The Univalence of A Generalized Integral Operator

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Abstract- For analytic function $f_{ij}, j=1, n$, in the open disk $U$, an integral operator $K_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}$ is introduced. In this paper we obtain the conditions of the univalence for the integral operator $K_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}$.

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The Univalence of a Generalized Integral Operator

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I. Introduction

Let $A$ be the class of functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. Let $S$ denote the subclass of $A$ consisting of the functions $f \in A$, which are univalent in $U$. We denote by $P$ the class of functions $p$ which are analytic in $U$, $p(0) = 1$ and $Re(p(z)) > 0$, for all $z \in U$. In this work, we introduce a new integral operator, which is given by

$$K_{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n}(z) = \int_0^z \prod_{j=1}^{n} \left( \frac{D^m f_j(u)}{u} \right)^{\alpha_j} \left( \frac{D^n f_j(u)}{u} \right)^{\beta_j} du$$

for $\alpha_j, \beta_j$ be complex numbers, $f_i \in A, f_j' \in P, j = 1, n$.

For $m = 1 \ beta_j = 0$ $j = 1, n$ we obtain the integral operator, which is defined in [4].

For $m = 1 \ alpha_j = 0$ $j = 1, n$ we have the integral operator, which is defined in [5].

II. Preliminary Results

In order to prove our main results we will need the following lemmas.

Lemma 2.1 [1] If the function $f$ is analytic in $U$ and

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} - 1 \right| \leq 1 \quad (2)$$

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for all \( z \in U \), then the function \( f \) is univalent in \( U \).

**Lemma 2.2** [3] Let \( \gamma \) be a complex number \( \Re \gamma > 0 \) and \( f \in A \). If

\[
1 - \left| \frac{z^{2 \Re \gamma}}{\Re \gamma} \right| \left| \frac{zf''(z)}{f'(z)} \right| \leq 1
\]

for all \( z \in U \), then for any complex number \( \delta, \Re \delta \geq \Re \gamma \), the function

\[
f_\delta(z) = \left[ \delta \int_0^z u^{\delta - 1} f'(u) du \right]^{1/\delta}
\]

is regular and univalent in \( U \).

**Lemma 2.3** (Schwarz [2]) Let \( f \) be the function regular in the disk \( U_R = \{ z \in C : |z| < R \} \) with \( |f(z)| < M, M \) fixed. If \( f \) has in \( z = 0 \) one zero multiply \( \geq m \) then

\[
|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in U_R)
\]

the equality (in the inequality (5) \( z \neq 0 \)) can hold only if

\[
f(z) = e^{i \theta} \frac{M}{R^m} z^m
\]

where \( \theta \) is constant.

**III. Main Results**

**Theorem 3.1:** Let \( \alpha_j, \beta_j \) be the complex numbers \( M_j, L_j \) positive real numbers, \( j = 1, n \) and the functions

\[
f_j \in A, f_j' \in P, f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + ... + a_{j}z^j \quad (j = 1, n)
\]

and

\[
\left| z \frac{D^{m+1}f_j(z)}{D^m f_j(z)} - 1 \right| \leq M_j \quad (j = 1, n : z \in U)
\]

\[
\left| z \frac{D^n f_j(z)^{n''}}{D^n f_j(z)^n} \right| \leq L_j \quad (j = 1, n : z \in U)
\]

and

\[
\sum_{j=1}^{n} [|\alpha_j| M_j + |\beta_j| L_j] \leq \frac{3\sqrt{3}}{2}
\]

Then the integral operator \( K_{\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n} \) defined by (1) is in the class \( S \).

**Proof:** The function \( K_{\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n}(z) \) is regular in \( U \) and

\[
K_{\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n}(0) = K'_{\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n}(0) - 1 = 0
\]
we have
\[ \frac{zK''_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)}{K'_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)} = \sum_{j=1}^{n} \left[ \alpha_j \left( \frac{D^{m+1} f_j(z)}{D^m f_j(z)} - 1 \right) \right] + \sum_{j=1}^{n} \left[ \beta_j z^{D^n f_j(z)} \right] \]
(9)
and hence we get
\[ (1 - |z|^2) \left| \frac{zK''_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)}{K'_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)} \right| \leq (1 - |z|^2) \sum_{j=1}^{n} \left[ |\alpha_j| \left| z \frac{D^{m+1} f_j(z)}{D^m f_j(z)} - 1 \right| + |\beta_j| z^{D^n f_j(z)} \right] \]
(10)
for all \( z \in U \).

By (6), (7) and Lemma 2.3, we obtain
\[ \left| z \frac{D^{m+1} f_j(z)}{D^m f_j(z)} - 1 \right| \leq M_j |z| \quad (j = 1, n : z \in U) \]
(11)
\[ \left| z^{D^n f_j(z)} \right| \leq L_j |z| \quad (j = 1, n : z \in U) \]
(12)
and from (10) we have
\[ (1 - |z|^2) \left| \frac{zK''_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)}{K'_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)} \right| \leq (1 - |z|^2) |z| \left\{ \sum_{j=1}^{n} |\alpha_j| M_j + |\beta_j| L_j \right\} \]
(13)
for all \( z \in U \).

Since
\[ \max_{|z|<1} (1 - |z|^2) |z| = \frac{2}{3\sqrt{3}} \]
from (8) and (13) we get
\[ (1 - |z|^2) \left| \frac{zK''_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)}{K'_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)} \right| \leq 1, \quad (z \in U) \]
and by Lemma 2.1, it results that the integral operator \( K_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n} \) is in the class \( S \).

**Theorem 3.2:** Let \( \alpha_j, \beta_j, \gamma \) be the complex numbers \( j = 1, n \), \( 0 < Re \gamma \leq 1 \) and the functions
\[ f_j \in A, f_j' \in P, f_j(z) = z + a_2 z^2 + a_3 z^3 + \ldots, j = 1, n \]
if
\[ \left| z \frac{D^{m+1} f_j(z)}{D^m f_j(z)} - 1 \right| \leq \frac{(2 Re \gamma + 1)^{\frac{2 Re + 1}{Re \gamma}}}{2} \quad (j = 1, n : z \in U) \]
(14)
where \( D^n f_j(z) \) and \( D^n f_j(z) \) are given by (14), (15) and Lemma 2.3, we have

\[
\frac{z K''_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)}{K'_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)} = \sum_{j=1}^{n} \left[ \alpha_j \left( \frac{z D^{m+1} f_j(z)}{D^m f_j(z)} - 1 \right) \right] + \sum_{j=1}^{n} \left[ \beta_j \frac{z [D^n f_j(z)]''}{[D^n f_j(z)]'} \right]
\]

and hence we get

\[
1 - |z|^{2 \Re \gamma} \frac{z K''_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)}{K'_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)} \leq 1 - |z|^{2 \Re \gamma} \sum_{j=1}^{n} |\alpha_j| \left| \frac{z D^{m+1} f_j(z)}{D^m f_j(z)} - 1 \right| + |\beta_j| \left| \frac{z [D^n f_j(z)]''}{[D^n f_j(z)]'} \right|
\]

for all \( z \in U \) by (14), (15) and Lemma 2.3 we have

\[
\left| \frac{z D^{m+1} f_j(z)}{D^m f_j(z)} - 1 \right| \leq \frac{(2 \Re \gamma + 1)}{2} \left| z \right| \quad (j = 1, n : z \in U) \quad (18)
\]

\[
\left| \frac{z [D^n f_j(z)]''}{[D^n f_j(z)]'} \right| \leq \frac{(2 \Re \gamma + 1)}{2} \left| z \right| \quad (j = 1, n : z \in U) \quad (19)
\]

and hence by (17) we get

\[
1 - |z|^{2 \Re \gamma} \frac{z K''_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)}{K'_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)} \leq 1 - |z|^{2 \Re \gamma} |z| \frac{(2 \Re \gamma + 1)}{2} \sum_{j=1}^{n} |\alpha_j| + |\beta_j| \quad \left| z \right| \leq 1 \quad (20)
\]

for all \( z \in U \).

\[
\max_{|z| \leq 1} \left[ 1 - |z|^{2 \Re \gamma} |z| \right] = \frac{2}{(2 \Re \gamma + 1)^{2 \Re \gamma}}
\]

From (16) and (20) we obtain that

\[
1 - |z|^{2 \Re \gamma} \frac{z K''_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)}{K'_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(z)} \leq 1 \quad (21)
\]
for all $z \in U$ and by Lemma 2.2 for $\delta = 1$ and $f = K_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}$ it results that the integral operator $K_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}$ defined by (1) belongs to the class $S$.

IV. Corollaries

**Corollary 4.1:** Let $\alpha_j$ be the complex numbers $M_j$ positive real numbers, $j = 1, n$ and the functions

$$f_j \in A, f'_j \in P, f_j(z) = z + a_{2j} z^2 + a_{3j} z^3 + \ldots, j = 1, n$$

if

$$\left| z \frac{D^{m+1} f_j(z)}{D^m f_j(z)} - 1 \right| \leq M_j \quad (j = 1, n \quad : \quad z \in U) \quad (22)$$

and

$$\sum_{j=1}^{n} |\alpha_j| M_j \leq \frac{3\sqrt{3}}{2} \quad (23)$$

then the function

$$G_{\alpha_1, \ldots, \alpha_n}(z) = \int_0^z \prod_{j=1}^{n} \left( \frac{D^m f_i(u)}{u} \right)^{\alpha_j} du$$

is in the class $S$.

**Corollary 4.2:** Let $\beta_j$ be the complex numbers $L_j$ positive real numbers, $j = 1, n$ and the functions

$$f_j \in A, f'_j \in P, f_j(z) = z + a_{2j} z^2 + a_{3j} z^3 + \ldots, j = 1, n$$

and

$$\left| z \frac{[D^n f_j(z)]''}{[D^n f_j(z)]'} \right| \leq L_j \quad (j = 1, n \quad : \quad z \in U) \quad (24)$$

and

$$\sum_{j=1}^{n} |\beta_j| L_j \leq \frac{3\sqrt{3}}{2} \quad (25)$$

then the function

$$H_{\beta_1, \ldots, \beta_n}(z) = \int_0^z \prod_{j=1}^{n} \left( (D^n f_i(u))' \right)^{\beta_j} du$$

belongs to the class $S$. 

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**Notes**

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**Corollary 4.1:** Let $\alpha_j$ be the complex numbers $M_j$ positive real numbers, $j = 1, n$ and the functions

$$f_j \in A, f'_j \in P, f_j(z) = z + a_{2j} z^2 + a_{3j} z^3 + \ldots, j = 1, n$$

if

$$\left| z \frac{D^{m+1} f_j(z)}{D^m f_j(z)} - 1 \right| \leq M_j \quad (j = 1, n \quad : \quad z \in U) \quad (22)$$

and

$$\sum_{j=1}^{n} |\alpha_j| M_j \leq \frac{3\sqrt{3}}{2} \quad (23)$$

then the function

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and

$$\sum_{j=1}^{n} |\beta_j| L_j \leq \frac{3\sqrt{3}}{2} \quad (25)$$

then the function

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belongs to the class $S$. 

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