Theorem 3.6. Let M be a semi-invariant submanifoldof a nearly hyperbolic cosymplectic manifold $\bar{M}$. If the horizontal distribution D is parallel, then

$$h(X, \phi Y) = h(Y, \phi X)$$

$for\ all\ X, Y \in D.$

Proof. Let $X, Y \in D$ and D is parallel then $\nabla_X \phi Y \in D\ and\ \nabla_Y \phi X \in D$. From (3.12), we have

$$h(Y, \phi X) + h(X, \phi Y) = 2\phi h(X, Y). \tag{3.22}$$

Replacing X by ϕX in (3.22) and using (2.1), we get

$$h(Y, X) + h(\phi X, \phi Y) = 2\phi h(\phi X, Y). \tag{3.23}$$

Again replacing $Y\ by\ \phi Y$ in (3.22) and using (2.1), we get

$$h(\phi Y, \phi X) + h(X, Y) = 2\phi h(X, \phi Y). \tag{3.24}$$

By virtue of (3.23) and (3.24), we have

$$\phi h(\phi X, Y) = \phi h(X, \phi Y). \tag{3.25}$$

Operating ϕ on both sides of (3.25), we get

$$h(\phi X, Y) = h(X, \phi Y).$$

Hence the theorem is proved.

Definition 3.7. A semi-invariant submanifold is said to be mixed totally geodesic [8] if $h(X, Y) = 0\ \ for\ all\ X \in D\ and\ \ Y \in D^{\perp}$.

Theorem 3.8. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then M is a mixed totally geodesic if and only if $A_{NX} \in D\ for\ all\ X \in D$.

Proof. Let $A_{NX} \in D\ for\ all\ X \in D$.

Now, $g(h(X, Y), N) = g(A_{NX}, Y) = 0\ \ for\ Y \in D^{\perp}$, which is equivalent to $h(X, Y) = 0$. Hence M is totally mixed geodesic.

Conversely, Let M is totally mixed geodesic, that is $h(X, Y) = 0\ for\ X \in D\ and\ \ Y \in D^{\perp}$.

Notes

© 2013 Global Journals Inc. (US)

Now, $g(h(X,Y),N) = g(A_{NX},Y)$ gives that $g(A_{NX},Y) = 0$. Consequently, we have $A_{NX} \in D$ for all $Y \in D^{\perp}$.
Hence the theorem is proved.

IV. Integrability Conditions for Distributions

Notes

Theorem 4.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then the distribution $D\oplus\langle\xi\rangle$ is integrable if

$$h(\phi X,Z) + h(\phi Z,X) = 0 \tag{4.1}$$

for any $X,Y,Z \in D\oplus\langle\xi\rangle$.

Proof. The torsion tensor $S(X,Y)$ of an almost contact structure (ϕ,ξ,η,g)
Using Gauss formula in (2.12), we have

$$N(X,Y) = 4\phi\big(\nabla_Y \emptyset X - \phi\nabla_Y X - \phi h(Y,X) + h(Y,\phi X)\big). \tag{4.2}$$

If $D\oplus\langle\xi\rangle$ is integrable, then $N(X,Y) = 0$. Hence

$$4\phi\nabla_Y\phi X - 4\nabla_Y X - 4\eta(\nabla_Y X)\xi - 4h(Y,X) + 4\phi h(Y,\phi X) = 0. \tag{4.3}$$

Comparing normal parts from both sides of (4.3), we get

$$\phi Q(\nabla_Y\phi X) - h(Y,X) = Ch(Y,\phi X) = 0. \tag{4.4}$$

Replacing Y *by* $\phi Z, Z \in D$ in (4.4), we have

$$\phi Q\big(\nabla_{\phi z}\phi X\big) - h(\phi Z,X) + Ch(\phi Y,\phi X) = 0. \tag{4.5}$$

Interchanging X *and* Z in (4.5), we obtain

$$\phi Q\big(\nabla_{\phi X}\phi Z\big) - h(\phi X,Z) + Ch(\phi X,\phi Y) = 0. \tag{4.6}$$

Subtracting (4.6) from (4.5), we get

$$\phi Q[\phi X,\phi Z] - h(\phi X,Z) + h(\phi Z,X) = 0. \tag{4.7}$$

Since $D\oplus\langle\xi\rangle$ is integrable so that $[\phi X,\phi Z] \in D\oplus\langle\xi\rangle$ *for* $X,Z \in D$.
Consequently (4.7) gives

$$h(\phi X,Z) + h(\phi Z,X) = 0.$$

Hence the theorem is proved.

Proposition 4.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosy mplectic manifold $\bar{M}$. Then

$$A_{\phi Y}Z - A_{\phi Z}Y = \phi P[Y,Z]$$

for any $Y,Z \in D^{\perp}$.

© 2013 Global Journals Inc. (US)

Proof. Let $X \in \chi(M)$ and $Y, Z \in D^{\perp}$. From (3.1) and (3.2), we have

$$2g(A_{\phi Z}Y, X) = g(h(Y,X), \phi Z) + g(h(X,Y), \phi Z).$$

Using (3.3) and (3.1), we get

$$2g(A_{\phi Z}Y, X) = -g(\bar{\nabla}_Y \phi X, Z) - g(\bar{\nabla}_X \phi Y, Z) + g((\bar{\nabla}_Y \phi)X + (\bar{\nabla}_X \phi)Y, Z).$$

Using (2.6) in above equation, we have

$$2g(A_{\phi Z}Y, X) = -g(\phi \bar{\nabla}_Y Z, X) + g(A_{\phi Y}Z, X). \tag{4.8}$$

Transvecting X from both sides of (4.8), we have

$$2A_{\phi Z}Y = \phi \bar{\nabla}_Y Z + A_{\phi Y}Z.$$

Interchanging $Y and\ Z$, we have

$$2A_{\phi Y}Z = \phi \bar{\nabla}_Z Y + A_{\phi Z}Y.$$

Subtracting above two equations, we get

$$A_{\phi Z}Y - A_{\phi Y}Z = \frac{1}{3}\phi[Y,Z].$$

Comparing the tangential parts from both sides of above equation, we get

$$A_{\phi Z}Y - A_{\phi Y}Z = \frac{1}{3}\phi P[Y,Z], \tag{4.9}$$

where $[Y, Z]$ is Lie bracket.
Hence the proposition is proved.

Theorem 4.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$.Then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 \tag{4.10}$$

$for\ all\ Y, Z \in D^{\perp}$.

Proof. Suppose that the distribution $D^{\perp}$ is integrable. Then $[Y,Z] \in D^{\perp}\ for\ any\ Y, Z \in D^{\perp}$. Therefore, $P[Y,Z] = 0$. From (4.9), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = 0.$$

Conversely, let (4.10) holds good. Then by virtue of (4.9), we get

$$\phi P[Y,Z] = 0$$

$for\ all\ Y, Z \in D^{\perp}$. Since $rank\ \phi = 2n$, therefore, either $P[Y,Z] = 0\ or\ P[Y,Z] = k\xi$.
But $P[Y,Z] = k\xi$ is not possible as P being a projection operator on D. Hence $P[Y,Z] = 0$.
This implies that $[Y,Z] \in D^{\perp} for\ all\ Z \in D^{\perp}$. Thus $D^{\perp}$ is integrable.
Hence the theorem is proved.

Notes

© 2013 Global Journals Inc. (US)

Notes

References

1. Ahmad, M., Semi-invariant submanifolds of nearly Kenmotsu manifold with semi-symmetric semi-metric connection, MathematickiVesnik 62 (2010), 189-198.
2. Ahmad, M. and Jun, J.B., On semi-invariant submanifolds of a nearly Kenmotsu manifold with a semi-symmetric non-metric connection, Journal of the Chungcheong Math. Soc. Vol. 23, no. 2, June (2010), 257-266.
3. Ahmad, M., Jun, J. B. and Siddiqi, M. D., Some properties of semi-invariant submanifolds of a nearly trans-Sasakian manifold admitting a quarter symmetric non-metric connection, JCCMS, vol. 25, No. 1 (2012), 73-90.
4. Ahmad, M. and Siddiqi, M.D., On nearly Sasakian manifold with a semi-symmetric semi-metric connection, Int. J. Math. Analysis, Vol. 4 (2010), 35, 1725-1732.
5. Ahmad, M. and Siddiqi, M.D., Semi-invariant submanifolds of Kenmotsu manifold immersed in a generalized almost r-contact structure admitting quarter-symmetric non-metric connection, Journ. Math.Comput. Sci. 2 (2012), No. 4, 982-998.
6. Ahmad, M. Rahman, S. and Siddiqui, M.D., Semi-invariant submanifolds of a nearly Sasakian manifold endowed with a semi-symmetric metric connection, Bull. Allahabad Math. Soc., vol. 25 (1), 2010, 23-33.
7. Bejancu, A., CR- submanifolds of a Kaehler manifold. I, Proc. Amer. Math. Soc. 69 (1978), 135-142.
8. Geometry of CR- submanifolds, D. Reidel Publishing Company, Holland, 1986.
9. Bejancu, A. and Papaghuic N., Semi-invariant submanifolds of a Sasakian manifold, An. St. Univ. Al. I. Cuza, Iasi 27 (1981), 163-170.
10. Blair, D.E., Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin, 1976.
11. Das, Lovejoy S., Ahmad, M. and Haseeb, A., Semi-invariant submanifolds of a nearly Sasakian manifold endowed with a semi-symmetric non-metric connection, Journal of Applied Analysis, vol. 17 (2011), no. 1, 119-130.
12. Friedmann, A. and Schouten, J.A., Uber die Geometric der halbsymmetrischenUbertragung Math. Z. 21 (1924), 211-223.
13. Joshi, N.K. and Dube, K.K., Semi-invariant submanifolds of an almost r-contact hyperbolic metric manifold, Demonstratio Math. 36 (2001), 135.143.
14. Matsumoto, K., On contact CR-submanifolds of Sasakian manifold, Intern. J. Math. Sci., 6 (1983), 313-326.
15. Shahid, M.H., On semi-invariant submanifolds of a nearly Sasakian manifold, Indian J. Pure and Applied Math. 95 (10) (1993), 571-580.
16. Upadhyay, M. D. and Dube, K.K., Almost contact hyperbolic (ϕ, ξ, η, g)-structure, Acta. Math. Acad. Scient. Hung. Tomus 28 (1976), 1-4.
17. Yano, K. and Kon, M., C ontact- CR submanifolds, Kodai Math. J. 5 (1982), 238-252.

© 2013 Global Journals Inc. (US)

This page is intentionally left blank

© 2013 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 4 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Strictly Practical Stabilization of Impulsive Functional Differential Equations by using Lyapunov Functions

By Sapna Rani & Dilbaj Singh
Lovely Professional University

Abstract - In this paper, we extend the concept of strict practical stability to impulsive functional differential equations by using Lyapunov functions and Razumikhin technique. As practical stability does not give us much information about the rate of decay of solution so we develop the idea for strict practical stability of functional differential equations with impulsive effect and obtained some conditions for strict practical uniform stability for functional differential equations with impulse by using piecewise continuous Lyapunov functions and Razumikhin technique.

Keywords : *strict practical stability, impulsive differential equations, lyapunov function.*

GJSFR-F Classification : *MSC 2010: 37L45, 12H20*



Strictly as per the compliance and regulations of :

© 2013. Sapna Rani & Dilbaj Singh. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

eJournal
physical existance Print Journal
epaper
open standard
3d virtual journal

1. Dilbaj Singh, S.K. Srivastava, Stability criteria of impulsive differential equations with comparison results, Advances in Differential Equations and Control Processes, Vol. 10, pp. 171-182, 2012.

Strictly Practical Stabilization of Impulsive Functional Differential Equations by using Lyapunov Functions

Sapna Rani [α] & Dilbaj Singh [σ]

Abstract - In this paper, we extend the concept of strict practical stability to impulsive functional differential equations by using Lyapunov functions and Razumikhin technique. As practical stability does not give us much information about the rate of decay of solution so we develop the idea for strict practical stability of functional differential equations with impulsive effect and obtained some conditions for strict practical uniform stability for functional differential equations with impulse by using piecewise continuous Lyapunov functions and Razumikhin technique.

Keywords : *strict practical stability, impulsive differential equations, lyapunov function.*

I. Introduction

Various physical processes undergo abrupt changes of state at certain moments of time between intervals of continuous equation. The duration of these changes is often negligible in comparison with that of the entire evolution process and thus the abrupt changes can be well approximated in terms of the instantaneous changes of state i.e. Impulses [7]. On the other hand Functional differential equations are important in scientific and technical professions and they are used to represent a rate of change of time varying phenomenon [5]. When both functional differential equations and Impulses are Involved, Impulsive functional differential system becomes a natural framework of mathematical modelling of varying physical phenomenon [5]. Impulsive functional differential systems are different from ordinary differential systems in the sense that the state undergo abrupt changes at certain moments and the derivation of the state variable depends not only on present state but also on past state. Stability is highly important in all physical application [1,2,3]. A stable equilibrium represents a behaviour usually which cannot be changed. Several stability criteria are obtained by many authors which shows that impulses do contribute to the stabilization of functional differential equations [2,3,5].

Theory of stability in the sense of Lyapunov is now well known and is widely used in concrete problems of real world. The desirable feature is to know the size of region of stability so that we can judge whether or not a given system is sufficiently stable to function properly and may be able to see how to improve its stability. On the other hand the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. So we need a notion of stability that is more suitable than stability and such concept is practical stability [4,6,8,9].

© 2013 Global Journals Inc. (US)

In this paper we establish stability result which provides sufficient conditions to maintain uniform strict practical stability of the trivial solution of a functional differential equation with impulse. The Lyapunov's second method of functions called Lyapunov function is employed in this work.

This paper is organized as: In section 2, we introduce some basic definitions and notations. In section 3, based upon Lyapunov functions and Razumikhin method, some conditions for strict uniform practical stability are obtained. Finally in section 4, some concluding remarks are given.

Notes

II. Preliminaries

Consider the following impulsive functional differential system:

$$\dot{x}(t)=f(t, x_t), \; t\geq t_0, \; t\neq\tau_k \tag{2.1}$$

$$\Delta x(t) = x(\tau_k) - x(\tau_k^{-}) = I_k(x(\tau_k^{-})), \quad t=\tau_k, \; k= 1,2,....$$

Where $f\in C([R^+\times D,R^n]$, D is an open set in $PC([-\tau, 0], R^n)$, where $\tau>0$, $PC([-\tau, 0], R^n) = \{\varphi :([-\tau, 0]\to R^n)$, $\varphi(t)$ is continuous everywhere except a finite number of points $\hat{t}$ at which $\varphi(\hat{t}^{+})$ and $\varphi(\hat{t}^{-})$ exist and $\varphi(\hat{t}^{+}) = \varphi(\hat{t})\}$. $f(t,0) = 0$ for all $t\in R^+, I_k(0) = 0$, for all $k\in Z$, $0 = \tau_0 < \tau_1 < \tau_2 < \ldots\ldots\ldots\ldots \tau_k \to \infty$ for $k \to \infty$ and $x(t^+) = \lim_{s\to t^+} x(s)$, $x(t^-) = \lim_{s\to t^-} x(s)$. For each $t \geq t_0$, $x_t \in D$ is defined by $x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$. For $\varphi\in$ PC $([-\tau, 0], R^n)$, $|\varphi|_1$ is defined by $|\varphi|_1 = sup_{-\tau\leq s\leq 0}\|\varphi\|$, $|\varphi|_2$ is defined by $|\varphi|_2 = inf_{-\tau\leq s\leq 0}\|\varphi\|$. We can see that $x(t) = 0$ is a solution of (2.1) which we call the zero solution or trivial solution.

For a given $\sigma \geq 0$ and $\varphi\in$ PC $([-\tau, 0], R^n)$, the initial value problem of equation (2.1) is

$$\dot{x}(t)=f(t, x_t), \; \sigma\geq t_0, \; t\neq\tau_k \tag{2.2}$$

$$\Delta x(t) = x(\tau_k) - x(\tau_k^{-})= I_k(x(\tau_k^{-})), \; t = \tau_k, \; k = 1,2,... \text{ Where } x_\sigma = \varphi.$$

Throughout this paper we let the following hypothesis hold:

(H_1) For each function $x(s)$: $[\sigma - \tau, \infty] \to R^n$, $\sigma \geq t_0$, which continuous everywhere except a finite number of points τ_k at which $x(\tau_k^{+})$ and $x(\tau_k^{-})$ exist and $x(\tau_k^{+}) = x(\tau_k)$, where $f(t, x_t)$ is continuous for almost all $t\in[\sigma,\infty)$ and at the discontinuous f is the right continuous.

(H_2) $f(t,\varphi)$ is lipschitzian in φ in each compact set in $PC([-\tau, 0], R^n)$.

(H_3) The functions $I_k: R^n\to R^n$, $k = 1,2,3$ ..... are such that for any $H > 0$, there exist a $\rho > 0$ such that if,

$x\in s(\rho) = \{x \in R^n: \|x\| < \rho\}$ implies that $\|x + I_k(x)\| < H$.

Under these hypothesis a unique solution of problem (2.2) exist throughout (σ, φ).

Let

$K= \{a \in[R^+, R^+]$: $a(t)$ is monotone strictly increasing and $a(0) = 0\}$

© 2013 Global Journals Inc. (US)

$$K_1 = \{w \in [R^+, R^+]: w(t) \in K \text{ and } 0 < w(s) < s,\ s > 0\}$$
$$PC_1(\rho) = \{\varphi \in PC([-\tau, 0], R^n): |\varphi|_1 < \rho\}$$
$$PC_2(\theta) = \{\varphi \in PC([-\tau, 0], R^n): |\varphi|_2 > \theta\}$$

We have the following definitions:

Definition: The trivial solution of (2.1) is said to be

(A_1) Strict practical stable, if for any $\sigma \geq t_0$ There exist (λ_1, A_1), $\lambda_1 \leq A_1$ such that $\varphi \in PC_1(\lambda_1)$ implies $\|x(t: \sigma, \varphi)\| < A_1$, $t \geq \sigma$, and for every $0 < \lambda_2 \leq \lambda_1$, there exist $0 < A_2 \leq \lambda_2$ such that $\varphi \in PC_2(\lambda_2)$ implies $\|x(t: \sigma, \varphi)\| > A_2$, $t \geq \sigma$.

(A_2) Strict Practically Uniformly Stable, if (A_1) holds for all $t \in R^+$.

Definition: The function $V: [t_0, \infty] \times S(\rho) \to R^+$ belongs to class v_0 if

I. The function V is continuous on each of the sets $[\tau_{k-1}, \tau_k) \times S(\rho)$ and for all, $t \geq t_0$, $V(t, 0) = 0$.

II. $V(t, x)$ is locally lipschitzian in $x \in S(\rho)$.

III. For each $k = 1, 2, \ldots$ there exist finite limits.

$$\lim_{(t,y)\to(\tau_k^-, x)} V(t, y) = V(\tau_k^-, x)$$

$$\lim_{(t,y)\to(\tau_k^+, x)} V(t, y) = V(\tau_k^+, x)$$

With $V(\tau_k^+, x) = V(\tau_k, x)$ satisfied.

Definition: Let $V \in v_0$,for $(t, x) \in [\tau_{k-1}, \tau_k) \times S(\rho)$, D^+V is defined as:

$$D^+V(t,x(t)) = \lim_{\delta\to 0^+} \sup \frac{1}{\delta}\{V(t+\delta, x(t+\delta)) - V(t, x(t))\}.$$

IV. Main Result

Now we consider the strict practical stability of the Impulsive functional differential equation (2.1) with following results:

Theorem: Assume that

(i) There exist (λ_1, A_1), $0 < \lambda_1 \leq A_1$ and $V_1 \in v_0$, such that $b_1(\|x(t)\|) \leq V_1(t,x(t)) \leq a_1(\|x(t)\|)$, $a_1, b_1 \in K$

(ii) For any solution $x(t)$ of (2.1), $V_1(t + s, x(t + s)) \leq V_1(t,x(t))$ for $s \in [-\tau, 0]$, implies that

$$D^+V_1(t, x(t)) \leq 0.$$

Also for all $K \in Z^+$ and $x \in S(\rho)$

$$V_1(\tau_k, x(\tau_k^-)) + I_k(x(\tau_k^-))) \leq (1 + d_k)V_1(\tau_k, x(\tau_k^-)), \text{ where } d_k \geq 0 \text{ and } \sum_{k=1}^{\infty} d_k < \infty$$

(iii) For any $0 < \lambda_2 \leq \lambda_1$ and $V_2 \in v_0$

$$b_2(\|x(t)\|) \leq V_2(t,x(t)) \leq a_2(\|x(t)\|), \ a_2, b_2 \in K$$

© 2013 Global Journals Inc. (US)

(iv) For any solution x(t) of (2.1), $V_2(t + s), x(t+ s)) \geq V_2(t,(x(t))$ for $s\in [-\tau,0]$, implies

$$D^+V_2(t, x(t)) \geq 0.$$

Also for all $K\in Z^+$ and $x\in S(\rho)$

$$V_2(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \geq (1 - c_k)\, V_2(\tau_k, x(\tau_k^-),$$

Where $0\leq c_k < 1$ and $\sum_{k=1}^{\infty} c_k < \infty$
Then the trivial solution of (2.1) is strict practical uniformly stable.
Proof: Since $\sum_{k=1}^{\infty} d_k < \infty$ and $\sum_{k=1}^{\infty} c_k < \infty$.
It follows that, $\prod_{k=1}^{\infty}(1 + d_k) =$Mand $\prod_{k=1}^{\infty}(1 - c_k) =$N, Obviously $1 \leq M < \infty$, $0 < N \leq 1$
Let $0 < A_1 <$ ρand $\sigma \geq t_0$ be given and $\sigma\in [\tau_k, \tau_{k+1}]$ for some $k \in Z$, Such that $Ma_1(\lambda_1) < b_1A_1$
Then we claim that $\varphi \in PC_1(\lambda_1)$ implies $\|x(t)\| < A_1$, $t \geq \sigma$
Obviously for any $t\in [\tau,\sigma]$, there exist $\theta\in [\tau,0]$, such that

$$V_1(t, x(t)) = V_1(\sigma + \theta, x(\sigma + \theta)) \leq a_1(\| x(\sigma + \theta) \|) =$$

$$a_1\| x_\sigma(\theta) \| = a_1\| \varphi(\theta) \| \leq a_1 (\lambda_1)$$

Then, we claim that

$$V_1(t, x(t)) \leq a_1, \quad \sigma\leq t< \tau_k \tag{3.1}$$

If the inequality (3.1), does not hold, then there exist a $\hat{t}\in(\sigma ,\tau_k)$ such that

$$V_1(\hat{t}, x(\hat{t})) > a_1 (\lambda_1) \geq V_1(\sigma, x(\sigma))$$

which implies that there exist a $\check{t}\in(\sigma, \hat{t}]$, such that

$$D^+V_1(\check{t}, x(\check{t})) >0 \tag{3.2}$$

and

$$V_1(\check{t} + s, x(\check{t} + s)) \leq V_1(\hat{t}, (x(\hat{t})) \text{ , where } s\in [-\tau, 0]$$

by condition (ii), which implies that $D^+V_1(\check{t}, x(\check{t}))\leq 0$. This contradicts inequality (3.2)
So inequality (3.1) holds.
From condition (ii), we have

$$V_1(\tau_k, x(\tau_k)) = V_1(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1 + d_k)\, V_1(\tau_k, x(\tau_k^-)) \leq (1 + dk)\, a_1(\lambda 1)$$

Next, we claim that

$$V_1(t, x(t)) \leq (1 + dk) a_1(\lambda 1),\ \tau_k \leq t \leq \tau_{k+1} \tag{3.3}$$

© 2013 Global Journals Inc. (US)

If inequality (3.3) does not hold then, there exist $\hat{s}\in(\tau_k,\ \tau_{k+1})$, such that

$$V_1(\hat{s}, x(\hat{s})) \geq (1+d_k)\, a_1(\lambda_1) \geq V_1(\tau_k, x(\tau_k))$$

which implies that there exist an $\check{s}\in(\tau_k,\ \tau_{k+1})$, such that

$$D^+V_1(\check{s}, x(\check{s})) > 0 \tag{3.4}$$

Notes

And

$$V_1(\check{s}+s), x(\check{s}+s)) \leq V_1(\hat{s}, (x(\hat{s})), \text{where } s\in[-\tau, 0]$$

by condition (ii), which implies that $D^+V_1(\hat{s}, x(\hat{s}))\leq 0$, This Contradicts inequality (3.4) so inequality (3.3) holds.
And from condition (ii), we have

$$V_1(\tau_{k+1}, x(\tau_{k+1}) = V_1(\tau_{k+1}, x(\tau_{k+1}^-) + I_k(x(\tau_{k+1}^-))) \leq (1+$$

$$d_k)\, V_1(\tau_{k+1}, x(\tau_{k+1}^-)) \leq (1+d_k)\,(1+d_k)\, a_1(\lambda 1).$$

By a simple induction, we can easily prove that in general form for, m = 0,1,2,3 .....

$$V_1(t, x(t)) \leq (1+d_{k+m}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \leq (1+d_k a_1(\lambda 1), \text{Where } \tau_{k+m}\leq t\leq \tau_{k+m+1}$$

Which together with inequality (3.1) and condition (i), we have

$$b_1(\|x(t)\|) \leq V_1(t, x(t)) \leq M\, a_1((\lambda_1) < b_1(A_1),\ t\geq\sigma$$

Thus we have

$$\|x(t)\| < A_1,\ t\geq\sigma$$

Now, let $0<\lambda_2\leq\lambda_1$ and choose $0<A_2<\lambda_2$, such that $a_2(A_2) < Nb_2(\lambda_2)$.
Next, we claim that $\varphi\in PC_2(\lambda_2)$ implies $\|x(t)\| > A_2,\ t\geq\sigma$
If it holds,then $\varphi\in PC_1(\lambda_1)\cap PC_2(\lambda_2)$ implies

$$A_2<\|x(t)\| < A_1,\ t\geq\sigma$$

Obviously for any, $t\in[\sigma-\tau, \sigma]$, there exist a $\theta\in[-\tau, 0]$, such that

$$V_2(t, x(t))=V_2(\sigma+\theta, x(\sigma+\theta))\geq b_2(\|x(\sigma+\theta)\|)=b_2\|x_\sigma(\theta)\|$$
$$= b_2\|\varphi(\theta)\| \geq b_2(\lambda_2)$$

Then, we claim that

$$V_2(t, x(t)) \geq b_2(\lambda_2),\ \sigma\leq t<\tau_k \tag{3.5}$$

© 2013 Global Journals Inc. (US)

If inequality (3.5) does not hold, then there exist $\bar{t}\in(\sigma, \tau_k)$, such that

$$V_2(\bar{t}, x(\bar{t})) < b_2(\lambda_2) \leq V_2(\sigma, x(\sigma))$$

Which implies that there exist a $t_1\in(\sigma, \bar{t})$, such that

$$D^+V_2(t_1, x(t_1)) < 0 \tag{3.6}$$

And

$$V_2(t_1 + s, x(t_1 + s) \geq V_2(t_1, x(t_1)), \quad s\in [-\tau, 0]$$

By condition (iv), we have, $D^+V_2(t_1, x(t_1)) \geq 0$, This contradicts inequality (3.6) So, inequality (3.5) holds.
From condition (iv), we have

$$V_2(\tau_k, x(\tau_k) = V_2(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \geq (1-c_k)\, V_2(\tau_k, x(\tau_k^-)) \geq (1-c_k)b_2(\lambda_2)$$

Next, we claim that

$$V_2(t, x(t)) \geq (1-c_k)b_2(\lambda_2), \quad \tau_k \leq t < \tau_{k+1} \tag{3.7}$$

If the inequality (3.7) does not hold then there exist an $\bar{r}\in(\tau_k, \tau_{k+1})$, such that

$$V_2(\bar{r}, x(\bar{r})) < (1- c_k)\, b_2(\lambda_2) \leq V_2(\tau_k, x(\tau_k))$$

Which implies that there exist an $\check{r}\in (\tau_k, \bar{r})$, such that

$$D^+V_2(\check{r}, x(\check{r})) < 0 \tag{3.8}$$

And

$$V_2(\check{r}+ s, x(\check{r}+ s) \geq V_2(\check{r}, x(\check{r})), s\in [-\tau, 0]$$

By condition (iv), we have, $D^+V_2(\check{r}, x(\check{r})) \geq 0$. Which contradicts inequality (3.8) So, inequality (3.7) holds
And from condition (iv), we have

$$V_2(\tau_{kk+1}, x(\tau_{kk+1})) = V_2(\tau_{k+1}, x(\tau_{k+1}^-) + I_k(x(\tau_{k+1}^-))) \geq (1- c_k)V_2(\tau_{k+1}, x(\tau_{k+1}^-))$$
$$\geq (1-c_{k+1})(1-c_k)\, b_2(\lambda_2).$$

And by a simple induction we can prove that, in general, for,m = 0,1,2,3….

$$V_2(t,x(t)) \geq (1-c_{k+m})\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\leq (1-c_k)b_2(\lambda_2), \quad \tau_{k+m} \leq t \leq \tau_{k+m+1}$$

Which together with inequality (3.5) and condition (iii), we have

$$a_2(\|x(t)\|) \leq V_2(t, x(t)) \leq Nb_2(\lambda_2) > a_2(A_2), \quad t \geq \sigma$$

Notes

© 2013 Global Journals Inc. (US)

Thus we have, $\|x(t)\| > A_2$, $t \geq \sigma$
Thus the zero solution of (2.1) is strict practical uniformly stable.
The proof of theorem is complete.

IV. Conclusion

Notes

In this paper, we investigated the strict practical stability criteria in the form of theorem for impulsive functional differential equations, which is more useful as compared to practical stability. It gives rate of decay of the solution, so it is finer concept which can give us more precise information. In future we can modify this theorem to get less restricted conditions to verify strict practical stability.

References Références Referencias

1. Dilbaj Singh, S.K. Srivastava, Stability criteria of impulsive differential equations with comparison results, Advances in Differential Equations and Control Processes, Vol. 10, pp. 171-182, 2012.
2. Dilbaj Singh, S.K. Srivastava, Strict stability criteria for impulsive functional differential equations, proceedings of the world congress on Engineering 2012 London, U.K., Vol I, pp. 169 – 171.
3. Dilbaj Singh, S.K. Srivastava, Uniform Strict Practical Stability Criteria for Impulsive Functional Differential Equations, Global Journal of Science Frontier Reserch (F), Vol. 13, pp. 1-8, 2013.
4. J. LaSalle and S. Lefschetz, Stability by Liapunov's Direct Method, with Applications, Academic Press, New York, NY, USA, 1961.
5. Kaien Liu, Guowei Yang, Strict Stability Criteria for Impulsive Functional Differential Systems, Journal of Inequalities and Applications, Article ID 243863, 8 pages, 2008
6. S.R. Bernfeld and V.Lakshmikantam, Practical Stability and Lyapunov Functions, University of Texas at Arlington, Texas, 1979.
7. Lakshmikantam, D. D. Bainov and P.S. Simonov, Theory of impulsive Differential equations, World Scientific, London, 1989.
8. Lakshmikantam, Y Zhang, Strict Practical Stability of Delay Differential Equations, Applied Mathematics and Computation, Vol. 122 no. 3, pp. 341–351, 2001.
9. Lakshmikantam, S. Leela, A. A. Martynyuk, Practical Stability of Non Linear Systems, World Scientific, 1990.

© 2013 Global Journals Inc. (US)

This page is intentionally left blank

© 2013 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 4 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Computation of a Summation Formula Clung To Recurrence Relation

By Salahuddin, M. P. Chaudhary & Upendra Kumar Pandit
P.D.M College of Engineering, India

Abstract - In this paper we have established a summation formula clung to contiguous relation and recurrence relation.

Keywords : *gaussian hypergeometric function, contiguous function, recurrence relation of gamma function, bailey summation theorem and legendre duplication formula.*

GJSFR-F Classification : *2010 MSC No: 33C60 , 33C70 , 33D15 , 33D50 , 33D60.*



Strictly as per the compliance and regulations of :

© 2013. Salahuddin, M. P. Chaudhary & Upendra Kumar Pandit. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Computation of a Summation Formula Clung To Recurrence Relation

Salahuddin [α], M. P. Chaudhary [σ] & Upendra Kumar Pandit [ρ]

Abstract - In this paper we have established a summation formula clung to contiguous relation and recurrence relation.

Keywords : *gaussian hypergeometric function, contiguous function, recurrence relation of gamma function, bailey summation theorem and legendre duplication formula.*

I. Basic Introduction

The Pochhammer symbol or **generalized factorial function** or **shifted factorial** or **falling factorial** is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{bmatrix} 1 & ; & n=0 \\ a(a+1)(a+2).....(a+n-1); & & n=1,2,3.... \end{bmatrix} \tag{1}$$

where $a \neq 0, -1, -2, ...$ and the notation Γ stands for Gamma function.

$$(b)_{-n} = \frac{\Gamma(b-n)}{\Gamma(b)} = \frac{(-1)^n}{(1-b)_n}\ ; \tag{2}$$

where $b \neq .....-3, -2, -1, 0, 1, 2, 3.....$ and $n = 1, 2, 3, ...$

If $m = 1, 2, , 3, 4 \ldots$ and $n = 0, 1, 2, 3, \ldots,$ then

$$(b)_{mn} = m^{mn}\left(\frac{b}{m}\right)_n\left(\frac{b+1}{m}\right)_n \cdots \left(\frac{b+m-2}{m}\right)_n\left(\frac{b+m-1}{m}\right)_n \tag{3}$$

Generalized Gaussian hypergeometric function of one variable is followed as

$${}_AF_B\left[\begin{array}{ccc} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

Author α : *P.D.M College of Engineering, Bahadurgarh, Haryana, India. E-mails : sludn@yahoo.com, vsludn@gmail.com*
Author σ : *International Scientific Research and Welfare Organization, New Delhi, India. E-mail : mpchaudhary_2000@yahoo.com*
Author ρ : *Former Research Scholar Department of Mathematics University of Delhi, Delhi-110007, India.*

© 2013 Global Journals Inc. (US)

or

$$ {}_AF_B\left[\begin{array}{ccc}(a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{ccc}(a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{4} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation[E. D. p.51(10), Andrews p.363(9.16)] is defined as follows

$$ (a-b)\ {}_2F_1\left[\begin{array}{cc}a,\ b\ ; & \\ c\ ; & z\end{array}\right] = a\ {}_2F_1\left[\begin{array}{cc}a+1,\ b\ ; & \\ c\ ; & z\end{array}\right] - b\ {}_2F_1\left[\begin{array}{cc}a,\ b+1\ ; & \\ c\ ; & z\end{array}\right] \tag{5} $$

Recurrence relation of gamma function is defined as follows

$$ \Gamma(z+1) = z\ \Gamma(z) \tag{6} $$

Legendre duplication formula[Bells & Wong p.26(2.3.1)] is defined as follows

$$ \sqrt{\pi}\ \Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right) \tag{7} $$

$$ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{8} $$

$$ = \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{9} $$

Bailey summation theorem [Prudnikov, p.491(7.3.7.8)]is defined as follows

$$ {}_2F_1\left[\begin{array}{cc}a, 1-a & ;\ \frac{1}{2}\\ c & ;\end{array}\right] = \frac{\Gamma(\frac{c}{2})\ \Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-1}\ \Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} \tag{10} $$

II. Main Result of Summation Formula

$$ {}_2F_1\left[\begin{array}{cc}a\ , & -a-51\ ;\ \frac{1}{2}\\ c & ;\end{array}\right] $$

$$ = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c+51}}\left[\frac{1}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+51}{2})}\left\{+39215629363216388314063441609359360000000\right.\right. $$

$$ -74021568132398532674161337237569536000000a $$

$$ \left.\left.+46295981506391174974374821359954329600000a^2\right.\right. $$

© 2013 Global Journals Inc. (US)

Notes

$$-1320609504346225933240581617829869568000a^3$$
$$+1811930446458959824391495038134193152 00a^4$$
$$-9311856861379155575957484232594867200a^5$$
$$-319305701534575139018589169352966400a^6+432906651408504871126269609925056 00a^7$$
$$+1459526532650544719250939889840 00a^8-8771917410859920237540675575520 0a^9$$
$$-4575088744242843615633002444 00a^{10}+1005808770049491450042585156 00a^{11}$$
$$+1508122770646685337604700500a^{12}-5142016025722708393563120 0a^{13}$$
$$-1509631951413201653983400a^{14}+4302016118331891960 0a^{15}+4606163067186269875 00a^{16}$$
$$+585472610368039680 0a^{17}-662319052263440 0a^{18}-74418821316840 0a^{19}$$
$$-6243725988500a^{20}-9352543200a^{21}+138868600a^{22}+795600a^{23}+1300a^{24}$$
$$+10161656819477326305691970258274877440000c$$
$$-1446367255959290974968755905793910374400 0ac$$
$$+7213434913697898840014034554316432998400a^2c$$
$$-1652471481542170230203835091179109662720a^3c$$
$$+1746582112809621535161394811697886126 08a^4c$$
$$-4969371876573656445853216824494834688a^5c$$
$$-4756041029223851855577462614864386 56a^6c$$
$$+2690942552956382033480148191746022 4a^7c+7668477053102870426726336078793 60a^8c$$
$$-4686074174974908708180859439020 8a^9c-1208658606098948851658543849776a^{10}c$$
$$+3578734555798042658974146062 4a^{11}c+1310981764885248713045228020a^{12}c$$
$$-4121598466147286396065248a^{13}c-6208563979723478843193 36a^{14}c$$
$$-6671743900559708043216a^{15}c+6558127104990267950 0a^{16}c+1896520433242735872a^{17}c$$
$$+1153021136901462 4a^{18}c-53516453726736a^{19}c-1025323839540a^{20}c-4458181728a^{21}c$$
$$-1725256a^{22}c+31824a^{23}c+52a^{24}c+1076616397008140759739536209336074240 0000c^2$$
$$-1222065339153341113055562146980665753600 0ac^2$$
$$+4964847384594138558862158629137140940800a^2c^2$$
$$-9173553475857202963004609719592828928 00a^3c^2$$
$$+7284818328107943034599645981002711040 0a^4c^2$$
$$-4613333979910927893660172636314624 00a^5c^2$$
$$-2344905737318877803070406580027392 00a^6c^2$$
$$+5607883301081174731722705232665600a^7c^2+4045327426057874772699839668992 00a^8c^2$$
$$-7562908655286410131260619449600a^9c^2-4823146433791448180907007104 00a^{10}c^2$$

© 2013 Global Journals Inc. (US)

$$+16497683403078933715422672 00a^{11}c^2+3158754417227862826724624 00a^{12}c^2$$
$$+3057010359220073337681600a^{13}c^2-7134279093442306908960 0a^{14}c^2$$
$$-1619095636189622448000a^{15}c^2-4986468790418544000a^{16}c^2+157146003545731200a^{17}c^2$$
$$+1822144519395200a^{18}c^2+4701184488000a^{19}c^2-34522488000a^{20}c^2-245044800a^{21}c^2$$
$$-436800a^{22}c^2+6529011725984880559625498384381509632000c^3$$
$$-607527203112514220836692567188777730048 0ac^3$$
$$+2032304372461996734448911257402664812544a^2c^3$$
$$-3020678620572310923886949723004789719 04a^3c^3$$
$$+1718072055568936330350721831353316147 2a^4c^3$$
$$+3045162780146217013937529575302471 68a^5c^3$$
$$-5928200045143129512117233626136985 6a^6c^3$$
$$+1515324587191551134306980047022 08a^7c^3+935587100140229553968545979586 56a^8c^3$$
$$+1127331108146289426687197406 72a^9c^3-8250765809298635820087200947 2a^{10}c^3$$
$$-8798814973420162230349031 04a^{11}c^3+3139362688575582221563283 2a^{12}c^3$$
$$+6860746929250194791690 88a^{13}c^3-6672368856121662545 28a^{14}c^3$$
$$-1399368420511416326 40a^{15}c^3-1285493802358113920a^{16}c^3+1721586727276416a^{17}c^3$$
$$+89439005591936a^{18}c^3+471090459840a^{19}c^3+340500160a^{20}c^3-3267264a^{21}c^3$$
$$-5824a^{22}c^3+2612866415715646813807239647295700992000c^4$$
$$-2023055775046023696675566334528454656000ac^4$$
$$+5597606675261636184600124256007290880 00a^2c^4$$
$$-6632956167167984828404965598970265600 0a^3c^4$$
$$+2455512005609343344145848292777984000a^4c^4$$
$$+1237535851513536362173688224473600 00a^5c^4$$
$$-8848826342631603836821350432640000a^6c^4$$
$$-1424035278140995663311034379520 00a^7c^4+1180714737565094411440833323520 00a^8c^4$$
$$+1958535340174562865381338400 00a^9c^4-7044583957774727005157880000a^{10}c^4$$
$$-1752821145922705335116640 00a^{11}c^4+9269200774452512894000 00a^{12}c^4$$
$$+6064650048474121104000 0a^{13}c^4+4374362730688922400 00a^{14}c^4$$
$$-4560723705938400000a^{15}c^4-81537114234576000a^{16}c^4-305264559600000a^{17}c^4$$
$$+1313672360000a^{18}c^4+12252240000a^{19}c^4+24024000a^{20}c^4$$
$$+7464487195938416781946417894200296079 36c^5$$

© 2013 Global Journals Inc. (US)

Notes

$$-48484234224919834080743341409765739724 8ac^5$$
$$+1109848016894475735519120702247379927 04a^2c^5$$
$$-1031723368727946254444011135645650124 8a^3c^5$$
$$+1950939528571816894404651335140638 72a^4c^5$$
$$+2364932148757800197825670685221888 0a^5c^5$$
$$-7967507845695813932055290677811 20a^6c^5 - 3126389268620074980188047934361 6a^7c^5$$
$$+8231557848426858833582541788 16a^8c^5 + 3005939207175143872612779072 0a^9c^5$$
$$-2326353187477596442959830 40a^{10}c^5 - 1522791942814168592697331 2a^{11}c^5$$
$$-8185281180894762424480 0a^{12}c^5 + 2562162110375475928320a^{13}c^5$$
$$+364163020462906579 20a^{14}c^5 + 3736483199729280 0a^{15}c^5 - 2170921221076608a^{16}c^5$$
$$-14204266876800a^{17}c^5 - 15116221120a^{18}c^5 + 98017920a^{19}c^5 + 192192a^{20}c^5$$
$$+1599367617240372884217170181803212 80000c^6$$
$$-8749212737111082488373415636390379 5200ac^6$$
$$+1651600970029225647429459878551224 3200a^2c^6$$
$$-1173671805520416747746112210311577 600a^3c^6$$
$$+1775391309788925186503726533836800a^4c^6$$
$$+2862264047025912796435298741452800a^5c^6 - 3631355003658491460239375728 6400a^6c^6$$
$$-3479560389180212846538284236800a^7c^6 + 2098728159618006248183255040 0a^8c^6$$
$$+2413106726268002114404454400a^9c^6 + 1081923974698429393090560 0a^{10}c^6$$
$$-7153662492242325835776 00a^{11}c^6 - 9335180202255799091200a^{12}c^6$$
$$+347359470239029248 00a^{13}c^6 + 131913011269079040 0a^{14}c^6 + 6809187280896000a^{15}c^6$$
$$-17303430144000a^{16}c^6 - 235243008000a^{17}c^6 - 512512000a^{18}c^6$$
$$+2657638401472618758221576819347292 1600c^7$$
$$-1225677240375840964909193396983588 4544ac^7$$
$$+1895875818441715846981287505091887 104a^2c^7$$
$$-9876339246552438748626259617723187 2a^3c^7$$
$$-1653746068979970114211115631509504a^4c^7$$
$$+2383166266278887193826963246940 16a^5c^7 + 5908658433080967378318473297 92a^6c^7$$
$$-2436036550682117691745516584 96a^7c^7 - 1564667020221574346476019712a^8c^7$$
$$+1158420607582797015031111 68a^9c^7 + 1563137249722283122348032a^{10}c^7$$
$$-1629693584699384763187 2a^{11}c^7 - 4194859863705361776 64a^{12}c^7$$

© 2013 Global Journals Inc. (US)

$$-1367035409521926144a^{13}c^7 + 22577009199833088a^{14}c^7 + 191257494405120a^{15}c^7$$
$$+274524856320a^{16}c^7 - 1344245760a^{17}c^7 - 2928640a^{18}c^7$$
$$+3507093480716382082057454435172352000c^8$$
$$-136190227379755457345546565797806080 0ac^8$$
$$+17094042739830324058091479337533440 0a^2c^8$$
$$-6067926395459215226369951249203200a^3c^8$$
$$-2382423862432601243329431773184 00a^4c^8 + 139581712425222863160597872640 00a^5c^8$$
$$+2239138355792106881873080320 00a^6c^8 - 1114032374517839103003156480 0a^7c^8$$
$$-1926068902089500539058688 00a^8c^8 + 3143176542139018125312000a^9c^8$$
$$+8056355222033739878400 0a^{10}c^8 + 4549225380767907840 0a^{11}c^8$$
$$-1004782518133954560 0a^{12}c^8 - 71472471662592000a^{13}c^8 + 91162266624000a^{14}c^8$$
$$+2285217792000a^{15}c^8 + 5601024000a^{16}c^8 + 373936129332143623419537730431877120c^9$$
$$-12188199359647766088654846316825 8048ac^9$$
$$+12244622122591681114343641738379264a^2c^9$$
$$-2546410607640370619633091721297 92a^3c^9$$
$$-1966367023335741556657324861030 4a^4c^9 + 56008479467022568667676278784 0a^5c^9$$
$$+1809574612500475422876065792 0a^6c^9 - 3106058308727262631265402 88a^7c^9$$
$$-1020747663135337611727052 8a^8c^9 + 2322307780121925353472 0a^9c^9$$
$$+2355636749511444439040a^{10}c^9 + 1396666394732799590 4a^{11}c^9$$
$$-117516116735553536a^{12}c^9 - 1384090399211520a^{13}c^9 - 2582047170560a^{14}c^9$$
$$+10156523520a^{15}c^9 + 24893440a^{16}c^9 + 3262443149751044008415240126464 0000c^{10}$$
$$-8881346851313890235603771562393600ac^{10}$$
$$+7007702800107856248737834729472 00a^2c^{10} - 5276222909482495066628397465600a^3c^{10}$$
$$-1131639476780264378939159347200a^4c^{10} + 1328631196452522577931796480 0a^5c^{10}$$
$$+8813451963415592102658048 00a^6c^{10} - 3076458866038702861516800a^7c^{10}$$
$$-3313785127571059654656 00a^8c^{10} - 1597994314386127257600a^9c^{10}$$
$$+4000308020175667200 0a^{10}c^{10} + 408320886900326400a^{11}c^{10} - 79710786355200a^{12}c^{10}$$
$$-12797219635200a^{13}c^{10} - 35846553600a^{14}c^{10} + 2350559102017734118725983508889600c^{11}$$
$$-5308492413293677107107666633687 04ac^{11} + 3204082190205603045447691979980 8a^2c^{11}$$
$$+1548691193163573591934406492 16a^3c^{11} - 4794024376510400642713701580 8a^4c^{11}$$
$$+1650831059349741270151987 2a^5c^{11} + 2918564816248263138685747 2a^6c^{11}$$
$$+1394154197632331673108 48a^7c^{11} - 6872246298970496630784a^8c^{11}$$

Notes

© 2013 Global Journals Inc. (US)

Notes

$$-68461810010010353664a^9c^{11}+31312605180100608 0a^{10}c^{11}+5905660917252096a^{11}c^{11}$$
$$+14157389955072a^{12}c^{11}-46535344128a^{13}c^{11}-130351104a^{14}c^{11}$$
$$+14075367280203107352188236595200 0c^{12}-26142919734994299344681277849600ac^{12}$$
$$+11618463325817878010549239808 00a^2c^{12}+19816175844567736562968166400a^3c^{12}$$
$$-15146808165260705389281280 00a^4c^{12}-13007910157465601064960000a^5c^{12}$$
$$+66749147310775606476800 0a^6c^{12}+74520790250798481408 00a^7c^{12}$$
$$-83890949279322931200a^8c^{12}-1363318831300608000a^9c^{12}-1212846415872000a^{10}c^{12}$$
$$+44208576921600a^{11}c^{12}+144472473600a^{12}c^{12}+70332555573175654177310349721 60c^{13}$$
$$-10627144706902857190454518087 68ac^{13}+32779414532156325397071396864a^2c^{13}$$
$$+10022692374039650387094405 12a^3c^{13}-35211188622667072624394240a^4c^{13}$$
$$-59359172675018752327680 0a^5c^{13}+10030581135956355645440a^6c^{13}$$
$$+18242041681740260966 4a^7c^{13}-343475889878532096a^8c^{13}-15530359717232640a^9c^{13}$$
$$-48184903925760a^{10}c^{13}+136026390528a^{11}c^{13}+444530688a^{12}c^{13}$$
$$+29381483038307735107010560000 0c^{14}-35630582416423246085750784000ac^{14}$$
$$+68794386734409115710259200 0a^2c^{14}+33783509590480862576640000a^3c^{14}$$
$$-57100745582772879360000 0a^4c^{14}-15474236362054041600000a^5c^{14}$$
$$+79181320218083328000a^6c^{14}+272732913008640000 0a^7c^{14}+5936389816320000a^8c^{14}$$
$$-97161707520000a^9c^{14}-381026304000a^{10}c^{14}+10259229421653434996201881600c^{15}$$
$$-98183872198774294537725542 4ac^{15}+94378071966287098145669 12a^2c^{15}$$
$$+83110564558287723823104 0a^3c^{15}-53750719883571101696 00a^4c^{15}$$
$$-26982524008999157760 0a^5c^{15}-203567309276577792a^6c^{15}+25409729750630400a^7c^{15}$$
$$+1047365104435 20a^8c^{15}-259097886720a^9c^{15}-1016070144a^{10}c^{15}$$
$$+29869828988705671689011200 0c^{16}-22087743903606947866214400ac^{16}$$
$$+34749165208324354867200a^2c^{16}+15256504595570570035200a^3c^{16}$$
$$+2601854431140249600a^4c^{16}-32104948692418560 00a^5c^{16}-12890056359936000a^6c^{16}$$
$$+136026390528000a^7c^{16}+666796032000a^8c^{16}+72163098146267657037414 40c^{17}$$
$$-40133249027490628553932 8ac^{17}-19946948191516738191 36a^2c^{17}$$
$$+20784195017805555302 4a^3c^{17}+943705218464612352a^4c^{17}-25157520819486720a^5c^{17}$$
$$-145384545976320a^6c^{17}+320062095360a^7c^{17}+1568931840a^8c^{17}$$
$$+14353770849318731776000 0c^{18}-57954785787541192704 00ac^{18}$$
$$-58868619943503462400a^2c^{18}+20460253945528320 00a^3c^{18}+150714381762560 00a^4c^{18}$$
$$-117356101632000a^5c^{18}-767033344000a^6c^{18}+23234440656339337216 00c^{19}$$

© 2013 Global Journals Inc. (US)

$$-649251921274223984 64ac^{19} - 916234973881565184a^2c^{19} + 13778265172869120a^3c^{19}$$
$$+124580748328960a^4c^{19} - 247065477120a^5c^{19} - 1614807040a^6c^{19}$$
$$+300929768772075520 00c^{20} - 5436453465096192 00ac^{20} - 92106736533504 00a^2c^{20}$$
$$+568250597376 00a^3c^{20} + 557108428800a^4c^{20} + 304216262815252480c^{21}$$
$$-32002791262126 08ac^{21} - 59990496772096a^2c^{21} + 108238209024a^3c^{21} + 1061158912a^4c^{21}$$
$$+23113970483200 00c^{22} - 11807804620800ac^{22} - 231525580800a^2c^{22} + 12408428953600c^{23}$$
$$-20535312384ac^{23} - 402653184a^2c^{23} + 41943040000c^{24} + 67108864c^{25}\Big\}+$$
$$+\frac{1}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+52}{2})}\Big\{-17464069942802715386614602906796032000000a$$
$$+20686287355803117929950207741475389440000a^2$$
$$-89378277350952189664003768283812331520 00a^3$$
$$+17781875231429143853642282718796858368 00a^4$$
$$-15295834999455270891171432131298479616 0a^5$$
$$+84580791261276992924025160195502361 6a^6$$
$$+62507388470410138827268293141233145 6a^7 - 16351780395261526276780909546342272a^8$$
$$-13877159130594298593242575586078 88a^9 + 29422980952667711908287591614240a^{10}$$
$$+21972722714969371470994180557 96a^{11} - 91546213857026072673125431 12a^{12}$$
$$-20090558959466593944440490 63a^{13} - 22907465907142301442089065a^{14}$$
$$+67806894845325092251212 6a^{15} + 18820430104162303836518a^{16}$$
$$+69913513840255423767a^{17} - 31060328718393810 55a^{18} - 48230575964749464a^{19}$$
$$-173954956930612a^{20} + 2022903890007a^{21} + 24154252265a^{22} + 87862086a^{23} + 2678a^{24}$$
$$-663a^{25} - a^{26} + 17464069942802730897824646237782016000000c$$
$$-54007367613351708706039996361205350400000ac$$
$$+41767493571718136116945131402825154560000a^2c$$
$$-13573255377452899961200986142034724864000a^3c$$
$$+20546717179456108606506652537709976576 00a^4c$$
$$-11835766148166118034864208368537495040 0a^5c$$
$$-31878105386048508220317844756794432 00a^6c$$
$$+53767945589634820729486018826559360 0a^7c+21561567877933197657113200287840 0a^8c$$
$$-11064344422506559947750729879696 00a^9c - 39005211881643018383038309708 00a^{10}c$$
$$+12993527595769132331897223012 00a^{11}c + 18242603793449626590118783300a^{12}c$$
$$-68134188058418541366674640 0a^{13}c - 19313434798930849743184200a^{14}c$$

Notes

© 2013 Global Journals Inc. (US)

Notes

$$+7547544375035398047600a^{15}c+60091256929518250007 00a^{16}c$$
$$+7542862590857742720 0a^{17}c-939894569306264 00a^{18}c-9694744240381200a^{19}c$$
$$-81018694656900a^{20}c-120521200800a^{21}c+1807184600a^{22}c+10342800a^{23}c$$
$$+16900a^{24}c+3332108025754864996621860032093356032 0000c^2$$
$$-61112319783077679498758852113706385408000ac^2$$
$$+3523916859244368212159736492025518981120 0a^2c^2$$
$$-89275948578721178852368549819688301772 80a^3c^2$$
$$+10266642141771117306133516450695729131 52a^4c^2$$
$$-33937321295214246160668222485847019008a^5c^2$$
$$-27182086039814644390155980616825488 64a^6c^2$$
$$+17346834336799238866980948672915187 2a^7c^2$$
$$+43889630745475193234758592147775 68a^8c^2-30487945465896598352737520311939 2a^9c^2$$
$$-73236132900824524921962747794 16a^{10}c^2+23810712057746792854377072602 4a^{11}c^2$$
$$+83161095798756193264712156 66a^{12}c^2-31210924941366650068334928a^{13}c^2$$
$$-40306787508500310428236 84a^{14}c^2-42456192021451570839048a^{15}c^2$$
$$+43460102747458570001 4a^{16}c^2+12315977161258028544a^{17}c^2+74364791296267472a^{18}c^2$$
$$-350917550607624a^{19}c^2-6666815293138a^{20}c^2-28956944016a^{21}c^2-11176308a^{22}c^2$$
$$+206856a^{23}c^2+338a^{24}c^2+28282653946454862828385645890424012800000c^3$$
$$-37847109234486369007419794151897563136000ac^3$$
$$+17134579371035444556951735602324491468800a^2c^3$$
$$-34348739862985467569978326427685814272 00a^3c^3$$
$$+29448904720092988909106397581109411840 0a^4c^3$$
$$-29872982992921124590857319491575808 00a^5c^3$$
$$-94057616732278352210820058186275840 0a^6c^3$$
$$+25431434993231939895777252470553600a^7c^3$$
$$+16513171743660878272187645669888 00a^8c^3-34397369483011709151574125734400a^9c^3$$
$$-20286632404921551187011159264 00a^{10}c^3+86464008832314800874790512 00a^{11}c^3$$
$$+13604789058704108240147536 00a^{12}c^3+12722508323197889416200000a^{13}c^3$$
$$-31290321118970293791840 0a^{14}c^3-69763486819972751520 00a^{15}c^3$$
$$-20901902309034032000a^{16}c^3+68360450584345920 0a^{17}c^3+7884558877395200a^{18}c^3$$
$$+20265613368000a^{19}c^3-149805656000a^{20}c^3-1061860800a^{21}c^3-1892800a^{22}c^3$$

© 2013 Global Journals Inc. (US)

$$+1440300849635277480118978884647308492800 0c^4$$
$$-150960224819365534135896082504816892313 60ac^4$$
$$+549850181306804818472775808469427211468 8a^2c^4$$
$$-875089957529533052576034038524658614272a^3c^4$$
$$+536437469536812592432516366462773411 84a^4c^4$$
$$+745901964781113133197915990274824192a^5c^4$$
$$-184010830303513208010308817367427584a^6c^4$$
$$+818367161622811724817415382305536a^7c^4$$
$$+295069311238367211284859400869888a^8c^4 + 468788101453269526684999426 56a^9c^4$$
$$-265547644209905563443206559264a^{10}c^4 - 269551643666341209691120948 8a^{11}c^4$$
$$+102893805882101328651547536a^{12}c^4 + 220191398659140564856200 0a^{13}c^4$$
$$-256183031419614817918 4a^{14}c^4 - 455189373853931951520a^{15}c^4$$
$$-41543219817731403 20a^{16}c^4 + 5748965064434592a^{17}c^4 + 290840371773952a^{18}c^4$$
$$+1529982133680a^{19}c^4 + 1104543440a^{20}c^4 - 10618608a^{21}c^4 - 18928a^{22}c^4$$
$$+498732460082223946428109653534069227520 0c^5$$
$$-42313376901729947265139758466165899264 00ac^5$$
$$+125463585064926798763757239214389985280 0a^2c^5$$
$$-157710034258206885829138153197028147200a^3c^5$$
$$+634191211917035513660628923286159360 0a^4c^5$$
$$+285724555386011051713443871587993600a^5c^5$$
$$-225165576987905063844560590803712 00a^6c^5$$
$$-325958511808137888230326702617600a^7c^5 + 304096559464959260798029765248 00a^8c^5$$
$$+478117034612337247952376988800a^9c^5 - 184497581315116051311406896 00a^{10}c^5$$
$$-446430201516143293713264000a^{11}c^5 + 253049621452272952667360 0a^{12}c^5$$
$$+157223273868315560505600a^{13}c^5 + 112019339012164034880 0a^{14}c^5$$
$$-119462069789270208 00a^{15}c^5 - 211771076704387200a^{16}c^5 - 790629695856000a^{17}c^5$$
$$+3422210792000a^{18}c^5 + 31855824000a^{19}c^5 + 62462400a^{20}c^5$$
$$+125855750279546574894574649931157261516 8c^6$$
$$-879365308715679527902577076638293426176ac^6$$
$$+213226342068718901576910141848419172352a^2c^6$$
$$-208887417390440831904152609630035476 48a^3c^6$$
$$+445350239425183456899743815003930624a^4c^6$$

© 2013 Global Journals Inc. (US)

Notes

$$+476970442495828897042416866157066 24a^5c^6$$
$$-173299232354517895275640971397580 8a^6c^6$$
$$-640835516165831680141567823907 84a^7c^6+1805944697836994302445981656832a^8c^6$$
$$+633748043179436102539476865 92a^9c^6-52745112795207530584067126 4a^{10}c^6$$
$$-32744884604360817430301760a^{11}c^6-17100054725544699526217 6a^{12}c^6$$
$$+55719083451714810167 04a^{13}c^6+78559241033904908992a^{14}c^6+78057691270086528a^{15}c^6$$
$$-4707823101229248a^{16}c^6-30755523839040a^{17}c^6-32707394720a^{18}c^6+212372160a^{19}c^6$$
$$+416416a^{20}c^6+241830426344143445812620039165050880000c^7$$
$$-140430911900394092446841334069303705600ac^7$$
$$+278342515450299241729726387709804544 00a^2c^7$$
$$-207550771145550821978660619233525760 0a^3c^7$$
$$+722454153797750835287655407288320 0a^4c^7$$
$$+507614192801360525343360477265920 0a^5c^7$$
$$-711619054898602252120168546304 00a^6c^7-626952044191716405270660464640 0a^7c^7$$
$$+422548014354573964422279168 00a^8c^7+44267148074098957188639744 00a^9c^7$$
$$+187037225255938857845760 00a^{10}c^7-13292516499724186103808 00a^{11}c^7$$
$$-171626158029500821504 00a^{12}c^7+65746142983045324800a^{13}c^7$$
$$+2448226396398182400a^{14}c^7+12606023079936000a^{15}c^7-32232026112000a^{16}c^7$$
$$-436879872000a^{17}c^7-951808000a^{18}c^7+364550363639774521307042066196332544 00c^8$$
$$-176691127417865776195061927021673185 28ac^8$$
$$+284991219785934353848477140700050227 2a^2c^8$$
$$-155490308698156476768736013566476288a^3c^8$$
$$-231574837489625776642035778643558 4a^4c^8$$
$$+376349415587111659860654974263296a^5c^8+614552347269208953427966926848a^6c^8$$
$$-389764985733751875918189023232a^7c^8-234525647004655720732486041 6a^8c^8$$
$$+187773430052693688565519872a^9c^8+24942416722340159185228 80a^{10}c^8$$
$$-267533933666265602519 04a^{11}c^8-679369622518788810752a^{12}c^8$$
$$-219445442315837337 6a^{13}c^8+36737957249190912a^{14}c^8+310595376199680a^{15}c^8$$
$$+445617469440a^{16}c^8-2184399360a^{17}c^8-4759040a^{18}c^8$$
$$+440406877603934096777026673311744 0000c^9$$
$$-178326131008147415788221034030694 4000ac^9$$
$$+232113864710755881803031756996608 000a^2c^9$$

© 2013 Global Journals Inc. (US)

$$-864406634285854196587884783206400 0a^3c^9$$
$$-3201107777856395066129853194240 00a^4c^9+19869691087509808561877532672000a^5c^9$$
$$+304557493863427189495126016000a^6c^9-160224530661878208426700800 00a^7c^9$$
$$-271058114268117146164480000a^8c^9+45743128324437974876160 00a^9c^9$$
$$+11550243977838018252800 0a^{10}c^9+5687883165675110400 0a^{11}c^9$$
$$-14511749989248512000a^{12}c^9-102960741531648000a^{13}c^9+132455504896000a^{14}c^9$$
$$+3300870144000a^{15}c^9+8090368000a^{16}c^9+433049472563672995745231040948469760c^{10}$$
$$-14624633125631298045179874490554 7776ac^{10}$$
$$+15168496614286038219626167337746432a^2c^{10}$$
$$-33429760249731875864504419260825 6a^3c^10$$
$$-24435417055526058339828463517696a^4c^{10}+72746188959478259574173093068 8a^5c^{10}$$
$$+22833231529388666525619544064a^6c^{10}-40703143038001266074284032 0a^7c^{10}$$
$$-13106081077916016150737920a^8c^{10}+31814940216382283710464a^9c^{10}$$
$$+30548440455509786501 12a^{10}c^{10}+18015970550045884416a^{11}c^{10}$$
$$-153108176727554048a^{12}c^{10}-1798208426606592a^{13}c^{10}-3353554620416a^{14}c^{10}$$
$$+13203480576a^{15}c^{10}+32361472a^{16}c^{10}+350546455225622995553985180467200 00c^{11}$$
$$-98357856795063653882506713759744 00ac^{11}$$
$$+79835433333631706652427073617920 0a^2c^{11}$$
$$-67323182368413397250225995776 00a^3c^{11}-12983256716353356905668804608 00a^4c^{11}$$
$$+16031712704360699496923136000a^5c^{11}+10242788923081222325895168 00a^6c^{11}$$
$$-38280653110299714895872 00a^7c^{11}-38944471879160091279360 0a^8c^{11}$$
$$-18531460853097873408 00a^9c^{11}+47307691166058086400a^{10}c^{11}$$
$$+481411625154969600a^{11}c^{11}-97959940915200a^{12}c^{11}-15123986841600a^{13}c^{11}$$
$$-42364108800a^{14}c^{11}+23552741637156584469560740020224 00c^{12}$$
$$-54582320621870124520454660869324 8ac^{12}+33794155269834290973635734667264a^2c^{12}$$
$$+14011372388038828834651688140 8a^3c^{12}-50947210104854667731378241536a^4c^{12}$$
$$+34417799286102523775877120a^5c^{12}+31335119871918556766765056a^6c^{12}$$
$$+14591272483288663882137 6a^7c^{12}-74351866737848277893 12a^8c^{12}$$
$$-73729701177942491136a^9c^{12}+340576396405260288a^{10}c^{12}+6393967917023232a^{11}c^{12}$$
$$+15324651503616a^{12}c^{12}-50413289472a^{13}c^{12}-141213696a^{14}c^{12}$$
$$+13209108250053065305305697484800 0c^{13}-25080727018788218156955048345600ac^{13}$$

Notes

© 2013 Global Journals Inc. (US)

Notes

$$+1141009818986979339736134451200a^2c^{13}+188043112362196161727168512000a^3c^{13}$$
$$-14973584596144425995534336000a^4c^{13}-125412485520989620961280000a^5c^{13}$$
$$+6650787127930542571520000a^6c^{13}+736946557321491578880000a^7c^{13}$$
$$-84070453438321459200a^8c^{13}-1360371592839168000a^9c^{13}-1201288617984000a^{10}c^{13}$$
$$+44208576921600a^{11}c^{13}+144472473600a^{12}c^{13}+62048116991485759738444827852800c^{14}$$
$$-955308033798859563215320252416ac^{14}+30129275830597180093530898432a^2c^{14}$$
$$+9038814861829694073085624320a^3c^{14}-325204452383804929203240960a^4c^{14}$$
$$-54244893399483568324608000a^5c^{14}+9320577112496207134720a^6c^{14}$$
$$+1685649009185431879680a^7c^{14}-322344216242061312a^8c^{14}-14412627627540480a^9c^{14}$$
$$-44710102794240a^{10}c^{14}+126310219776a^{11}c^{14}+412778496a^{12}c^{14}$$
$$+24446080104562253319110656000000c^{15}-30121275192514357897671475200ac^{15}$$
$$+5949409810310026477174784000a^2c^{15}+28752678828946713634406400a^3c^{15}$$
$$-4949447012184230985728000a^4c^{15}-1329985293155783147520000a^5c^{15}$$
$$+690697168623501312000a^6c^{15}+235896966453657600000a^7c^{15}+5121755578368000a^8c^{15}$$
$$-84206813184000a^9c^{15}-330222796800a^{10}c^{15}+807321805190159745564016640000c^{16}$$
$$-78305675731255665527409868800ac^{16}+774034696479690574633369600a^2c^{16}$$
$$+66763806738731096840601600a^3c^{16}-4401822778388480786432000a^4c^{16}$$
$$-2183067353227121786880000a^5c^{16}-160045814544924672000000a^6c^{16}+20633616468541440a^7c^{16}$$
$$+85040625745920a^8c^{16}-210517032960a^9c^{16}-825556992a^{10}c^{16}$$
$$+22287020496886847399526400000c^{17}-16665542994690293052211200000ac^{17}$$
$$+2885286146541488701440000a^2c^{17}+1158704206851822059520000a^3c^{17}$$
$$+140283054011842560000a^4c^{17}-2450507423809536000a^5c^{17}-9827187621888000a^6c^{17}$$
$$+1040201809920000a^7c^{17}+509902848000a^8c^{17}+5116743183106645604433920000c^{18}$$
$$-287184942520039832027136000ac^{18}-1403015552680256339968000a^2c^{18}$$
$$+149541088213486534656000a^3c^{18}+676438446397063168000a^4c^{18}-181591497297100800a^5c^{18}$$
$$-1049334736486400a^6c^{18}+231155957760a^7c^{18}+1133117440a^8c^{18}$$
$$+969113750273523712000000000c^{19}-394179706270148198400000ac^{19}$$
$$-39879941009965056000000a^2c^{19}+1397449692807168000000a^3c^{19}+102878952816640000a^4c^{19}$$
$$-802962800640000a^5c^{19}-5248122880000a^6c^{19}+1496430120547018342400000c^{20}$$
$$-4205581734077320396800ac^{20}-592824246953050112000a^2c^{20}+8950947523854336000a^3c^{20}$$
$$+809292036833280000a^4c^{20}-1605925601280000a^5c^{20}-10496245760000a^6c^{20}$$

© 2013 Global Journals Inc. (US)

$$+18518937847201792000c^{21} - 335979518676172800ac^{21} - 5690809542246400a^2c^{21}$$
$$+351774179328800a^3c^{21} + 344876646400a^4c^{21} + 179146036021821440c^{22}$$
$$-1890050686058496ac^{22} - 35428864360448a^2c^{22} + 63958941696a^3c^{22} + 627048448a^4c^{22}$$
$$+1304260771840000c^{23} - 6673976524800ac^{23} - 130862284800a^2c^{23} + 6717597286400c^{24}$$
$$-11123294208ac^{24} - 218103808a^2c^{24} + 21810380800c^{25} + 33554432c^{26}\Big\}\Big] \qquad (11)$$

III. Derivation of Main Formula

Substituting $b = -a - 51, z = \frac{1}{2}$ in given result (5),we get

$$(2a+51)\ {}_2F_1\left[\begin{matrix} a \\ c \end{matrix}\ ,\ \begin{matrix} -a-51 \\ \end{matrix}\ ;\ \frac{1}{2}\right]$$
$$= a\ {}_2F_1\left[\begin{matrix} a+1 \\ c \end{matrix}\ ,\ \begin{matrix} -a-51 \\ \end{matrix}\ ;\ \frac{1}{2}\right] + (a+51)\ {}_2F_1\left[\begin{matrix} a \\ c \end{matrix}\ ,\ \begin{matrix} -a-50 \\ \end{matrix}\ ;\ \frac{1}{2}\right]$$

Now involving the result which is established in Ref[5], we can establish the main result.

References Références Referencias

1. Andrews, L.C.(1992) ; *Special Function of mathematics for Engineers,second Edition*, McGraw-Hill Co Inc., New York.
2. Bells, Richard, Wong, Roderick ; *Special Functions , A Graduate Text.* Cambridge Studies in Advanced Mathematics, 2010.
3. Mathai, A.M.,Haubold , Hans J.(2008) ; *Special Functions for Applied Scientists*, Springer, New York.
4. Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O.I.; *Integrals and Series Vol. 3: More Special Functions.* Nauka, Moscow, 1986. Translated from the Russian by G.G. Gould, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.
5. Salahuddin, Chaudhary, M.P ; A New Summation Formula Clung to Contigious relation, *Global Journal of Science Frontier Research(F)*, 13(2013),27- 39.

Notes

© 2013 Global Journals Inc. (US)

Global Journals Inc. (US) Guidelines Handbook 2013

www.GlobalJournals.org

FELLOW OF ASSOCIATION OF RESEARCH SOCIETY IN SCIENCE (FARSS)

- 'FARSS' title will be awarded to the person after approval of Editor-in-Chief and Editorial Board. The title 'FARSS" can be added to name in the following manner. eg. Dr. John E. Hall, Ph.D., FARSS or William Walldroff Ph. D., M.S., FARSS
- Being FARSS is a respectful honor. It authenticates your research activities. After becoming FARSS, you can use 'FARSS' title as you use your degree in suffix of your name. This will definitely will enhance and add up your name. You can use it on your Career Counseling Materials/CV/Resume/Visiting Card/Name Plate etc.
- 60% Discount will be provided to FARSS members for publishing research papers in Global Journals Inc., if our Editorial Board and Peer Reviewers accept the paper. For the life time, if you are author/co-author of any paper bill sent to you will automatically be discounted one by 60%
- FARSS will be given a renowned, secure, free professional email address with 100 GB of space eg.johnhall@globaljournals.org. You will be facilitated with Webmail, SpamAssassin, Email Forwarders, Auto-Responders, Email Delivery Route tracing, etc.
- FARSS member is eligible to become paid peer reviewer at Global Journals Inc. to earn up to 15% of realized author charges taken from author of respective paper. After reviewing 5 or more papers you can request to transfer the amount to your bank account or to your PayPal account.
- Eg. If we had taken 420 USD from author, we can send 63 USD to your account.
- FARSS member can apply for free approval, grading and certification of some of their Educational and Institutional Degrees from Global Journals Inc. (US) and Open Association of Research,Society U.S.A.
- After you are FARSS. You can send us scanned copy of all of your documents. We will verify, grade and certify them within a month. It will be based on your academic records, quality of research papers published by you, and 50 more criteria. This is beneficial for your job interviews as recruiting organization need not just rely on you for authenticity and your unknown qualities, you would have authentic ranks of all of your documents. Our scale is unique worldwide.
- FARSS member can proceed to get benefits of free research podcasting in Global Research Radio with their research documents, slides and online movies.
- After your publication anywhere in the world, you can upload you research paper with your recorded voice or you can use our professional RJs to record your paper their voice. We can also stream your conference videos and display your slides online.
- FARSS will be eligible for free application of Standardization of their Researches by Open Scientific Standards. Standardization is next step and level after publishing in a journal. A team of research and professional will work with you to take your research to its next level, which is worldwide open standardization.

© Copyright by Global Journals Inc.(US) | Guidelines Handbook

- FARSS is eligible to earn from their researches: While publishing his paper with Global Journals Inc. (US), FARSS can decide whether he/she would like to publish his/her research in closed manner. When readers will buy that individual research paper for reading, 80% of its earning by Global Journals Inc. (US) will be transferred to FARSS member's bank account after certain threshold balance. There is no time limit for collection. FARSS member can decide its price and we can help in decision.

MEMBER OF ASSOCIATION OF RESEARCH SOCIETY IN SCIENCE (MARSS)

- 'MARSS' title will be awarded to the person after approval of Editor-in-Chief and Editorial Board. The title 'MARSS" can be added to name in the following manner. eg. Dr. John E. Hall, Ph.D., MARSS or William Walldroff Ph. D., M.S., MARSS
- Being MARSS is a respectful honor. It authenticates your research activities. After becoming MARSS, you can use 'MARSS' title as you use your degree in suffix of your name. This will definitely will enhance and add up your name. You can use it on your Career Counseling Materials/CV/Resume/Visiting Card/Name Plate etc.
- 40% Discount will be provided to MARSS members for publishing research papers in Global Journals Inc., if our Editorial Board and Peer Reviewers accept the paper. For the life time, if you are author/co-author of any paper bill sent to you will automatically be discounted one by 60%
- MARSS will be given a renowned, secure, free professional email address with 30 GB of space eg.johnhall@globaljournals.org. You will be facilitated with Webmail, SpamAssassin, Email Forwarders, Auto-Responders, Email Delivery Route tracing, etc.
- MARSS member is eligible to become paid peer reviewer at Global Journals Inc. to earn up to 10% of realized author charges taken from author of respective paper. After reviewing 5 or more papers you can request to transfer the amount to your bank account or to your PayPal account.
- MARSS member can apply for free approval, grading and certification of some of their Educational and Institutional Degrees from Global Journals Inc. (US) and Open Association of Research,Society U.S.A.
- MARSS is eligible to earn from their researches: While publishing his paper with Global Journals Inc. (US), MARSS can decide whether he/she would like to publish his/her research in closed manner. When readers will buy that individual research paper for reading, 40% of its earning by Global Journals Inc. (US) will be transferred to MARSS member's bank account after certain threshold balance. There is no time limit for collection. MARSS member can decide its price and we can help in decision.

AUXILIARY MEMBERSHIPS

ANNUAL MEMBER

- Annual Member will be authorized to receive e-Journal GJSFR for one year (subscription for one year).
- The member will be allotted free 1 GB Web-space along with subDomain to contribute and participate in our activities.
- A professional email address will be allotted free 500 MB email space.

PAPER PUBLICATION

- The members can publish paper once. The paper will be sent to two-peer reviewer. The paper will be published after the acceptance of peer reviewers and Editorial Board.

© Copyright by Global Journals Inc.(US) | Guidelines Handbook

Process of submission of Research Paper

The Area or field of specialization may or may not be of any category as mentioned in 'Scope of Journal' menu of the GlobalJournals.org website. There are 37 Research Journal categorized with Six parental Journals GJCST, GJMR, GJRE, GJMBR, GJSFR, GJHSS. For Authors should prefer the mentioned categories. There are three widely used systems UDC, DDC and LCC. The details are available as 'Knowledge Abstract' at Home page. The major advantage of this coding is that, the research work will be exposed to and shared with all over the world as we are being abstracted and indexed worldwide.

The paper should be in proper format. The format can be downloaded from first page of 'Author Guideline' Menu. The Author is expected to follow the general rules as mentioned in this menu. The paper should be written in MS-Word Format (*.DOC,*.DOCX).

The Author can submit the paper either online or offline. The authors should prefer online submission.Online Submission: There are three ways to submit your paper:

(A) (I) First, register yourself using top right corner of Home page then Login. If you are already registered, then login using your username and password.

(II) Choose corresponding Journal.

(III) Click 'Submit Manuscript'. Fill required information and Upload the paper.

(B) If you are using Internet Explorer, then Direct Submission through Homepage is also available.

(C) If these two are not conveninet , and then email the paper directly to dean@globaljournals.org.

Offline Submission: Author can send the typed form of paper by Post. However, online submission should be preferred.

© Copyright by Global Journals Inc.(US)| Guidelines Handbook

Preferred Author Guidelines

MANUSCRIPT STYLE INSTRUCTION (Must be strictly followed)

Page Size: 8.27" X 11'"

- Left Margin: 0.65
- Right Margin: 0.65
- Top Margin: 0.75
- Bottom Margin: 0.75
- Font type of all text should be Swis 721 Lt BT.
- Paper Title should be of Font Size 24 with one Column section.
- Author Name in Font Size of 11 with one column as of Title.
- Abstract Font size of 9 Bold, "Abstract" word in Italic Bold.
- Main Text: Font size 10 with justified two columns section
- Two Column with Equal Column with of 3.38 and Gaping of .2
- First Character must be three lines Drop capped.
- Paragraph before Spacing of 1 pt and After of 0 pt.
- Line Spacing of 1 pt
- Large Images must be in One Column
- Numbering of First Main Headings (Heading 1) must be in Roman Letters, Capital Letter, and Font Size of 10.
- Numbering of Second Main Headings (Heading 2) must be in Alphabets, Italic, and Font Size of 10.

You can use your own standard format also.
Author Guidelines:

1. General,

2. Ethical Guidelines,

3. Submission of Manuscripts,

4. Manuscript's Category,

5. Structure and Format of Manuscript,

6. After Acceptance.

1. GENERAL

Before submitting your research paper, one is advised to go through the details as mentioned in following heads. It will be beneficial, while peer reviewer justify your paper for publication.

Scope

The Global Journals Inc. (US) welcome the submission of original paper, review paper, survey article relevant to the all the streams of Philosophy and knowledge. The Global Journals Inc. (US) is parental platform for Global Journal of Computer Science and Technology, Researches in Engineering, Medical Research, Science Frontier Research, Human Social Science, Management, and Business organization. The choice of specific field can be done otherwise as following in Abstracting and Indexing Page on this Website. As the all Global

Journals Inc. (US) are being abstracted and indexed (in process) by most of the reputed organizations. Topics of only narrow interest will not be accepted unless they have wider potential or consequences.

2. ETHICAL GUIDELINES

Authors should follow the ethical guidelines as mentioned below for publication of research paper and research activities.

Papers are accepted on strict understanding that the material in whole or in part has not been, nor is being, considered for publication elsewhere. If the paper once accepted by Global Journals Inc. (US) and Editorial Board, will become the copyright of the Global Journals Inc. (US).

Authorship: The authors and coauthors should have active contribution to conception design, analysis and interpretation of findings. They should critically review the contents and drafting of the paper. All should approve the final version of the paper before submission

The Global Journals Inc. (US) follows the definition of authorship set up by the Global Academy of Research and Development. According to the Global Academy of R&D authorship, criteria must be based on:

1) Substantial contributions to conception and acquisition of data, analysis and interpretation of the findings.

2) Drafting the paper and revising it critically regarding important academic content.

3) Final approval of the version of the paper to be published.

All authors should have been credited according to their appropriate contribution in research activity and preparing paper. Contributors who do not match the criteria as authors may be mentioned under Acknowledgement.

Acknowledgements: Contributors to the research other than authors credited should be mentioned under acknowledgement. The specifications of the source of funding for the research if appropriate can be included. Suppliers of resources may be mentioned along with address.

Appeal of Decision: The Editorial Board's decision on publication of the paper is final and cannot be appealed elsewhere.

Permissions: It is the author's responsibility to have prior permission if all or parts of earlier published illustrations are used in this paper.

Please mention proper reference and appropriate acknowledgements wherever expected.

If all or parts of previously published illustrations are used, permission must be taken from the copyright holder concerned. It is the author's responsibility to take these in writing.

Approval for reproduction/modification of any information (including figures and tables) published elsewhere must be obtained by the authors/copyright holders before submission of the manuscript. Contributors (Authors) are responsible for any copyright fee involved.

3. SUBMISSION OF MANUSCRIPTS

Manuscripts should be uploaded via this online submission page. The online submission is most efficient method for submission of papers, as it enables rapid distribution of manuscripts and consequently speeds up the review procedure. It also enables authors to know the status of their own manuscripts by emailing us. Complete instructions for submitting a paper is available below.

Manuscript submission is a systematic procedure and little preparation is required beyond having all parts of your manuscript in a given format and a computer with an Internet connection and a Web browser. Full help and instructions are provided on-screen. As an author, you will be prompted for login and manuscript details as Field of Paper and then to upload your manuscript file(s) according to the instructions.



To avoid postal delays, all transaction is preferred by e-mail. A finished manuscript submission is confirmed by e-mail immediately and your paper enters the editorial process with no postal delays. When a conclusion is made about the publication of your paper by our Editorial Board, revisions can be submitted online with the same procedure, with an occasion to view and respond to all comments.

Complete support for both authors and co-author is provided.

4. MANUSCRIPT'S CATEGORY

Based on potential and nature, the manuscript can be categorized under the following heads:

Original research paper: Such papers are reports of high-level significant original research work.

Review papers: These are concise, significant but helpful and decisive topics for young researchers.

Research articles: These are handled with small investigation and applications

Research letters: The letters are small and concise comments on previously published matters.

5.STRUCTURE AND FORMAT OF MANUSCRIPT

The recommended size of original research paper is less than seven thousand words, review papers fewer than seven thousands words also.Preparation of research paper or how to write research paper, are major hurdle, while writing manuscript. The research articles and research letters should be fewer than three thousand words, the structure original research paper; sometime review paper should be as follows:

Papers: These are reports of significant research (typically less than 7000 words equivalent, including tables, figures, references), and comprise:

(a)Title should be relevant and commensurate with the theme of the paper.

(b) A brief Summary, "Abstract" (less than 150 words) containing the major results and conclusions.

(c) Up to ten keywords, that precisely identifies the paper's subject, purpose, and focus.

(d) An Introduction, giving necessary background excluding subheadings; objectives must be clearly declared.

(e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition; sources of information must be given and numerical methods must be specified by reference, unless non-standard.

(f) Results should be presented concisely, by well-designed tables and/or figures; the same data may not be used in both; suitable statistical data should be given. All data must be obtained with attention to numerical detail in the planning stage. As reproduced design has been recognized to be important to experiments for a considerable time, the Editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned un-refereed;

(g) Discussion should cover the implications and consequences, not just recapitulating the results; conclusions should be summarizing.

(h) Brief Acknowledgements.

(i) References in the proper form.

Authors should very cautiously consider the preparation of papers to ensure that they communicate efficiently. Papers are much more likely to be accepted, if they are cautiously designed and laid out, contain few or no errors, are summarizing, and be conventional to the approach and instructions. They will in addition, be published with much less delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and to make suggestions to improve briefness.

It is vital, that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

Format

Language: The language of publication is UK English. Authors, for whom English is a second language, must have their manuscript efficiently edited by an English-speaking person before submission to make sure that, the English is of high excellence. It is preferable, that manuscripts should be professionally edited.

Standard Usage, Abbreviations, and Units: Spelling and hyphenation should be conventional to The Concise Oxford English Dictionary. Statistics and measurements should at all times be given in figures, e.g. 16 min, except for when the number begins a sentence. When the number does not refer to a unit of measurement it should be spelt in full unless, it is 160 or greater.

Abbreviations supposed to be used carefully. The abbreviated name or expression is supposed to be cited in full at first usage, followed by the conventional abbreviation in parentheses.

Metric SI units are supposed to generally be used excluding where they conflict with current practice or are confusing. For illustration, 1.4 l rather than 1.4 × 10-3 m3, or 4 mm somewhat than 4 × 10-3 m. Chemical formula and solutions must identify the form used, e.g. anhydrous or hydrated, and the concentration must be in clearly defined units. Common species names should be followed by underlines at the first mention. For following use the generic name should be constricted to a single letter, if it is clear.

Structure

All manuscripts submitted to Global Journals Inc. (US), ought to include:

Title: The title page must carry an instructive title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) wherever the work was carried out. The full postal address in addition with the e-mail address of related author must be given. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining and indexing.

Abstract, used in Original Papers and Reviews:

Optimizing Abstract for Search Engines

Many researchers searching for information online will use search engines such as Google, Yahoo or similar. By optimizing your paper for search engines, you will amplify the chance of someone finding it. This in turn will make it more likely to be viewed and/or cited in a further work. Global Journals Inc. (US) have compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Key Words

A major linchpin in research work for the writing research paper is the keyword search, which one will employ to find both library and Internet resources.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy and planning a list of possible keywords and phrases to try.

Search engines for most searches, use Boolean searching, which is somewhat different from Internet searches. The Boolean search uses "operators," words (and, or, not, and near) that enable you to expand or narrow your affords. Tips for research paper while preparing research paper are very helpful guideline of research paper.

Choice of key words is first tool of tips to write research paper. Research paper writing is an art.A few tips for deciding as strategically as possible about keyword search:

- One should start brainstorming lists of possible keywords before even begin searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in research paper?" Then consider synonyms for the important words.
- It may take the discovery of only one relevant paper to let steer in the right keyword direction because in most databases, the keywords under which a research paper is abstracted are listed with the paper.
- One should avoid outdated words.

Keywords are the key that opens a door to research work sources. Keyword searching is an art in which researcher's skills are bound to improve with experience and time.

Numerical Methods: Numerical methods used should be clear and, where appropriate, supported by references.

Acknowledgements: Please make these as concise as possible.

References

References follow the Harvard scheme of referencing. References in the text should cite the authors' names followed by the time of their publication, unless there are three or more authors when simply the first author's name is quoted followed by et al. unpublished work has to only be cited where necessary, and only in the text. Copies of references in press in other journals have to be supplied with submitted typescripts. It is necessary that all citations and references be carefully checked before submission, as mistakes or omissions will cause delays.

References to information on the World Wide Web can be given, but only if the information is available without charge to readers on an official site. Wikipedia and Similar websites are not allowed where anyone can change the information. Authors will be asked to make available electronic copies of the cited information for inclusion on the Global Journals Inc. (US) homepage at the judgment of the Editorial Board.

The Editorial Board and Global Journals Inc. (US) recommend that, citation of online-published papers and other material should be done via a DOI (digital object identifier). If an author cites anything, which does not have a DOI, they run the risk of the cited material not being noticeable.

The Editorial Board and Global Journals Inc. (US) recommend the use of a tool such as Reference Manager for reference management and formatting.

Tables, Figures and Figure Legends

Tables: Tables should be few in number, cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g. Table 4, a self-explanatory caption and be on a separate sheet. Vertical lines should not be used.

Figures: Figures are supposed to be submitted as separate files. Always take in a citation in the text for each figure using Arabic numbers, e.g. Fig. 4. Artwork must be submitted online in electronic form by e-mailing them.

Preparation of Electronic Figures for Publication

Even though low quality images are sufficient for review purposes, print publication requires high quality images to prevent the final product being blurred or fuzzy. Submit (or e-mail) EPS (line art) or TIFF (halftone/photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Do not use pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings) in relation to the imitation size. Please give the data for figures in black and white or submit a Color Work Agreement Form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution (at final image size) ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs) : >350 dpi; figures containing both halftone and line images: >650 dpi.

Color Charges: It is the rule of the Global Journals Inc. (US) for authors to pay the full cost for the reproduction of their color artwork. Hence, please note that, if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a color work agreement form before your paper can be published.

Figure Legends: Self-explanatory legends of all figures should be incorporated separately under the heading 'Legends to Figures'. In the full-text online edition of the journal, figure legends may possibly be truncated in abbreviated links to the full screen version. Therefore, the first 100 characters of any legend should notify the reader, about the key aspects of the figure.

6. AFTER ACCEPTANCE

Upon approval of a paper for publication, the manuscript will be forwarded to the dean, who is responsible for the publication of the Global Journals Inc. (US).

6.1 Proof Corrections

The corresponding author will receive an e-mail alert containing a link to a website or will be attached. A working e-mail address must therefore be provided for the related author.

Acrobat Reader will be required in order to read this file. This software can be downloaded

(Free of charge) from the following website:

www.adobe.com/products/acrobat/readstep2.html. This will facilitate the file to be opened, read on screen, and printed out in order for any corrections to be added. Further instructions will be sent with the proof.

Proofs must be returned to the dean at dean@globaljournals.org within three days of receipt.

As changes to proofs are costly, we inquire that you only correct typesetting errors. All illustrations are retained by the publisher. Please note that the authors are responsible for all statements made in their work, including changes made by the copy editor.

6.2 Early View of Global Journals Inc. (US) (Publication Prior to Print)

The Global Journals Inc. (US) are enclosed by our publishing's Early View service. Early View articles are complete full-text articles sent in advance of their publication. Early View articles are absolute and final. They have been completely reviewed, revised and edited for publication, and the authors' final corrections have been incorporated. Because they are in final form, no changes can be made after sending them. The nature of Early View articles means that they do not yet have volume, issue or page numbers, so Early View articles cannot be cited in the conventional way.

6.3 Author Services

Online production tracking is available for your article through Author Services. Author Services enables authors to track their article - once it has been accepted - through the production process to publication online and in print. Authors can check the status of their articles online and choose to receive automated e-mails at key stages of production. The authors will receive an e-mail with a unique link that enables them to register and have their article automatically added to the system. Please ensure that a complete e-mail address is provided when submitting the manuscript.

6.4 Author Material Archive Policy

Please note that if not specifically requested, publisher will dispose off hardcopy & electronic information submitted, after the two months of publication. If you require the return of any information submitted, please inform the Editorial Board or dean as soon as possible.

6.5 Offprint and Extra Copies

A PDF offprint of the online-published article will be provided free of charge to the related author, and may be distributed according to the Publisher's terms and conditions. Additional paper offprint may be ordered by emailing us at: editor@globaljournals.org .



© Copyright by Global Journals Inc.(US)| Guidelines Handbook

Before start writing a good quality Computer Science Research Paper, let us first understand what is Computer Science Research Paper? So, Computer Science Research Paper is the paper which is written by professionals or scientists who are associated to Computer Science and Information Technology, or doing research study in these areas. If you are novel to this field then you can consult about this field from your supervisor or guide.

TECHNIQUES FOR WRITING A GOOD QUALITY RESEARCH PAPER:

1. Choosing the topic: In most cases, the topic is searched by the interest of author but it can be also suggested by the guides. You can have several topics and then you can judge that in which topic or subject you are finding yourself most comfortable. This can be done by asking several questions to yourself, like Will I be able to carry our search in this area? Will I find all necessary recourses to accomplish the search? Will I be able to find all information in this field area? If the answer of these types of questions will be "Yes" then you can choose that topic. In most of the cases, you may have to conduct the surveys and have to visit several places because this field is related to Computer Science and Information Technology. Also, you may have to do a lot of work to find all rise and falls regarding the various data of that subject. Sometimes, detailed information plays a vital role, instead of short information.

2. Evaluators are human: First thing to remember that evaluators are also human being. They are not only meant for rejecting a paper. They are here to evaluate your paper. So, present your Best.

3. Think Like Evaluators: If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

4. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

6. Use of computer is recommended: As you are doing research in the field of Computer Science, then this point is quite obvious.

7. Use right software: Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

8. Use the Internet for help: An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. Pick a good study spot: To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.



© Copyright by Global Journals Inc.(US)| Guidelines Handbook

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. Think technically: Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.



© Copyright by Global Journals Inc.(US)| Guidelines Handbook

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.



- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.



© Copyright by Global Journals Inc.(US)| Guidelines Handbook

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

Administration Rules Listed Before Submitting Your Research Paper to Global Journals Inc. (US)

Please carefully note down following rules and regulation before submitting your Research Paper to Global Journals Inc. (US):

Segment Draft and Final Research Paper: You have to strictly follow the template of research paper. If it is not done your paper may get rejected.

- The **major constraint** is that you must independently make all content, tables, graphs, and facts that are offered in the paper. You must write each part of the paper wholly on your own. The Peer-reviewers need to identify your own perceptive of the concepts in your own terms. NEVER extract straight from any foundation, and never rephrase someone else's analysis.

- Do not give permission to anyone else to "PROOFREAD" your manuscript.

- Methods to avoid Plagiarism is applied by us on every paper, if found guilty, you will be blacklisted by all of our collaborated research groups, your institution will be informed for this and strict legal actions will be taken immediately.)
- To guard yourself and others from possible illegal use please do not permit anyone right to use to your paper and files.



© Copyright by Global Journals Inc.(US)| Guidelines Handbook

Criterion for Grading a Research Paper (Compilation)
by Global Journals Inc. (US)

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals Inc. (US).

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

© Copyright by Global Journals Inc.(US) | Guidelines Handbook

INDEX

A

Acceptability · 78
Approximation · 60, 63, 95, 99, 114

C

Circumference · 34, 36
Cylindrical · 81, 83

D

Denominator · 22, 114
Discretization · 67, 68, 69, 71

E

Eigenfunction · 1, 6, 7, 26

G

Gaussian · 150

H

Homogeneous · 8, 60, 71
Horizontal · 60, 130, 135
Hypergeometric · 118

L

Lipschitzian · 142, 144
Lyapunov · 140, 142, 144, 145, 146, 147, 148, 149

N

Negligible · 74, 140

P

Periodically · 16
Perpendicular · 88, 91
Pochhammer · 118, 150

R

Recurrence · 118, 119, 150, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164

S

Singularities · 16, 18, 26, 36
Symmetrically · 28, 34

T

Trigonometric · 1, 3, 8, 20, 41, 58

W

Wendroff · 60, 69, 71, 72, 73, 74, 75, 76, 77, 78



Global Journal of Science Frontier Research

Visit us on the Web at www.GlobalJournals.org | www.JournalofScience.org
or email us at helpdesk@globaljournals.org

ISSN 9755896



© Global Journals