Mixed Quadrature Rule for Double Integrals of Simpsons $\frac{1}{3}rd$ and Gauss Legendre Two Point Rule in Two Variables

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Mixed Quadrature Rule for Double Integrals of Simpsons $\frac{1}{3}rd$ and Gauss Legendre Two Point Rule in Two Variables

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Abstract: In this article we are concerned with mixed quadrature rule of higher degree of precision for double integrals with double variables. The rule is numerically tested taking some suitable texts and the error bound is determined.

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I. Introduction

A Mixed quadrature rule of higher degree of precision has been formed by [2], [3], [4], [5], [6], [7]. This rule is meant for single integral. In the same vein here we develop a mixed quadrature rule of degree of precision-7 for double integrals taking the convex combination of Simpson’s $\frac{1}{3}rd$ and Gauss-Legendre-2 point rule each of degree of precision 5. This paper is designed as follows. Section -2 contains formulation of quadrature of constituent rules and the corresponding errors in-2 variables. Section-3 is devoted to construction of mixed quadrature rule. The error analysis is done in section-4. In section-5 the rule is numerically verified by taking suitable examples. The conclusions are drawn in section-6.

II. Formulation of Quadrature Rule in Two Variables X and Y

For approximate evaluation of real definite integral

$$I(f) = \int_{-1}^{1} \int_{-1}^{1} f(x, y) dx dy$$

(2.1)

The Simpson’s $\frac{1}{3}rd$ rule is

$$I(f) \approx R^{*} (f) = \frac{1}{9} \left[ \{f(-1,-1)+4f(0,-1)+f(1,-1)\} 
+ 4\{f(-1,0)+4f(0,0)+f(1,0)\} 
+ \{f(-1,1)+4f(0,1)+f(1,1)\} \right]$$

(2.1)

And the Gauss-Legendre’s two point rule is

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\[ I(f) \equiv R_{GL_2}(f) = f\left(-\frac{1}{\sqrt{3}}, - \frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, - \frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \] (2.3)

where each rule of equation (2.2) and equation (2.3) is of precision 3.

Hence

\[ I(f) = R_{s_1^3}(f) + E_{s_1^3} (f) \] (2.4)

\[ I(f) = R_{GL_2}(f) + E_{GL_2}(f) \] (2.5)

Where \( E_{s_1^3}(f) \) and \( E_{GL_2}(f) \) are error in approximating the integrals \( I(f) \) by equation (2.2) and equation (2.3) respectively.

Now assuming \( f(x,y) \) to be sufficiently differentiable in \(-1 \leq x \leq 1, -1 \leq y \leq 1\)

We can write equation (2.1) using Maclaurin’s expansion

\[
I(f) = \int_{-1}^{1} \int_{-1}^{1} \left[ f(0,0) + \{ x f_{1,0}(0,0) + y f_{0,1}(0,0) \} \\
+ \frac{1}{2!} \{ x^2 f_{2,0}(0,0) + 2xy f_{1,1}(0,0) + y^2 f_{0,2}(0,0) \} \\
+ \frac{1}{3!} \{ x^3 f_{3,0}(0,0) + 3x^2y f_{2,1}(0,0) + 3xy^2 f_{1,2}(0,0) + y^3 f_{0,3}(0,0) \} \\
+ \frac{1}{4!} \left\{ x^4 f_{4,0}(0,0) + 4x^3y f_{3,1}(0,0) + 6x^2y^2 f_{2,2}(0,0) \right\} \\
+ \frac{1}{5!} \left\{ x^5 f_{5,0}(0,0) + 5x^4y f_{4,1}(0,0) + 10x^3y^2 f_{3,2}(0,0) \right\} \\
+ \frac{1}{6!} \left\{ x^6 f_{6,0}(0,0) + 6x^5y f_{5,1}(0,0) + 15x^4y^2 f_{4,2}(0,0) \right\} \\
+ \frac{1}{7!} \left\{ x^7 f_{7,0}(0,0) + 7x^6y f_{6,1}(0,0) + 21x^5y^2 f_{5,2}(0,0) \right\} \\
+ \frac{1}{8!} \left\{ x^8 f_{8,0}(0,0) + 8x^7y f_{7,1}(0,0) + 28x^6y^2 f_{6,2}(0,0) \right\} \\
+ \frac{1}{9!} \left\{ x^9 f_{9,0}(0,0) + 9x^8y f_{8,1}(0,0) + 36x^7y^2 f_{7,2}(0,0) \right\} \\
+ \frac{1}{10!} \left\{ x^{10} f_{10,0}(0,0) + 10x^9y f_{9,1}(0,0) + 45x^8y^2 f_{8,2}(0,0) \right\} \\
+ \frac{1}{11!} \left\{ x^{11} f_{11,0}(0,0) + 11x^{10}y f_{10,1}(0,0) + 66x^9y^2 f_{9,2}(0,0) \right\} \\
+ \frac{1}{12!} \left\{ x^{12} f_{12,0}(0,0) + 12x^{11}y f_{11,1}(0,0) + 90x^{10}y^2 f_{10,2}(0,0) \right\} \\
+ \frac{1}{13!} \left\{ x^{13} f_{13,0}(0,0) + 13x^{12}y f_{12,1}(0,0) + 114x^{11}y^2 f_{11,2}(0,0) \right\} \\
+ \frac{1}{14!} \left\{ x^{14} f_{14,0}(0,0) + 14x^{13}y f_{13,1}(0,0) + 150x^{12}y^2 f_{12,2}(0,0) \right\} \right] \\
dx\,dy \] (2.6)

Integrating equation (2.6) we have

\[ I(f) = 4f_{0,0}(0,0) + \frac{2}{3} [f_{2,0}(0,0) + f_{0,2}(0,0)] + \frac{1}{30} [f_{4,0}(0,0) + f_{0,4}(0,0)] \]

\[ + \frac{1}{9} f_{2,2}(0,0) + \frac{1}{180} [f_{4,2}(0,0) + f_{2,4}(0,0)] + \frac{4}{71} [f_{6,0}(0,0) + f_{0,6}(0,0)] + \ldots \] (2.7)

Again using Maclaurin’s expansion

\[ f(-1,-1) = f_{0,0} - [f_{1,0}(0,0) + f_{0,1}(0,0)] + \frac{1}{2!} [f_{2,0}(0,0) + 2f_{1,1}(0,0) + f_{0,2}(0,0)] \]

\[ - \frac{1}{3!} [f_{3,0}(0,0) + 3f_{2,1}(0,0) + 3f_{1,2}(0,0) + f_{0,3}(0,0)] \]

\[ + \frac{1}{4!} [f_{4,0}(0,0) + 4f_{3,1}(0,0) + 6f_{2,2}(0,0) + 4f_{1,3}(0,0) + f_{0,4}(0,0)] \]
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\[- \frac{1}{5!} \left[ f_{5,0}(0,0) + 5f_{4,1}(0,0) + 10f_{3,2}(0,0) + 10f_{2,3}(0,0) + 5f_{1,4}(0,0) + f_{0,5}(0,0) \right] \]
\[+ \frac{1}{6!} \left[ f_{6,0}(0,0) + 6f_{5,1}(0,0) + 15f_{4,2}(0,0) + 20f_{3,3}(0,0) + 15f_{2,4}(0,0) + 6f_{1,5}(0,0) + f_{0,6}(0,0) \right] \]
\[+ \text{.............} \]

(2.8)

Using equation (2.8) and \( f(0, -1), f(1, -1), f(-1, 0), f(1, 0), f(0, 1), f(-1, 1) \) and \( f(1, 1) \) in equation (2.2)

\[ R_{s_3}(f) = 4f_{0,0}(0,0) + \frac{2}{3} \left\{ f_{2,0}(0,0) + f_{0,2}(0,0) \right\} + \frac{1}{18} \left\{ f_{4,0}(0,0) + f_{0,4}(0,0) \right\} \]
\[+ \frac{1}{9} f_{2,2}(0,0) + \frac{1}{108} \left\{ f_{4,2}(0,0) + f_{2,4}(0,0) \right\} + \frac{11}{9 \times 6!} \left\{ f_{6,0}(0,0) + f_{0,6}(0,0) \right\} \]

(2.9)

a) Error in Simpson’s 1/3rd rule:

Using equation (2.6), equation (2.9) in equation (2.4) error associated with Simpson’s 1/3rd rule is

\[ E_{s_3}(f) = I(f) - R_{s_3}(f) \]
\[ E_{s_3}(f) = - \frac{1}{45} \left\{ f_{4,0}(0,0) + f_{0,4}(0,0) \right\} - \frac{1}{270} \left\{ f_{4,2}(0,0) + f_{2,4}(0,0) \right\} - \]
\[\frac{41}{63 \times 6!} \left\{ f_{6,0}(0,0) + f_{0,6}(0,0) \right\} \]

(2.10)

As the error contains at least fourth derivative of the integral function, it vanishes for all polynomials of degree \( \leq 3 \) in \( x \) and \( y \). Thus the formula becomes exact for all polynomials of degree up to 3 i.e. degree of precision of the formula is three.

Similarly using Maclaurin’s expansion

We have

\[ f\left( -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) \]
\[= f(0,0) - \frac{1}{\sqrt{3}} \left[ f_{1,0}(0,0) + f_{0,1}(0,0) \right] \]
\[+ \frac{1}{2!} \left[ \left(-\frac{1}{\sqrt{3}}\right)^2 f_{2,0}(0,0) + 2 \left(-\frac{1}{\sqrt{3}}\right)^2 f_{1,1}(0,0) + \left(-\frac{1}{\sqrt{3}}\right)^2 f_{0,2}(0,0) \right] \]
\[+ \frac{1}{3!} \left[ \left(-\frac{1}{\sqrt{3}}\right)^3 f_{3,0}(0,0) + 3 \left(-\frac{1}{\sqrt{3}}\right)^3 f_{2,1}(0,0) + 3 \left(-\frac{1}{\sqrt{3}}\right)^3 f_{1,2}(0,0) + \left(-\frac{1}{\sqrt{3}}\right)^3 f_{0,3}(0,0) \right] \]
\[ \frac{1}{4!} \left[ \left( -\frac{1}{\sqrt{3}} \right)^4 f_{4,0} + 4 \left( -\frac{1}{\sqrt{3}} \right)^4 f_{3,1} + 6 \left( -\frac{1}{\sqrt{3}} \right)^4 f_{2,2} + 4 \left( -\frac{1}{\sqrt{3}} \right)^4 f_{1,3} + \left( -\frac{1}{\sqrt{3}} \right)^4 f_{0,4} \right] \]

\[ + \frac{1}{5!} \left[ \left( -\frac{1}{\sqrt{3}} \right)^5 f_{5,0} + 5 \left( -\frac{1}{\sqrt{3}} \right)^5 f_{4,1} + 10 \left( -\frac{1}{\sqrt{3}} \right)^5 f_{3,2} + 5 \left( -\frac{1}{\sqrt{3}} \right)^5 f_{2,3} + \left( -\frac{1}{\sqrt{3}} \right)^5 f_{1,4} + \left( -\frac{1}{\sqrt{3}} \right)^5 f_{0,5} \right] \]

\[ + \frac{1}{6!} \left[ \left( -\frac{1}{\sqrt{3}} \right)^6 f_{6,0} + 6 \left( -\frac{1}{\sqrt{3}} \right)^6 f_{5,1} + 15 \left( -\frac{1}{\sqrt{3}} \right)^6 f_{4,2} + 6 \left( -\frac{1}{\sqrt{3}} \right)^6 f_{3,3} + \left( -\frac{1}{\sqrt{3}} \right)^6 f_{2,4} + \left( -\frac{1}{\sqrt{3}} \right)^6 f_{1,5} + \left( -\frac{1}{\sqrt{3}} \right)^6 f_{0,6} \right] \]

\[ \begin{array}{c}
\text{Now } f\left( -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), f\left( \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), f\left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), f\left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \text{can be found out same ways as } (2.10) \\
\text{Now substituting the value of } \end{array} \]

\[ f\left( -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), f\left( \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), f\left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), f\left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \]

in equation (2.3)

\[ R_{GL2}(f) = 4f(0,0) + \frac{2}{3} \left[ f_{2,0}(0,0) + f_{0,2}(0,0) \right] + \frac{1}{9 \times 3!} \left[ f_{4,0}(0,0) + f_{0,4}(0,0) \right] \]

\[ + \frac{20}{9 \times 6!} \left[ f_{4,2}(0,0) + f_{2,4}(0,0) \right] + \frac{1}{9} f_{2,2}(0,0) + \frac{4}{27 \times 6!} \left[ f_{6,0}(0,0) + f_{0,6}(0,0) \right] \]

\[ (2.12) \]

b) Error in Gauss-Legendre's two point rule

Now the error associated with the Gauss- Legendre 2-point rule is obtained substituting equation (2.6) and (2.11) in equation (2.5)

\[ E_{GL2}(f) = I(f) - R_{GL2}(f) \]

\[ = \frac{2}{135} \left[ f_{4,0}(0,0) + f_{0,4}(0,0) \right] + \frac{1}{405} \left[ f_{4,2}(0,0) + f_{2,4}(0,0) \right] + \frac{1}{1701} \left[ f_{6,0}(0,0) + f_{0,6}(0,0) \right] \]

\[ (2.13) \]
In this case also, the error contains at least fourth derivative of the integral function. Thus the degree of the precision is 3.

**III. Mixed Quadrature Rule**

Now multiplying (2) by equation (2.10) and (3) by equation (2.13) and adding them, we get

\[
I(f) = \frac{1}{5} \left[ 2R_{\frac{1}{3}}(f) + 3R_{GL2}(f) \right] + \frac{1}{5} \left[ 2E_{\frac{1}{3}}(f) + 3E_{GL2}(f) \right]
\]  

(3.1)

Where

\[
I_{mix}(f) = R_{\frac{1}{3}GL2}(f) + E_{\frac{1}{3}GL2}(f)
\]

(3.2)

Where

\[
R_{\frac{1}{3}GL2}(f) = \frac{1}{5} \left[ 2R_{\frac{1}{3}}(f) + 3R_{GL2}(f) \right]
\]

(3.3)

Where \( R_{\frac{1}{3}GL2}(f) \) and \( E_{\frac{1}{3}GL2}(f) \) are mixed quadrature rule and its error obtained by Simpson’s \( \frac{1}{3} \)rd and Gauss-Legendre’s 2 point rule respectively.

The truncation error generated by this approximation is given by

\[
E_{\frac{1}{3}GL2} = \frac{1}{5} \left[ 2E_{\frac{1}{3}}(f) + 3R_{GL2}(f) \right] = \frac{1}{189 \times 5!} \left[ f_{0,0}(0,0) + f_{0,0}(0,0) \right]
\]

(3.4)

The rule (3.3) may be called as mixed quadrature rule integrate exactly all polynomial of degree \( \leq 5 \) in x and y.

**IV. Error Analysis**

a) **Theorem 4.1**

Let \( f(x,y) \) be sufficiently differentiable function in the closed interval \([-1,1]\). The bounds of truncational error \( E_{\frac{1}{3}GL2}(f) \) associated with the rule \( R_{\frac{1}{3}GL2}(f) \) is given by

\[
\left| E_{\frac{1}{3}GL2}(f) \right| \leq \frac{1}{189 \times 5!} \left[ f_{0,0}(0,0) + f_{0,0}(0,0) \right]
\]

**Proof:**

The proof obviously follows from the equation (3.4)

**Theorem 4.2**

The bounds for the truncational error \( \left| E_{\frac{1}{3}GL2}(f) \right| \) is given by

\[
\left| E_{\frac{1}{3}GL2}(f) \right| \leq \frac{2M}{225} |\eta_2 - \eta_1|
\]

Where \( \eta, \eta_2 \in [-1,1] \)

Where \( M = \text{Max} \left| f_{0,0}(0,0) + f_{0,0}(0,0) \right| \quad -1 \leq x \leq 1, -1 \leq y \leq 1 \)
**Proof:**

We have

\[ E_{s_{1/2}}(f) = -\frac{1}{45} [f_{4,0}(n_1) + f_{0,4}(n_1)] \]

\[ E_{GL2}(f) = \frac{2}{135} [f_{4,0}(n_2) + f_{0,4}(n_2)] \]

Where

\[ n_1, n_2 \in [-1,1] \]

Hence

\[ E_{s_{1/2}GL2}(f) = \frac{1}{5} \left| 2E_{s_{1/2}}(f) + 3E_{GL2}(f) \right| \]

\[ = \frac{2}{225} \left[ f_{4,0}(n_2,0) + f_{0,4}(0,n_2) - f_{4,0}(n_1,0) - f_{0,4}(0,n_1) \right] \]

\[ = \frac{2}{225} \int \int_{n_1,n_2} \left[ f_{5,0}(x,*) + f_{0,5}(*,y) \right] \]

Hence

\[ |E_{s_{1/2}GL2}(f)| \leq \frac{2M|n_1 - n_2|}{225} \]

Where

\[ M = \max \left| f_{5,0}(x,*) + f_{0,5}(*,y) \right| \quad -1 \leq x \leq 1, \quad -1 \leq y \leq 1 \]

Which, gives only the truncation error bound on \( n_1, n_2 \) are closed to each other.

**b) Corollary**

The error bound for the truncated error \( |E_{s_{1/2}GL2}(f)| \leq \frac{4M}{225} \)

When \( |n_1 - n_2| \leq 2 \quad [1] \)

**V. Numerical Verification**

The exact value of the integrals

\[ I_1 = \int \int_{-1}^{1} e^{x+y} \, dxdy = 5.524391382167262 \]

\[ I_1 = \int \int_{-1}^{1} e^{-(x+y)^2} \, dxdy = 2.230985141404134 \]

\[ I_3 = \int \int_{0}^{1} \frac{\sin^2(x+y)}{(x+y)} \, dxdy = 0.613260369981918 \]
VI. Conclusion

From the Table -1, we find that

\[ |E_{s_3}(f)| \leq |E_{GLS}(f)| \] (i)

\[ |E_{s_3GL2}(f)| \leq |E_{s_1}(f)| \] (ii)

From (i) and (ii)

\[ |E_{s_3GL2}(f)| \leq |E_{s_1}(f)| \leq |E_{GLS}(f)| \]

It is evident that the mixed quadrature rule \( R_{s_3GL2}(f) \) of degree of precision 5 provides us better result than constituent rule \( R_{s_3}(f), R_{GL2}(f) \) each of degree of precision three. Hence, the mixed quadrature rule is more efficient and numerically better convergent than exact result.

References Références Referencias
