Sum And Product Of K-Regular Fuzzy Matrices

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\[ A \{ 1^k \} = \{ X/ A^k X A = A^k \}. \]

In general, right k-regular is different from left k-regular. Hence a right k-g-inverse need not be a left k-g-inverse.

1) Lemma

For A, B ∈ F_n, and a positive integer k, the following hold.
(i) If A is right k-regular and R(B) ⊆ R(A) then B = XA for some X ∈ F_n.
(ii) If A is left k-regular and C(B) ⊆ C(A) then B = AY for some Y ∈ F_n.

2) Lemma

For A, B ∈ F_n, with R(A) = R(B) and R(A^k) = R(B^k) then, A is right k-regular ⇔ B is right k-regular.

Proof

For A, B ∈ F_n, with C(A) = C(B) and C(A^k) = C(B^k) then, A is left k-regular ⇔ B is left k-regular.

III. K-REGULARITY OF SUM OF K-REGULAR FUZZY MATRICES

In this section, we discuss the k-regularity of the sum of k-regular fuzzy matrices.

1) Lemma

For A, B ∈ F_n, we have the following.
if R(A) = R(B) and AB = BA then R(A^k) = R(B^k).
if C(A) = C(B) and AB = BA then C(A^k) = C(B^k).

Proof

(i) By Lemma (2.1), R(A) ⊆ R(B) ⇒ A = XB for some X ∈ F_n.
⇒ A^2 = XAB
⇒ A^2 = X(B^2).
Thus proceeding we get,
R(A) ⊆ R(B) ⇒ A^k = X^kB^k ⇒ R(A^k) ⊆ R(B^k)
Therefore, R(A) ⊆ R(B) and AB = BA ⇒ R(A^k) ⊆ R(B^k)
Similarly, we can prove that R(B) ⊆ R(A) and AB = BA ⇒ R(B^k) ⊆ R(A^k).
Therefore, R(A^k) = R(B^k).
Hence the proof.
(ii) Proof of (ii) is similar to that of (i) and hence omitted.

2) Theorem

For A, B ∈ F_n, if R(B) ⊆ R(A) ⊆ R(A+B) and AB = BA then, A is right k-regular ⇔ A+B is right k-regular.

Proof

Since AB = BA,
(A+B)^k = A^k + A^{k-1}B + ... + AB + B^k.
Since R(B) ⊆ R(A) by Lemma (2.1), B = XA for some X ∈ F_n.
Since R(A) ⊆ R(A+B), we get R(A) = R(A+B).
Therefore by Lemma (3.1), R(A^k) = R((A+B)^k).
Therefore by Theorem (2.1), it follows that, A is right k-regular ⇔ A+B is right k-regular.

2) Theorem

For A, B ∈ F_n, if C(B) ⊆ C(A) ⊆ C(A+B) and AB = BA then, A is left k-regular ⇔ A+B is left k-regular.
Proof:
Proof is similar to Theorem (3.1) and hence omitted.

1) Remark

For $k=1$, the above theorems reduce to the following:

3) Theorem

For $A, B \in F_n$, if $R(B) \subseteq R(A) \subseteq R(A+B)$ and $C(B) \subseteq C(A) \subseteq C(A+B)$, then $A$ is a regular fuzzy matrix $\iff$ $A+B$ is a regular fuzzy matrix.

4) Theorem

For $A, B \in F_n$, if $A, B$ are right $k$-regular, $R(A)=R(B)$ and $AB=BA$ with $A \{1^k \} \cap B \{1^k \} \neq \phi$ then, $A+B$ is right $k$-regular.

Proof:

Let $X$ be a right $k$-$g$-inverse of $A$ and $B$. That is, $A^k X A^k = A^k$ and $B^k X B^k = B^k$ (3.1).

Since $AB=BA$, we have $(A+B)^k = A^k + A^{k-1} B^k + \ldots + A B^{k+1} + B^k$ (3.2).

By Lemma (2.1), $R(A) \subseteq R(A+X) \Rightarrow B = VA$ (3.3) for some $V \in F_n$ and $R(A) \subseteq R(B) \Rightarrow A = UB$ (3.4) for some $U \in F_n$.

We claim that $X$ is the right $k$-$g$-inverse of $A+B$.

$(A+B)^k (X(A+B) = (A^k + A^{k-1} B^k + \ldots + A B^{k+1} + B^k) (X(A+B) = (A^k X A^k + A^{k-1} B X A^k + \ldots + A B^{k+1} X A^k + B^k X A^k + A^k X B^k + \ldots + A B^{k+1} X B^k + B^k X B^k)$

By using Equations (3.1) to (3.4), we have

$(A+B)^k X (A+B) = A^k + A^{k-1} B^k + \ldots + A B^{k+1} + B^k = (A+B)^k$.

Hence the theorem.

2) Remark

The converse of the above theorem need not to be true. This is illustrated in the following Example 3.1: Let

\[
\begin{pmatrix}
1 & 0.5 & 0 \\
0.5 & 0 & 0.5 \\
0.5 & 0.5 & 0
\end{pmatrix}
\]

For $X = \begin{pmatrix}
1 & 0 & 0.5 \\
0 & 0.5 & 0.5 \\
0 & 0 & 0
\end{pmatrix}$, we have

\[
(A+B)^k X (A+B) = (A+B)^k, \text{Hence } A+B \text{ is } 2 \text{-regular, } X \text{ is a right } 2 \text{-} g \text{-inverse of } A+B.
\]

\[
A = \begin{pmatrix}
1 & 0.5 & 0 \\
0 & 0 & 0.5 \\
0.5 & 0 & 0
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
1 & 0.5 & 0 \\
0 & 0 & 0.5 \\
0.5 & 0 & 0
\end{pmatrix}
\]

\[
\text{such that } A+B = \begin{pmatrix}
1 & 0.5 & 0 \\
0 & 0 & 0.5 \\
0.5 & 0.5 & 0
\end{pmatrix}
\]

For $X = \begin{pmatrix}
1 & 0 & 0.5 \\
0 & 0.5 & 0.5 \\
0 & 0.5 & 0
\end{pmatrix}$, $A^2 X A^2 = A^2$ and $B^3 X B^3 = B^3$ holds.

Thus $A$ is 2-regular and $B$ is 3-regular also $A B = B A$ but $R(B) \not\subseteq R(A)$.

5) Theorem

For $A, B \in F_n$, if $A, B$ are left $k$-regular, $C(A)=C(B)$ and $AB=BA$ with $A \{1^k \} \cap B \{1^k \} \neq \phi$ then, $A+B$ is left $k$-regular.

Proof:

This can be proved as that of Theorem (3.4) and hence omitted.

IV. K-REGULARITY OF PRODUCT OF K-REGULAR FUZZY MATRICES

In this section, we discuss the $k$-regularity of the product of $k$-regular fuzzy matrices.

1) Lemma

For $A \in F_n$, the following hold.

(i) if $R(A) \subseteq R(A^k)$ then $R(A) = R(A^k)$.

(ii) if $C(A) \subseteq C(A^k)$ then $C(A) = C(A^k)$.

Proof:

(i) By Lemma (2.1), $R(A) \subseteq R(A^k) \Rightarrow A^2 X A^2$ for some $X \in F_n$.

\[
\Rightarrow A^2 = X A^3 \Rightarrow R(A^3) \subseteq R(A^3)
\]

By Lemma (2.1), $R(A^2) \subseteq R(A) \Rightarrow A^2 = X A^3$ for some $Y \in F_n$.

\[
\Rightarrow A^2 = X A^2 \Rightarrow R(A^2) \subseteq R(A^2)
\]

Hence $R(A^3) = R(A^3)$. Thus proceeding in this way, we get $R(A) = R(A^k) \Rightarrow R(A) = R(A^k)$.

(ii) Proof of (ii) is similar to that of (i) and hence omitted.

2) Lemma

For $A \in F_n$, if $R(A) = R(A^k)$ then the following statements are equivalent:

(i) $A$ is regular

(ii) $A$ is right $k$-regular

(iii) $A^k$ is regular

(iv) $A^k$ is right $k$-regular
Proof:  
If A is regular, then AXA=A for some X in F_n, for k≥1, premultiplying by A^{-1} on both sides, we get A^kXA=A^k. Therefore A is right k-regular for all k≥1. Thus (i)⇒(ii).
Since R(A) ⊆ R(A^2), by Lemma (2.1),
A=YA^k\hspace{1cm}(4.1)
for some Y∈F_n.
If A is right k-regular then,
A^kXA=A^k\hspace{1cm}(4.2)
for some X∈F_n.
Premultiplying by Y on both sides in (4.2) and using (4.1), we get YA^kXA=YA^k ⇒ AXA=A. Therefore A is regular. Thus (ii)⇒(i).
If A is regular then,
A^kZA^k=A^k\hspace{1cm}(4.3)
for some Z∈F_n.  
This can be written as A^kWA =A^k where W=ZA^{-1}, hence A is right k-regular and W is a right k-g-inverse of A. Thus (iii)⇒(ii).
By using (4.1) and (4.2), A^kXYA^k = A^k ⇒ A^kVA^k = A^k where V=X. Therefore A^k is regular. Thus (ii)⇒(iii).
Hence (i)⇔(ii)⇔(iii).
Next, let us prove that (iii)⇒(iv).
Premultiplying by (A^k)^{-1} on both sides in (4.3), we get (A^k)^{-1}ZA^k = (A^k)^{-1}. Thus A^k is right k-regular. Hence (iii)⇒(iv).
If (A^k)^{-1} is right k-regular then (A^k)^{-1}UA^k = (A^k)^{-1}\hspace{1cm}(4.4)
for some U∈F_n.
By using (4.1) and (4.4), we get A^k UA^k = A^k. Thus (iv)⇒(iii).
Hence the theorem.

3) Lemma

For A∈F_n, if C(A)=C(A^2) then the following statements are equivalent:

(i) A is regular
(ii) A is left k-regular
(iii) A^k is regular
(iv) A^k is left k-regular

Proof:
This can be proved as that of Lemma (4.2) and hence omitted.

1) Theorem

For A, B∈F_n, if R(A)⊆R(AB) and AB=BA then, A is right k-regular ⇔ AB is right k-regular.
Proof:
Since R(A)⊆R(AB) and AB=BA, R(A)=R(AB). Therefore by Lemma (3.1), R(A^k)=R((AB)^k).
Therefore by Theorem (2.1), it follows that, A is right k-regular ⇔ AB is right k-regular.

2) Theorem

For A, B∈F_n, if C(A)⊆C(AB) and AB=BA then, A is left k-regular ⇔ AB is left k-regular.
Proof:
This can be proved as that of Theorem (4.1) and hence omitted.

1) Remark

For k=1, the above theorems reduce to the following:

3) Theorem

For the fuzzy matrices A, B∈F_n, if R(A)=R(AB) then, A is a regular fuzzy matrix ⇔ AB is a regular fuzzy matrix.

2) Remark

In particular for A=B, the Theorem (4.1) and (4.2) reduces to the following:

(i) if R(A)=R(A^2) then, A is right k-regular ⇔ A^2 is right k-regular.
(ii) if C(A)=C(A^2) then, A is left k-regular ⇔ A^2 is left k-regular.
The condition R(A)=R(A^2) is a necessary condition in (i) and the condition C(A)=C(A^2) is a necessary condition in (ii). This is illustrated in the following:

Example 4.1:

Let us consider A=[1 0.5 0]
0 0 0.5 . For this A, A^2=
0.5 0 0.

1 0.5 0.5
0 0 0
0.5 0 0.5

A^3=
1 0.5 0.5
0 0.5 0
0.5 0.5 0.

A^3XA=A^3 holds. Therefore A
0.5 0.5 0.5
is 3-regular. A^5=A^4. R(A)⊆R(A^2).
(A^3)^2XA=(A^2)^2. Thus A^2 is 2-regular. Hence A^2 is right 2-regular but A is not a right 2-regular. Therefore the condition R(A)=R(A^2) is necessary.

3) Remark

By Lemma (4.1), R(A)=R(A^2) ⇒ R(A)=R(A^4).
Therefore by lemma (4.2), A is regular ⇔ A^k is right k-regular ⇔ A^k is right k-regular. Thus for R(A)=R(A^3), k-regularity coincides with regularity.

V. REFERENCES

