On Dynamical Systems Induced by the Adele Ring

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1. Introduction

Continued from [10], in this paper, we consider how primes (or prime numbers) act on operator algebras. In particular, instead of acting each $p$-adic number fields $\mathbb{Q}_p$ to operator algebras, for every prime $p$, we act the Adele ring $\mathbb{A}_\mathbb{Q}$ on operator algebras. In [10], we act $p$-adic number fields $\mathbb{Q}_p$ on a given von Neumann algebra $M$, and construct a corresponding dynamical system generating its crossed product algebra. We have studied fundamental properties of such dynamical systems and crossed product $W^*$-algebras. Also, by applying free probability, we considered free-distributional data of certain operators. Here, based on results of [10], we act the Adele ring $\mathbb{A}_\mathbb{Q}$ on $M$.

The relations between primes and operator algebras have been studied in various different approaches. The main purposes of finding such relations are (i) to provide new tools for studying operator algebras, (ii) to apply operator-algebraic techniques (for example, free probability) to study number theory, and hence, (iii) to establish bridges between number theory and operator algebra theory. In [4], we studied how primes act “on” certain von Neumann algebras. Also, the primes as operators in certain von Neumann algebras have been studied, too, in [5] and [8]. In [6] and [7], we have studied primes as linear functionals acting on arithmetic functions. i.e., each prime induces a free-probabilistic structure on arithmetic functions. In such a case, one can understand arithmetic functions as Krein-space operators (for fixed primes), via certain representations (See [11] and [12]). These studies are all moti-vated by well-known number-theoretic results under free probability techniques.

Arveson studied histories as a group of actions induced by real numbers $\mathbb{R}$ on (type I subfactors of) $B(H)$, satisfying certain additional conditions, where $H$ is an infinite dimensional separable Hilbert space (e.g., [1], [2] and cited papers therein). By understanding the field $\mathbb{R}$ as an additive group $(\mathbb{R}, +)$, he defined an $\mathcal{E}_0$-group $\Gamma_\mathbb{R}$ of $\ast$-homomorphisms acting on $B(H)$ indexed by $\mathbb{R}$. By putting additional conditions on $\Gamma_\mathbb{R}$, he defined a history $\Gamma$ acting on $B(H)$. We mimic Arveson’s con-

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construction to establish our dynamical systems and corresponding crossed product algebras (e.g., [8], [9] and [10]).

In [9], by framing (e.g., also see [8]), a group $\Gamma$ to groupoids generated by partial isometries, we studied possible distortions $\Gamma_G$ of a history $\Gamma$. It shows that whenever a history $\Gamma$ acts on $H$, a family of partial isometries distorts (or reduces, or restricts) the “original” historical property (in the sense of Arveson) of $\Gamma$. And such distortions are completely characterized by groupoid actions, sometimes called the $E_0$-groupoid actions induced by partial isometries on $B(H)$. The above framed ($E_0$-)groupoids $\Gamma_G$ induce corresponding $C^\ast$-subalgebras $C^\ast(\Gamma_G)$ of $B(H)$, investigated by dynamical system theory and free probability (e.g., [15], [16] and [17]).

Independently, $p$-adic analysis provides an important tool for studying geometry at small distance (e.g., [18]). It is not only interesting in various mathematical fields but also in physics (e.g., [3], [4], [5] and [18]). The $p$-adic number fields (or $p$-prime fields) $\mathbb{Q}_p$ and the Adele ring $\mathbb{A}_Q$ play key roles in modern number theory, analytic number theory, $L$-function theory, and algebraic geometry (e.g., [3], [13] and [14]). Also, analysis on such Adelic structures gives a way for understanding small-distance-measured geometry (e.g., [18]) and vector analysis under non-Archimedean metric (e.g., [5]). Thus, prime fields and the Adele ring are interesting topics both in mathematics and in other scientific fields.

We attempt to combine the above two topics; dynamical systems and $p$-adic analysis under Adelic settings; which seem independent from each other.

II. Definitions and Background

In this section, we introduce basic definitions and backgrounds of the paper.

a) $p$-Adic Number Fields $\mathbb{Q}_p$ and The Adele Ring $\mathbb{A}_Q$. Fundamental theorem of arithmetic says that every positive integer in $\mathbb{N}$ except 1 can be expressed as a usual multiplication of primes (or prime numbers), equivalently, all positive integers which are not 1 are prime-factorized under multiplication. i.e., the primes are the building blocks of all positive integers except for 1. Thus, it is trivial that primes are playing key roles in both classical and advanced number theory.

The Adele ring $\mathbb{A}_Q$ is one of the main topics in advanced number theory connected with other mathematical fields like algebraic geometry and $L$-function theory, etc.

Throughout this paper, we denote the set of all natural numbers (which are positive integers) by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, and the set of all rational numbers by $\mathbb{Q}$.

Let’s fix a prime $p$. Define the $p$-norm $|\cdot|_p$ on the rational numbers $\mathbb{Q}$ by

$$|q|_p = |p^{\text{ord}_p q}|_p \overset{\text{def}}{=} \frac{1}{p^{\text{ord}_p q}},$$

whenever $q = p^{\text{ord}_p q} \in \mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$, for some $r \in \mathbb{Z}$, with an additional identity:

$$|0|_p \overset{\text{def}}{=} 0 \text{ (for all primes } p).$$

For example,

$$|\frac{-24}{5}|_2 = |2^3 \cdot (-\frac{3}{5})|_2 = \frac{1}{2^3} = \frac{1}{8}.$$

It is easy to check that

(i) $|q|_p \geq 0$, for all $q \in \mathbb{Q}$,

(ii) $|q_1 q_2|_p = |q_1|_p \cdot |q_2|_p$, for all $q_1, q_2 \in \mathbb{Q}$

(iii) $|q_1 + q_2|_p \leq \max\{|q_1|_p, |q_2|_p\}$,

for all $q_1, q_2 \in \mathbb{Q}$. In particular, by (iii), we verify that

(iii)$'$ $|q_1 + q_2|_p \leq |q_1|_p + |q_2|_p,$

for all $q_1, q_2 \in \mathbb{Q}$. Thus, by (i), (ii) and (iii)$'$, the $p$-norm $|\cdot|_p$ is indeed a norm. However, by (iii), this norm is “non-Archimedean.”
i. **Definition 2.1.** We define a set \( \mathbb{Q}_p \) by the norm-closure of the normed space \( (\mathbb{Q}, |\cdot|_p) \), for all primes \( p \). We call \( \mathbb{Q}_p \), the \( p \)-adic number field.

For a fixed prime \( p \), all elements of \( \mathbb{Q}_p \) are formed by

\[
p^r \left( \sum_{k=0}^{\infty} a_k p^k \right), \quad 0 \leq a_k < p,
\]

(2.1.1)

for all \( k \in \mathbb{N} \), and for all \( r \in \mathbb{Z} \). For example,

\[-1 = (p - 1)p^0 + (p - 1)p + (p - 1)p^2 + \cdots.\]

The subset of \( \mathbb{Q}_p \), consisting of all elements formed by

\[
\sum_{k=0}^{\infty} a_k p^k, \quad 0 \leq a_k < p \text{ in } \mathbb{N},
\]

is denoted by \( \mathbb{Z}_p \). i.e., for any \( x \in \mathbb{Q}_p \), there exist \( r \in \mathbb{Z} \), and \( x_0 \in \mathbb{Q}_p \), such that

\[x = p^r x_0.\]

Notice that if \( x \in \mathbb{Z}_p \), then \( |x|_p \leq 1 \), and vice versa. i.e.,

\[\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \}.\]

(2.1.2)

So, the subset \( \mathbb{Z}_p \) of (2.1.2) is said to be the unit disk of \( \mathbb{Q}_p \). Remark that

\[\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset p^3\mathbb{Z}_p \supset \cdots.\]

It is not difficult to verify that

\[\mathbb{Z}_p \subset p^{-1}\mathbb{Z}_p \subset p^{-2}\mathbb{Z}_p \subset p^{-3}\mathbb{Z}_p \subset \cdots;\]

and hence

\[\mathbb{Q}_p = \bigcup_{k=-\infty}^{\infty} p^k \mathbb{Z}_p, \text{ set-theoretically.}\]

(2.1.3)

Consider the boundary \( U_p \) of \( \mathbb{Z}_p \). By construction, the boundary \( U_p \) of \( \mathbb{Z}_p \) is identical to \( \mathbb{Z}_p \setminus p\mathbb{Z}_p \), i.e.,

\[U_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p = \{ x \in \mathbb{Z}_p : |x|_p = 1 = p^0 \}.\]

(2.1.4)

Similarly, the subsets \( p^k U_p \) are the boundaries of \( p^k \mathbb{Z}_p \) satisfying

\[p^k U_p = p^k \mathbb{Z}_p \setminus p^{k+1} \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}.\]

We call the subset \( U_p \) of \( \mathbb{Q}_p \) in (2.1.4) the unit circle of \( \mathbb{Q}_p \). And all elements of \( U_p \) are said to be units of \( \mathbb{Q}_p \).

Therefore, by (2.1.3) and (2.1.4), we obtain that

\[\mathbb{Q}_p = \bigcup_{k=-\infty}^{\infty} p^k U_p, \text{ set-theoretically,}\]

(2.1.5)

where \( \sqcup \) means the disjoint union. By [18], whenever \( q \in \mathbb{Q}_p \) is given, there always exist \( a \in \mathbb{Q} \), \( k \in \mathbb{Z} \), such that

\[q = a + p^k \mathbb{Z}_p, \text{ for } a, k \in \mathbb{Z}.\]

**Fact** (See [18]) The \( p \)-adic number field \( \mathbb{Q}_p \) is a Banach space. And it is locally compact. In particular, the unit disk \( \mathbb{Z}_p \) is compact in \( \mathbb{Q}_p \). \( \square \)

Define now the addition on \( \mathbb{Q}_p \) by

\[
(\sum_{n=-\infty}^{-N_1} a_n p^n) + (\sum_{n=-\infty}^{-N_2} b_n p^n) = \sum_{n=-\max\{N_1, N_2\}}^{\infty} c_n p^n,
\]

(2.1.6)

for \( N_1, N_2 \in \mathbb{N} \), where the summands \( c_n p^n \) satisfies that

\[
c_n p^n \overset{\text{def}}{=} \begin{cases} 
(a_n + b_n) p^n & \text{if } a_n + b_n < p \\
p^{n+1} & \text{if } a_n + b_n = p \\
2n p^{n+1} + r_n p^n & \text{if } a_n + b_n = s_n p + r_n,
\end{cases}
\]
for all \( n \in \{-\max\{N_1, N_2\}, \ldots, 0, 1, 2, \ldots\} \). Clearly, if \( N_1 > N_2 \) (resp., \( N_1 < N_2 \)), then, for all \( j = -N_1, \ldots, -(N_1 - N_2 + 1) \) (resp., \( j = -N_2, \ldots, -(N_2 - N_1 + 1) \)),

\[
c_j = a_j \quad \text{(resp., } c_j = b_j)\].

And define the multiplication “on \( \mathbb{Z}_p \)” by

\[
(\sum_{k_1=0}^{\infty} a_{k_1} p^{k_1})(\sum_{k_2=0}^{\infty} b_{k_2} p^{k_2}) = \sum_{n=-N}^{\infty} c_n p^n,
\]

where

\[
c_n = \sum_{k_1+k_2=n} \left( r_{k_1,k_2} i_{k_1,k_2} + s_{k_1-1,k_2} i_{k_1-1,k_2} + s_{k_1,k_2-1} i_{k_1,k_2-1} + s_{k_1-1,k_2-1} i_{k_1-1,k_2-1} \right),
\]

where

\[
a_{k_1} b_{k_2} = s_{k_1,k_2} + r_{k_1,k_2},
\]

by the division algorithm, and

\[
i_{k_1,k_2} = \begin{cases} 
1 & \text{if } a_{k_1} b_{k_2} < p \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
i_{k_1,k_2} = 1 - i_{k_1,k_2},
\]

for all \( k_1, k_2 \in \mathbb{N} \), and hence, “on \( \mathbb{Q}_p \),” the multiplication is extended to

\[
(\sum_{k_1=-N_1}^{\infty} a_{k_1} p^{k_1})(\sum_{k_2=-N_2}^{\infty} b_{k_2} p^{k_2}) = (p^{-N_1})(p^{-N_2})(\sum_{k_1=0}^{\infty} a_{k_1} p^{k_1}\mathbb{Z}_p)(\sum_{k_2=0}^{\infty} b_{k_2} p^{k_2}\mathbb{Z}_p).
\]

Then, under the addition (2.1.6) and the multiplication (2.1.7), the algebraic triple \((\mathbb{Q}_p, +, \cdot)\) becomes a field, for all primes \( p \). Thus the \( p \)-prime fields \( \mathbb{Q}_p \) are algebraically fields.

**Fact** Every \( p \)-acid number field \( \mathbb{Q}_p \), with the binary operations (2.1.6) and (2.1.7), is indeed a field. \( \Box \)

Moreover, the Banach filed \( \mathbb{Q}_p \) is also a (unbounded) Haar-measure space \((\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \rho_p)\), for all primes \( p \), where \( \sigma(\mathbb{Q}_p) \) means the \( \sigma \)-algebra of \( \mathbb{Q}_p \), consisting of all measurable subsets of \( \mathbb{Q}_p \). Moreover, this measure \( \rho_p \) satisfies that

\[
\rho_p \left( a + p^k \mathbb{Z}_p \right) = \rho_p \left( p^k \mathbb{Z}_p \right) = \frac{1}{p^k},
\]

for all \( a \in \mathbb{Q} \), and \( k \in \mathbb{Z} \), where \( \mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\} \). Also, one has

\[
\rho_p (a + p^k \mathbb{U}_p) = \rho_p (p^k \mathbb{Z}_p) = \rho_p (p^k \mathbb{Z}_p \setminus p^{k+1} \mathbb{Z}_p) = \frac{1}{p^k} - \frac{1}{p^{k+1}},
\]

for all \( a \in \mathbb{Q} \). Similarly, we obtain that

\[
\rho_p (a + p^k \mathbb{U}_p) = \rho (p^k \mathbb{U}_p) = \frac{1}{p^k} - \frac{1}{p^{k+1}},
\]

for all \( a \in \mathbb{N} \), and \( k \in \mathbb{Z} \) (See Chapter IV of [18]).

**Fact** The Banach field \( \mathbb{Q}_p \) is an unbounded Haar-measure space, where \( \rho_p \) satisfies (2.1.8) and (2.1.9), for all primes \( p \). \( \Box \)

The above three facts show that \( \mathbb{Q}_p \) is a unbounded Haar-measured, locally compact Banach field, for all primes \( p \).
ii. Definition 2.2. Let \( P = \{ \text{all primes} \} \cup \{ \infty \} \). The Adele ring \( \mathbb{A}_Q = (\mathbb{A}_Q, +, \cdot) \) is defined by the set

\[
\{(x_p)_{p \in P} : x_p \in \mathbb{Q}_p, \text{almost all } x_p \in \mathbb{Z}_p, x_\infty \in \mathbb{R}\},
\]

with identification \( \mathbb{Q}_\infty = \mathbb{R} \), and \( \mathbb{Z}_\infty = [0, 1] \), the closed interval in \( \mathbb{R} \), equipped with

\[
(x_p)_p + (y_p)_p = (x_p + y_p)_p, \quad \text{and} \quad (x_p)_p(y_p)_p = (x_p y_p)_p,
\]

for all \( (x_p)_p, (y_p)_p \in \mathbb{A}_Q \).

Indeed, this algebraic structure \( \mathbb{A}_Q \) forms a ring. Also, by the algebraic construction and the product topology, the Adele ring \( \mathbb{A}_Q \) is also a locally compact Banach space equipped with the product measure. Set-theoretically,

\[
\mathbb{A}_Q \subseteq \prod_{p \in P} \mathbb{Q}_p = \mathbb{R} \times \left( \prod_{p \text{prime}} \mathbb{Q}_p \right).
\]

In fact, the Adele ring \( \mathbb{A}_Q \) is a weak direct product \( \prod_{p \in \mathbb{P}} \mathbb{Q}_p \) of \( \{ \mathbb{Q}_p \}_{p \in \mathbb{P}} \), i.e.,

\[
\mathbb{A}_Q = \prod_{p \in P} \mathbb{Q}_p,
\]

i.e., whenever \( (x_p)_p \in \mathbb{A}_Q \), almost all \( x_q \) are in \( \mathbb{Q}_q \), for primes \( q \), except for finitely many \( x_p \).

The product measure \( \rho \) of the Adele ring \( \mathbb{A}_Q \) is given:

\[
\rho = \times_{p \in P} \rho_p,
\]

with identification \( \rho_\infty = \rho_\mathbb{R} \), the usual distance-measure (induced by \( |\cdot|_\infty \)) on \( \mathbb{R} \).

**Fact** The Adele ring \( \mathbb{A}_Q \) is a unbounded-measured locally compact Banach ring.

\( \square \)

\( b) \) **Dynamical Systems Induced by Algebraic Structures:** In this section, we briefly discuss about dynamical systems induced by algebraic structures. Let \( X \) be an arbitrary algebraic structures, i.e., \( X \) is a semigroup, or a group, or a groupoid, or an algebra, etc (maybe equipped with topology).

Let \( M \) be an algebra over \( \mathbb{C} \), and assume there exists a well-defined action \( \alpha \) of \( X \) acting on \( M \), i.e., \( \alpha(x) \) is a well-defined function on \( X \), satisfying that:

\[
\alpha(x_1 \cdot x_2) = \alpha(x_1) \circ \alpha(x_2) \text{ on } M,
\]

for all \( x_1, x_2 \in X \), where \( x_1 \cdot x_2 \) means the operation on \( X \), and \( \circ \) means the usual functional composition. For convenience, we denote \( \alpha(x) \) simply by \( \alpha_x \), for all \( x \in X \).

Then the triple \( (X, M, \alpha) \) is called the dynamical system induced by \( X \) on \( M \) via \( \alpha \). For such a dynamical system \( (X, M, \alpha) \), one can define a crossed product algebra

\[
\mathbb{M}_X = M \times_\alpha X,
\]

by the algebra generated by \( M \) and \( \alpha(X) \), satisfying that:

\[
(m_1 \alpha_{x_1})(m_2 \alpha_{x_2}) = (m_1 \alpha_{x_1}(m_2)) \alpha_{x_1x_2} \text{ in } \mathbb{M}_X,
\]

where \( \alpha_{x_j} = \alpha(x_j) \), for all \( m_1 \alpha_{x_j} \in M_X \), for \( j = 1, 2 \).

If \( M \) is a *-algebra, then one may have an additional condition:

\[
(m \alpha_x)^* = \alpha_x(m^*) \alpha_x^* \text{ in } \mathbb{M}_X,
\]

for all \( m \alpha_x \in \mathbb{M}_X \).

Of course, one can consider the cases where \( M \) is equipped with topology. More precisely, in \( \mathbb{M}_X \), we may put a topology from the topology on \( M \), making \( \alpha(X) \) be continuous.
In this paper, we are interested in cases where given algebras $M$ are von Neumann algebras. In such cases, we call the corresponding topological dynamical systems, $W^*$-dynamical systems, and the corresponding crossed product algebra, the crossed product $W^*$-algebras.

c) Free Probability: For more about free probability theory, see [16] and [17]. In this section, we briefly introduce Speicher’s combinatorial free probability (e.g., [16]), which is the combinatorial characterization of the original Voiculescu’s analytic free probability (e.g., [17]).

Let $B \subset A$ be von Neumann algebras with $1_B = 1_A$ and assume that there exists a conditional expectation $E_B : A \to B$ satisfying that:

(i) $E_B(b) = b$, for all $b \in B$,
(ii) $E_B(b \ a \ b') = E_B(a) b'$, for all $b, b' \in B$ and $a \in A$,
(iii) $E_B$ is bounded (or continuous), and
(iv) $E_B(a^*) = E_B(a^*)$, for all $a \in A$.

Then the pair $(A, E_B)$ is called a $B$-valued (amalgamated) $W^*$-probability space (with amalgamation over $B$).

For any fixed $B$-valued random variables $a_1, \ldots, a_s$ in $(A, E_B)$, we can have the $B$-valued free distributional data of them;

- $(i_1, \ldots, i_n)$-th $B$-valued joint *-moments:
  \[ E_B \left( b_1 a_{i_1}^{r_1} \ldots b_n a_{i_n}^{r_n} \right) \]

- $(j_1, \ldots, j_m)$-th $B$-valued joint *-cumulants:
  \[ k_m^B \left( b_1 a_{j_1}^{r_1} \ldots b_m a_{j_m}^{r_m} \right) \]

which provide the equivalent $B$-valued free distributional data of $a_1, \ldots, a_s, \ b_1, \ldots, b_n, \ b_1', \ldots, b_m' \in B$ are arbitrary and $r_1, \ldots, r_n, r_1', \ldots, t_m \in \{1, \ldots, r\}$. By the Möbius inversion, indeed, they provide the same, or equivalent, $B$-valued free distributional data of $a_1, \ldots, a_s$, i.e., they satisfy

\[ E_B \left( b_1 a_{i_1}^{r_1} \ldots b_n a_{i_n}^{r_n} \right) = \sum_{\pi \in NC(n)} k_m^B \left( b_1 a_{i_1}^{r_1} \ldots b_n a_{i_n}^{r_n} \right) \]

and

\[ k_m^B \left( b_1 a_{j_1}^{r_1} \ldots b_m a_{j_m}^{r_m} \right) = \sum_{\theta \in NC(m)} E_{B, \theta} \left( b_1 a_{j_1}^{r_1} \ldots b_m a_{j_m}^{r_m} \right) \mu(\theta, 1_m), \]

where $NC(k)$ is the lattice of all noncrossing partitions over $\{1, \ldots, k\}$, for $k \in \mathbb{N}$, and $k_m^B(...)$ and $E_{B, \theta}(...) \mu$ are the partition-depending moment and the partition-depending cumulant, and where $\mu$ is the Möbius functional in the incidence algebra $I_2$.

Recall that the partial ordering on $NC(k)$ is defined by

\[ \pi \leq \theta \overset{def}{=} \forall \text{blocks } V \in \pi, \exists \text{blocks } B \text{ in } \theta \text{ s.t. } V \subseteq B, \]

for all $k \in \mathbb{N}$. Under such a partial ordering $\leq$, the set $NC(k)$ is a lattice with its maximal element $1_k = \{1, \ldots, k\}$ and its minimal element $0_k = \{(1), (2), \ldots, (k)\}$. The notation $\{\ldots\}$ inside partitions $\{\ldots\}$ means the blocks of the partitions. For example, $1_k$ is the one-block partition and $0_k$ is the $k$-block partition, for $k \in \mathbb{N}$. Also, recall that the incidence algebra $I_2$ is the collection of all functionals

\[ \xi : \cup_{k \in \mathbb{N}} (NC(k) \times NC(k)) \to \mathbb{C}, \]

satisfying $\xi(\pi, 0) = 0$, whenever $\pi > 0$, with its usual function addition $(\cdot)$ and its convolution $(\ast)$ defined by

\[ \xi_1 \ast \xi_2(\pi, \theta) \overset{def}{=} \sum_{\pi \leq \sigma \leq \theta} \xi_1(\pi, \sigma) \xi_2(\sigma, \theta), \]
for all $\xi_1, \xi_2 \in I_2$. Then this algebra $I_2$ has the zeta functional $\zeta$, defined by

$$\zeta(\pi, \theta) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \pi \leq \theta \\ 0 & \text{otherwise.} \end{cases}$$

The M"{o}bius functional $\mu$ is the convolution-inverse of $\zeta$ in $I_2$. So, it satisfies

$$\sum_{\pi \in NC(k)} \mu(\pi, 1_k) = 0, \quad \text{and} \quad \mu(0_k, 1_k) = (-1)^{k-1} c_{k-1}, \tag{4.1.1}$$

for all $k \in \mathbb{N}$, where $c_m \overset{\text{def}}{=} \frac{1}{m+1} \left( \frac{2m}{m} \right)$ is the $m$-th Catalan number, for all $m \in \mathbb{N}$.

The amalgamated freeness is characterized by the amalgamated $*$-cumulants. Let $(A, E_B)$ be given as above. Two $W^*$-subalgebras $A_1$ and $A_2$ of $A$, having their common $W^*$-subalgebra $B$ in $A$, are free over $B$ in $(A, E_B)$, if and only if all their "mixed" $*$-cumulants vanish. Two subsets $X_1$ and $X_2$ of $A$ are free over $B$ in $(A, E_B)$, if $vN(X_1, B)$ and $vN(X_2, B)$ are free over $B$ in $(A, E_B)$, where $vN(S_1, S_2)$ means the von Neumann algebra generated by $S_1$ and $S_2$. In particular, two $B$-valued random variable $x_1$ and $x_2$ are free over $B$ in $(A, E_B)$, if $\{x_1\}$ and $\{x_2\}$ are free over $B$ in $(A, E_B)$.

Suppose two $W^*$-subalgebras $A_1$ and $A_2$ of $A$, containing their common $W^*$-subalgebra $B$, are free over $B$ in $(A, E_B)$. Then we can construct a $W^*$-subalgebra $vN(A_1, A_2) = B[A_1 \cup A_2]_B^*$ of $A$ generated by $A_1$ and $A_2$. Such $W^*$-subalgebra of $A$ is denoted by $A_1 \ast_B A_2$. If there exists a family $\{A_i : i \in I\}$ of $W^*$-subalgebras of $A$, containing their common $W^*$-subalgebra $B$, satisfying $A = \ast_{i \in I} A_i$, then we call $A$, the $B$-valued free product algebra of $\{A_i : i \in I\}$.

Assume now that the $W^*$-subalgebra $B$ is $*$-isomorphic to $C = C \cdot 1_A$. Then the conditional expectation $E_B$ becomes a linear functional on $A$. By $\varphi$, denote $E_B$. Then, for $a_1, \ldots, a_n \in (A, \varphi)$,

$$k_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, \ldots, a_n) \mu(\pi, 1_n)$$

by the M"{o}bius inversion

$$= \sum_{\pi \in NC(n)} \left( \prod_{V \in \pi} \varphi_V(a_1, \ldots, a_n) \right) \mu(\pi, 1_n)$$

since the images of $\varphi$ are in $C$.

For example, if $\pi = \{ (1, 3), (2), (4, 5) \}$ in $NC(5)$, then

$$\varphi_\pi(a_1, \ldots, a_5) = \varphi(a_1 \varphi(a_2) a_3) \varphi(a_4 a_5) = \varphi(a_1 a_3) \varphi(a_2) \varphi(a_4 a_5).$$

Remember here that, if $\varphi$ is an arbitrary conditional expectation $E_B$, and if $B \not\subseteq C \cdot 1_A$, then the above second equality does not hold in general.

So, we have

$$k_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \left( \prod_{V \in \pi} \varphi_V(a_1, \ldots, a_n) \mu(0_{|V|}, 1_{|V|}) \right) \tag{4.1.2}$$

by the multiplicativity of $\mu$.

### III. $p$-Adic $W^*$-Dynamical Systems

In this section, we introduce $W^*$-dynamical systems induced by $p$-adic number fields $\mathbb{Q}_p$, for $p \in \mathbb{P}$. They are defined by a certain semigroup(-or-monoidal) dynamical systems induced by semigroups (resp., monoids) $\sigma(\mathbb{Q}_p) = (\sigma(\mathbb{Q}_p), \cap)$. Throughout this section, we fix a von Neumann subalgebra $M$ of $B(H)$, and a prime $p$. 
Lemma 3.2. Let $X$ be a measurable subset of the unit circle $U_p$ in $Q_p$, for primes $p$. Then there exists $0 \leq r_X \leq 1$ in $\mathbb{R}$.

Ref

such that

$$\rho_p(X) = r_X \left( 1 - \frac{1}{p} \right).$$

$\square$

By (3.1.3), we can obtain the following theorem.

**Theorem 3.3.** (See [10]) Let $\chi_S$ be a characteristic function for $S \in \sigma(Q_p)$. Then there exist $N \in \mathbb{N} \cup \{\infty\}$, and $k_1, ..., k_N \in \mathbb{Z}, r_1, ..., r_N \in \mathbb{R}$, such that

$$\int_{Q_p} \chi_S \; d\rho_p = \sum_{j=1}^{N} r_j \left( \frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}} \right).$$

(3.1.4)

The above formula (3.1.4) characterizes the identically-distributedness under the integral in $\mathcal{M}_p$. By (3.1.4), one can obtain the following corollary.

**Corollary 3.4.** Let $g = \sum_{s \in \sigma(Q_p)} t_s \chi_S$ be an element of the $p$-prime von Neumann algebra $\mathcal{M}_p$. Then there exist

$$r_j \in [0, 1] \text{ in } \mathbb{R}, \; k_j \in \mathbb{Z}, \; \text{and } t_j \in \mathbb{C},$$

and

$$h = \sum_{j=-\infty}^{\infty} (t_j r_j p^{k_j}) \chi_{t\rho_p}$$

(3.1.5)

such that $g$ and $h$ are identically distributed under the integral $\int_{Q_p} \bullet \; d\rho_p$. $\square$

**b) $p$-Adic Semigroup $W^*$-Dynamical Systems:** Now, let $M$ be a fixed von Neumann algebra in the operator algebra $B(H)$ on the Hilbert space $H$, and $Q_p$, a fixed $p$-adic number field, and let $\mathcal{M}_p = L^\infty(Q_p, \rho_p)$ be the $p$-prime von Neumann algebra in the sense of Section 3.1.

Let $\mathcal{H}_p$ be the tensor product Hilbert space $H \otimes H_p$ of the $p$-prime Hilbert space $H_p$ and the Hilbert space $H$, where $\otimes$ means the topological tensor product of Hilbert spaces. i.e.,

$$\mathcal{H}_p = H \otimes H_p.$$  

Understand the algebra $\sigma(Q_p)$ of $Q_p$ as a monoid $(\sigma(Q_p), \cap)$. It is not difficult to check indeed $\sigma(Q_p)$ is a semigroup under the intersection $(\cap)$, with $(\cap)$-identity $Q_p \in \sigma(Q_p)$, i.e., it is a well-defined monoid.

Define an action $\alpha$ of the monoid $\sigma(Q_p)$, acting on the von Neumann algebra $M$ in $B(\mathcal{H}_p)$ by

$$\alpha(S)(m) \overset{\text{def}}{=} \chi_S \cdot m \cdot \chi_S^* = \chi_S \cdot m \cdot \chi_S,$$

(3.2.1)

for all $S \in \sigma(Q_p)$, and $m \in M$, in $B(\mathcal{H}_p)$, by understanding

$$\chi_S = \chi_S \otimes 1_M, \text{ and } m = 1_{\mathcal{M}_p} \otimes m \text{ in } B(\mathcal{H}_p),$$

where $1_Q$ is the identity map $\chi_Q$ on $Q_p$, and $1_M$ is the identity element of $M$.

**Lemma 3.5.** (See [10]) The action $\alpha$ of $\sigma(Q_p)$ in the sense of (3.2.1) acting on a von Neumann algebra $M$ is a monoid action, and hence, the triple $(M, \sigma(Q_p), \alpha)$ forms a monoidal dynamical system. $\square$

Indeed, the morphism $\alpha$ of (3.2.1) satisfies that:

$$\alpha(S_1 \cap S_2) = \alpha(S_1) \circ \alpha(S_2) \text{ on } M,$$

for all $S_1, S_2 \in \sigma(Q_p)$.

Remark that all elements $f$ of the $p$-prime von Neumann algebra $\mathcal{M}_p = L^\infty(Q_p, \rho_p)$ is generated by the algebra $\sigma(Q_p)$ of $Q_p$, in the sense that: every element $f \in \mathcal{M}_p$ has its expression, $\sum_{S \in \text{Supp}(f)} t_S \chi_S$. So, the action $\alpha$ of (3.2.1) is extended to the linear morphism, also denoted by $\alpha$, from $\mathcal{M}_p$ into $B(\mathcal{H}_p)$, acting on $M$, with
\[ \alpha(f)(m) = \alpha \left( \sum_{S \in \text{Supp}(f)} t_S \chi_S \right)(m) \]  
(3.2.2)

\[ \overline{\text{def}} \sum_{S \in \text{Supp}(f)} t_S \alpha(S)(m) = \sum_{S \in \text{Supp}(f)} t_S (\chi_S m \chi_S), \]
for all \( f \in \mathcal{M}_p \).

**Definition 3.2.** Let \( \sigma(\mathbb{Q}_p) \) be the \( \sigma \)-algebra of the \( p \)-adic number field \( \mathbb{Q}_p \), understood as a monoid \( (\sigma(\mathbb{Q}_p), \cap) \), and let \( \alpha \) be the action of \( \sigma(\mathbb{Q}_p) \) on a von Neumann algebra \( M \) in the sense of (3.2.1). Then the mathematical triple \( (M, \sigma(\mathbb{Q}_p), \alpha) \) is called the \( p \)-adic (monoidal) \( W^* \)-dynamical system. For this \( p \)-adic \( W^* \)-dynamical system, define the crossed product \( W^* \)-algebra

\[ \mathcal{M}_p \overset{\text{def}}{=} M \times_\alpha \sigma(\mathbb{Q}_p) \]  
(3.2.5)

by the von Neumann subalgebra of \( B(\mathcal{H}_p) \) generated by \( M \) and \( \chi(\sigma(\mathbb{Q}_p)) \) satisfying (3.2.2) (See Section 2.2 above).

The von Neumann algebra \( \mathcal{M}_p \) is called the \( p \)-adic dynamical \( W^* \)-algebra induced by the \( p \)-adic \( W^* \)-dynamical system \( (M, \sigma(\mathbb{Q}_p), \alpha) \).

Note that, all elements of the \( p \)-adic dynamical \( W^* \)-algebra \( \mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p) \) induced by the \( p \)-adic \( W^* \)-dynamical system \( \mathcal{Q}(M, p) \) have their expressions,

\[ \sum_{S \in \sigma(\mathbb{Q}_p)} m_S \chi_S, \text{ with } m_S \in M \]
(possibly infinite sums under topology). Define the support \( \text{Supp}(T) \) of a fixed element \( T = \sum_{S \in \sigma(\mathbb{Q}_p)} m_S \chi_S \) in \( \mathcal{M}_p \) by

\[ \text{Supp}(T) \overset{\text{def}}{=} \{ S \in \sigma(\mathbb{Q}_p) : m_S \neq 0_M \}. \]

Now, let \( m_1 \chi_{S_1}, m_2 \chi_{S_2} \in \mathcal{M}_p \), with \( m_1, m_2 \in M, S_1, S_2 \in \sigma(\mathbb{Q}_p) \). Then

\[ (m_1 \chi_{S_1})(m_2 \chi_{S_2}) = m_1 \chi_{S_1} m_2 \chi_{S_1} \chi_{S_2} = m_1 \chi_{S_1} m_2 \chi_{S_1} \chi_{S_2} \]

since \( \chi_S = 1_M \otimes \chi_S \) (in \( B(\mathcal{H}_p) \)) are projections \( (\chi_S^2 = \chi_S = \chi_S^*), \) for all \( S \in \sigma(\mathbb{Q}_p) \)

\[ = m_1 \alpha_{S_1}(m_2) \chi_{S_1} \chi_{S_2} = m_1 \alpha_{S_1}(m_2) \chi_{S_1 \cap S_2}. \]

**Notation** For convenience, if there is no confusion, we denote \( \alpha_S(m) \) by \( m^S \), for all \( S \in \sigma(\mathbb{Q}_p) \), and \( m \in M \). \( \square \)

More generally, one has that:

\[ \prod_{j=1}^N (m_j \chi_{S_j}) = m_1 m_2 s_3 \ldots m_N s_{1 \cap S_2 \cap \ldots \cap S_N}^{S_1 \cap \ldots \cap S_N - 1} \chi_{S_1 \cap \ldots \cap S_N} \]

\[ = \left( \prod_{j=1}^N m_j^{\chi_{S_j}^*} \right) \left( \chi_{S_1 \cap \ldots \cap S_N} \right) \]

for all \( N \in \mathbb{N} \). Also, we obtain that

\[ (m \chi_S)^* = \chi_S m^* \chi_S \chi_S = (m^*)^S \chi_S, \]
(3.2.4)

for all \( m \chi_S \in \mathcal{M}_p \), with \( m \in M \), and \( S \in \sigma(\mathbb{Q}_p) \).

So, let

\[ T_k = \sum_{S_k \in \text{Supp}(T_k)} m_{S_k} \chi_{S_k} \in \mathcal{M}_p, \text{ for } k = 1, 2. \]
Then
\[ T_1 T_2 = \sum_{(S_1, S_2) \in \text{Supp}(T_1) \times \text{Supp}(T_2)} m_{S_1} S_1 m_{S_2} S_2 \]
\[ = \sum_{(S_1, S_2) \in \text{Supp}(T_1) \times \text{Supp}(T_2)} m_{S_1} S_1 m_{S_2} S_2 \chi_{S_1 \cap S_2}, \]
by (3.2.3).

Also, if \( T = \sum_{S \in \text{Supp}(T)} m_S S \) in \( M_p \), then
\[ T^* = \sum_{S \in \text{Supp}(T)} (m_S)^* S, \]
by (3.2.4).

So, one can have that if
\[ T_k = \sum_{S_k \in \text{Supp}(T_k)} m_{S_k} S_k \in M_p, \text{ for } k = 1, \ldots, n, \]
for \( n \in \mathbb{N} \), then
\[ T_1^{r_1} T_2^{r_2} \cdots T_n^{r_n} = \prod_{j=1}^{n} \sum_{S_j \in \text{Supp}(T_j)} [m_{S_j}^{r_j}] S_j \chi_{S_j} \]
where
\[ [m_{S_j}^{r_j}] S_j \text{ def } \{ \begin{array}{ll} m_{S_j} & \text{if } r_j = 1 \\ (m_{S_j}^*) S_j & \text{if } r_j = *, \end{array} \]
for \( j = 1, \ldots, n \)
\[ = \sum_{(S_1, \ldots, S_n) \in \prod_{j=1}^{n} \text{Supp}(T_j)} \left( \prod_{j=1}^{n} \left( [m_{S_j}^{r_j}] S_j \chi_{S_j} \right) \right), \]
\[ \text{for all } (r_1, \ldots, r_n) \in \{1, *\}^n. \]

Lemma 3.6. Let \( T_k = \sum_{S_k \in \text{Supp}(T_k)} m_{S_k} S_k \) be elements of the \( p \)-adic semigroup \( W^* \)-algebra \( M_p = M \times_{\alpha} \sigma(\mathbb{Q}_p) \) in \( B(\mathcal{H}_p) \), for \( k = 1, \ldots, n, \) for \( n \in \mathbb{N} \). Then
\[ \prod_{j=1}^{n} T_j^{r_j} = \sum_{(S_1, \ldots, S_n) \in \prod_{j=1}^{n} \text{Supp}(T_j)} \left( \left( \prod_{j=1}^{n} \left( [m_{S_j}^{r_j}] S_j \chi_{S_j} \right) \right) ^{\frac{1}{\prod_{j=1}^{n} S_j}} \right), \]
for all \( r_1, \ldots, r_n \in \{1, *\} \), where \([m_{S_j}^{r_j}] S_j\) are in the sense of (3.2.7).

Proof. The proof of (3.2.9) is done by (3.2.8) with (3.2.7). \( \Box \)

c) Structure Theorems of \( M \times_{\alpha} \sigma(\mathbb{Q}_p) \). Let \( M_p = M \times_{\alpha} \sigma(\mathbb{Q}_p) \) be the \( p \)-adic \( W^* \)-algebra induced by the \( p \)-adic \( W^* \)-dynamical system \( (M, \sigma(\mathbb{Q}_p), \alpha) \). In this section, we consider a structure theorem for this crossed product von Neumann algebra \( M_p \).

Define the usual tensor product \( W^* \)-subalgebra
\[ \mathcal{M}_0 = M \otimes_{\mathbb{C}} M_p \text{ of } B(\mathcal{H}_p), \]
where \( \mathcal{M}_p = L^\infty(\mathbb{Q}_p, \rho_p) \) is the \( p \)-prime von Neumann algebra in the sense of Section 3.1, and where \( \otimes_{\mathbb{C}} \) is the topological tensor product of topological operator algebras over \( \mathbb{C} \). By definition, clearly, one can verify that \( \mathcal{M}_p \) is a \( W^* \)-subalgebra of \( \mathcal{M}_0 \) in \( B(\mathcal{H}_p), \) i.e.,
Subalgebra \( \mathcal{M}_p \subseteq \mathcal{M}_0 \).

Now, define the “conditional” tensor product \( W^*-\)algebra

\[
\mathcal{M}_0^p = M \otimes_{\alpha} \mathfrak{M}_p,
\]

induced by an action \( \alpha \) of \( \mathfrak{M}_p \) acting on \( M \) (in the sense of (3.2.4)), by a \( W^* \)-subalgebra of \( \mathcal{M}_0 \) dictated by the \( \alpha \)-relations:

\[
(m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2}) = (m_1m_2^S) \otimes \chi_{S_1\chi_{S_2}},
\]

and

\[
(m \otimes \chi_S)^* = (m^*)^S \otimes \chi^*_S,
\]

for all \( m_1, m_2, m \in M \), and \( S_1, S_2, S \in \sigma(Q_p) \), i.e., the \( W^* \)-subalgebra \( \mathcal{M}_0^p \) of \( \mathcal{M}_0 \) satisfying the \( \alpha \)-relations (3.3.1) and (3.3.2) is the conditional tensor product \( W^*-\)algebra \( M \otimes_{\alpha} \mathfrak{M}_p \).

**Theorem 3.7.** (See [10]) Let \( \mathcal{M}_p = M \times_{\alpha} \sigma(Q_p) \) be the \( p \)-adic \( W^*-\)algebra induced by the \( p \)-adic \( W^* \)-dynamical system \( \mathcal{Q}(M, p) \), and let \( \mathcal{M}_0 = M \otimes_{\alpha} \mathfrak{M}_p \) be the conditional tensor product \( W^*-\)algebra of \( M \) and the \( p \)-prime von Neumann algebra \( \mathfrak{M}_p \) satisfying the \( \alpha \)-relations (3.3.1) and (3.3.2). Then these von Neumann algebras \( \mathcal{M}_p \) and \( \mathcal{M}_0^p \) are \( * \)-isomorphic in \( B(\mathcal{H}_p) \), i.e.,

\[
(3.3.3)
\]

\[
\mathcal{M}_p = M \times_{\alpha} \sigma(Q_p) \xrightarrow{\text{iso}} M \otimes_{\alpha} \mathfrak{M}_p = \mathcal{M}_0^p,
\]

in \( B(\mathcal{H}_p) \). \( \square \)

**IV. Free Probability on \( p \)-Adic Dynamical \( W^*-\)Algebras**

In this section, we consider free probability on the \( p \)-adic dynamical \( W^*-\)algebra

\[
\mathcal{M}_p = M \times_{\alpha} \sigma(Q_p)
\]

induced by the \( p \)-adic \( W^*-\)dynamical system \( (M, \sigma(Q_p), \alpha) \).

By Section 3.3, the von Neumann subalgebra \( \mathcal{M}_p \) is \( * \)-isomorphic to the conditional tensor product \( W^*-\)algebra \( \mathcal{M}_0^p = M \otimes_{\alpha} \mathfrak{M}_p \) of a fixed von Neumann subalgebra \( M \) of \( B(H) \) and the \( p \)-prime von Neumann algebra \( \mathfrak{M}_p = L^\infty(Q_p, \rho_p) \), in \( B(\mathcal{H}_p) \), for \( p \in \mathcal{P} \). So, throughout this section, we understand \( \mathcal{M}_p \) and \( \mathcal{M}_0^p \), alternatively.

By understanding \( \mathcal{M}_p \) as \( \mathcal{M}_0^p \), we construct a well-defined conditional expectation

\[
(4.1)
\]

\[
E_p : \mathcal{M}_0^p \xrightarrow{\text{iso}} \mathcal{M}_p \rightarrow M_p,
\]

where

\[
M_p = M \otimes_{\alpha} \mathbb{C} \{\chi_S : S \in \sigma(Q_p), S \subseteq U_p\},
\]

where \( U_p \) is the unit circle of \( Q_p \), which is the boundary \( Z_p - pZ_p \) of the unit disk \( Z_p \) of \( Q_p \), satisfying that:

\[
E_p(m\chi_S) = E_p(m \otimes \chi_S)^{\text{def}} = m \chi_{S \cap U_p},
\]

for all \( m \in M \), and \( S \in \sigma(Q_p) \).

Define now a morphism

\[
F_p : M_p \rightarrow M_p
\]

by a linear transformation satisfying that:

\[
F_p(m\chi_S) = m\left(r_S\chi_{U_p}\right),
\]

for all \( S \in \sigma(Q_p) \), where \( r_S \in [0, 1] \) in \( \mathbb{R} \), making

\[
(4.1')
\]
Define now a linear functional
\[ \gamma : M_p \to \mathbb{C} \]
by
\[ \gamma \overset{\text{def}}{=} \left( \otimes \int_{\mathbb{Q}_p} \bullet \, d\rho_p \right) \circ F_p, \tag{4.2} \]
where \( F_p \) is in the sense of (4.1)'. More precisely, it satisfies that:
\[ \gamma(m \otimes \chi_S) \overset{\text{def}}{=} (m) \int_{\mathbb{Q}_p} (r_S \chi_{U_p}) \, d\rho_p = r_S (m) \left( 1 - \frac{1}{p} \right). \]
And then define a linear functional
\[ \gamma_p : M_p \overset{\ast \text{-iso}}{\to} M_p^0 \to \mathbb{C} \]
by
\[ \gamma_p = \gamma \circ E_p, \tag{4.3} \]
where \( \gamma \) and \( E_p \) are in the sense of (4.2) and (4.1), respectively. i.e., for all \( m \in M \), and \( S \in \sigma(\mathbb{Q}_p) \),
\[ \gamma_p(m \chi_S) = \gamma(E_p(m \chi_S)) = \gamma \left( m \chi_S \right) \]
\[ = \gamma \left( m \otimes \chi_S \right) \]
\[ = (m) \int_{\mathbb{Q}_p} (r_S \chi_{U_p}) \, d\rho_p = r_S (m) \left( 1 - \frac{1}{p} \right), \]
for some \( r \in [0, 1] \), satisfying (4.1)".

Then the pair \((M_p, \gamma_p)\) is a \(W^*\)-probability space in the sense of Section 2.3. We consider the free distributional data of certain elements of \((M_p, \gamma_p)\).

Let \( M_p = M \times_{\sigma} \mathbb{Q}_p \) be the \( p \)-adic dynamical \( W^*\)-algebra in \( B(\mathcal{H}_p) \), understood also as its \( * \)-isomorphic von Neumann algebra, \( M_p^0 = M \otimes_{\sigma} \mathbb{M}_p \). Let \( \gamma_p = \gamma \circ E_p \) be the linear functional in the sense of (4.3) on \( M_p^0 = M_p \), where \( \gamma \) is in the sense of (4.2) and \( E_p \) is in the sense of (4.1), with (4.1)". i.e., \( \gamma_p \) is a linear functional on \( M_p \), satisfying that:
\[ \gamma_p(m \chi_S) = \gamma(E_p(m \chi_S)) = \gamma \left( m \chi_S \right) \]
\[ = \gamma \left( m \otimes \chi_S \right) \]
\[ = (m) \int_{\mathbb{Q}_p} (r_S \chi_{U_p}) \, d\rho_p = r_S (m) \left( 1 - \frac{1}{p} \right), \]
for some \( r \in [0, 1] \), satisfying (4.1)"", for all \( m \in M \), and \( S \in \sigma(\mathbb{Q}_p) \).

By [10], the morphism \( \gamma_p = \gamma \circ E_p : M_p \to \mathbb{C} \) of (4.3) is indeed a well-defined bounded linear functional on \( M_p \overset{\ast \text{-iso}}{=} M_p^0 \).

\textbf{Definition 4.1.} The pair \((M_p, \gamma_p)\) is called the \( p \)-adic dynamical \( W^*\)-probability space.

The following lemmas are obtained by the straightforward computations.

\textbf{Lemma 4.1. (See [10])} Let \( m \chi_S \) be a free random variable in the \( p \)-adic dynamical \( W^*\)-probability space \((M_p, \gamma_p)\), with \( m \in M \), and \( S \in \sigma(\mathbb{Q}_p) \). Then
\[ \gamma_p(m \chi_S^n) = r_S \psi(m \chi_S^{n-1}) \left( 1 - \frac{1}{p} \right), \tag{4.4} \]
for all \( n \in \mathbb{N} \), where \( r_S \in [0, 1] \) satisfies (4.1)\(^{\prime}\). □

**Lemma 4.2.** (See [10]) Let \( m_1 \chi_{S_1}, \ldots, m_n \chi_{S_n} \) be free random variables in the \( p \)-adic dynamical \( W^* \)-probability space \((M_p, \gamma_p)\), with \( m_k \in M, S_k \in \sigma(\mathbb{Q}_p) \), for \( k = 1, \ldots, n \), for \( n \in \mathbb{N} \). Then there exists \( r_0 \in [0, 1] \), such that:

\[
\gamma_p \left( \prod_{j=1}^{n} m_j \chi_{S_j} \right) = r_0 \left( \left( \prod_{j=1}^{n} m_j \chi_{S_j} \right) \left( 1 - \frac{1}{p} \right) \right). 
\]

(4.5)

By (4.4) and (4.5), we obtain the following free distributional data of free random variables in the \( p \)-adic dynamical \( W^* \)-probability space, and let

\[
T_k = \sum_{S_k \in \text{supp}(T_k)} m_{S_k} \chi_{S_k}, \text{ for } k = 1, \ldots, n,
\]

be free random variables in \((M_p, \gamma_p)\), for \( n \in \mathbb{N} \). Then

\[
\gamma_p \left( \prod_{j=1}^{n} T_j \right) = \sum_{(S_1, \ldots, S_n) \in \prod_{j=1}^{n} \text{supp}(T_j)} r_{(S_1, \ldots, S_n)} \left( \left( \prod_{j=1}^{n} m_j \chi_{S_j} \right) \left( 1 - \frac{1}{p} \right) \right).
\]

(4.6)

So, by the Möbius inversion of Section 2.3, one can obtain that:

\[
k_n \left( \left( \prod_{i=1}^{r_1} m_i \right)^{S_1}, \ldots, \left( \prod_{i=1}^{r_n} m_i \right)^{S_n} \right) = \sum_{\pi \in \text{NC}(n)} \gamma_p \left( \left( \prod_{i=1}^{r_1} m_i \right)^{S_1}, \ldots, \left( \prod_{i=1}^{r_n} m_i \right)^{S_n} \right) \mu(\pi, 1_n)
\]

\[
= \sum_{\pi \in \text{NC}(n)} \left( \prod_{V \in \pi} \gamma_p \left( \left( \prod_{i=1}^{r_1} m_i \right)^{S_1}, \ldots, \left( \prod_{i=1}^{r_n} m_i \right)^{S_n} \right) \mu(0_{|V|}, 1_{|V|}) \right)
\]

by the Möbius inversion (See Section 4.1)

\[
= \sum_{\pi \in \text{NC}(n)} \left( \prod_{V=(i_1, \ldots, i_k) \in \pi} r_V \left( \left( \prod_{i=1}^{k} \left( \prod_{t=1}^{r_{i_t}} m_{i_t} \right)^{S_{i_t}} \right) \left( 1 - \frac{1}{p} \right) \right) \mu(0, 1_k) \right)
\]

(4.7)

by (4.6), where \( r_V \in [0, 1] \) satisfy (4.1)\(^{\prime}\).

By (4.7), we obtain the following inner free structure of the \( p \)-adic dynamical \( W^* \)-algebra \( M_p \), with respect to \( \gamma_p \).

**Theorem 4.4.** (See [10]) Let \( m_1 \chi_{S}, m_2 \chi_{S} \) be free random variables in the \( p \)-adic dynamical \( W^* \)-probability space \((M_p, \gamma_p)\), with \( m_1, m_2 \in M \), and \( S \in \sigma(\mathbb{Q}_p) \setminus \{\emptyset\} \). Also, assume that \( S \) is not a measure-zero element in \( \sigma(\mathbb{Q}_p) \). Then \( \{m_1, m_1^S\} \) and \( \{m_2, m_2^S\} \) are free in the \( W^* \)-probability space \((M, \psi)\), if and only if \( m_1 \chi_{S} \) and \( m_2 \chi_{S} \) are free in \((M_p, \gamma_p)\). □

Now, let \( m_1 \chi_{S} \) and \( m_2 \chi_{U_p} \) be \( m_1, m_2 \in M \), and \( S \in \sigma(\mathbb{Q}_p) \). Assume that \( S \cap U_p \) is empty. Since \( S \cap U_p = \emptyset \), all mixed cumulants of \( m_1 \chi_{S} \) and \( m_2 \chi_{U_p} \) have \( r_V = 0 \), for some \( V \in \pi \) in (4.7), for all \( \pi \in \text{NC}(n) \). Therefore, one obtains the following inner freeness condition of \((M_p, \gamma_p)\).

**Theorem 4.5.** (See [10]) Let \( S_1 \neq S_2 \in \sigma(\mathbb{Q}_p) \) such that \( S_1 \cap S_2 = \emptyset \). Then the subsets \( \{m \chi_{S_1} : m \in M\} \) and \( \{a \chi_{S_2} : a \in M\} \) are free in \((M_p, \gamma_p)\). □
One may do the same process by fixing \( p^k U_p \) instead of fixing \( U_p \), for \( k \in \mathbb{Z} \). Recall that \( p^k U_p \) are the boundaries \( p^k \mathbb{Z}_p \setminus p^{k+1} \mathbb{Z}_p \) of \( p^k \mathbb{Z}_p \), for all \( k \in \mathbb{Z} \) (See Section 2.1), i.e., for a fixed \( k \in \mathbb{Z} \), define

\[
M_{p,k} \overset{def}{=} M \otimes_\alpha \mathbb{C} \left[ \{\chi_{p^k U_p}\} \right]^{\ast \ast_{iso}} M,
\]

Then \( M_p = M \otimes_\alpha \mathbb{C} \left[ \{\chi_{U_p}\} \right] \) of (4.1) is identical to \( M_{p,0} \) in the sense of (4.8).

Similar to (4.1), construct a conditional expectation

\[
E_{p,k} : M_p = M_{p,0} \rightarrow M_{p,k}
\]

by a linear morphism satisfying that:

\[
E_{p,k} (m \chi_S) = m^0 \chi_{S \cap p^k U_p},
\]

with

\[
\chi_{S \cap p^k U_p} = r \chi_{p^k U_p},
\]

where \( r \in [0, 1] \) satisfying (4.9)

\[
\int_{Q_p} \chi_{S \cap p^k U_p} \, d\rho_p = r \int_{Q_p} \chi_{p^k U_p} \, d\rho_p = r \left( \frac{1}{p^k} - \frac{1}{p^{k+1}} \right).
\]

Then, just like (4.1), \( E_{p,k} \) is a well-defined conditional expectation from \( M_p \) onto \( M_{p,k} = M \).

And then, for \( k \in \mathbb{Z} \), define a linear functional

\[
\gamma_k : M_{p,k} \rightarrow \mathbb{C}
\]

by

\[
\gamma_k \left( m \chi_{p^k U_p} \right) \overset{def}{=} \psi(m) \int_{Q_p} \left( \chi_{p^k U_p} \right) \, d\rho_p
\]

\[
= \psi(m) \left( \frac{1}{p^k} - \frac{1}{p^{k+1}} \right).
\]

Then one has a well-defined linear functional

\[
\gamma_{p,k} : M_p \rightarrow \mathbb{C}
\]

defined by

\[
\gamma_{p,k} \overset{def}{=} \gamma_k \circ E_{p,k}, \text{ for all } k \in \mathbb{Z}.
\]

Note that our linear functional \( \gamma_p \) in the sense of (4.3) is identified with \( \gamma_{p,0} \) of (4.11).

**Observation 4.1** Let’s replace \( M_p = M_{p,0} \) of (4.1) to \( M_{p,k} \), for \( k \in \mathbb{Z} \). Then the formulae (4.4), (4.5), (4.6) and (4.7) can be re-obtained by replacing factors \( \left( 1 - \frac{1}{p^k} \right) \) to \( \left( \frac{1}{p^k} - \frac{1}{p^{k+1}} \right) \). So, the freeness of the above two theorems are same under \( (M_p, \gamma_{p,k}) \)-settings.

For instance, if \( m_j \chi_{S_j} \in (\mathcal{M}_p, \gamma_{p,k}) \), for \( j = 1, ..., n \), for \( n \in \mathbb{N} \), then

\[
\gamma_{p,k} \left( \prod_{j=1}^n m_j \chi_{S_j} \right) = r_0 \left( \psi \left( \prod_{j=1}^n m_j \right) \right) \left( \frac{1}{p^k} - \frac{1}{p^{k+1}} \right),
\]

for some \( r_0 \in [0, 1] \), satisfying (4.9)’. \( \square \)

The above **Observation 4.1** shows that we have systems of \( W^* \)-probability spaces

\[
\left\{ (\mathcal{M}_p, \gamma_{p,k}) \right\}_{k \in \mathbb{Z}},
\]

sharing similar free probability with \( (\mathcal{M}_p, \gamma_p = \gamma_{p,0}) \).
V Adelic $W^*$-Dynamical Systems

In this section, we consider a $W^*$-dynamical system induced by the $\sigma$-algebra $\sigma(A_Q)$ of the Adele ring $A_Q$. Similar to the $p$-adic cases of Sections 3 and 4, one may understand $\sigma(A_Q)$ as a monoid

$$\sigma(A_Q) = (\sigma(A_Q), \cap),$$

equipped with its binary operation $\cap$, the set intersection. Like in Section 3, we fix a von Neumann algebra $M$ embedded in an operator algebra $B(H)$ on a Hilbert space $H$.

We will define a suitable action, also denoted by $\alpha$, of the monoid $\sigma(A_Q)$ acting on $M$ in $B(H_Q)$, where $H_Q = H \otimes H_Q$.

Before proceeding, we introduce weak tensor product structures in Section 5.1.

$a)$ Weak Tensor Product Structures: Let $X_i$ be arbitrary sets, for $i \in \Lambda$, where $\Lambda$ means any countable index set. Let

$$g_i : X_i \to X_i \quad (5.1.1)$$

be well-defined functions, for all $i \in \Lambda$.

Now, let $X$ be the Cartesian product $\prod_{i \in \Lambda} X_i$ of $\{X_i\}_{i \in \Lambda}$. Define the subset $\mathcal{X}$ of $X$ by

$$\mathcal{X} = \left\{ (x_i)_{i \in \Lambda} \in X \mid \begin{array}{l}
finitely many x_i \in X_i, \text{ and} \\
a\text{almost of all other } x_j \in g_j(X_j), \\
\text{for } i, j \in \Lambda. 
\end{array} \right\}, \quad (5.1.2)$$
determined by a system $g = \{g_i\}_{i \in \Lambda}$ of (5.1.1). We denote this subset $\mathcal{X}$ of (5.1.2) in $X$ by

$$\mathcal{X} = \prod_{i \in \Lambda} g_i X_i.$$

It is clear that $\mathcal{X}$ is a subset of $X$. If $g_i$ are bijections, for all $i \in \Lambda$, then $\mathcal{X}$ is equipotent (or bijective) to $X$. However, in general, $\mathcal{X}$ is a subset of $X$.

**Definition 5.1.** The subset $\mathcal{X} = \prod_{i \in \Lambda} g_i X_i$ of $X = \prod_{i \in \Lambda} X_i$, in the sense of (5.1.2), is called the weak tensor product set of $\{X_i\}_{i \in \Lambda}$ induced by a system $g = \{g_i\}_{i \in \Lambda}$ of functions $g_i$.

Let $Q_p$ be our $p$-adic number fields, for all $p \in \mathcal{P}$. Define a function

$$g_p : Q_p \to Q_p$$

by

$$g_p \left( p^{-N} \left( \sum_{j=0}^{\infty} a_j p^j \right) \right) \overset{\text{def}}{=} \sum_{j=0}^{\infty} a_j p^j, \quad (5.1.3)$$

for all $p^{-N} \sum_{j=0}^{\infty} a_j p^j \in Q_p$ (with $N \in \mathbb{N} \cup \{0\}$), for all $p \in \mathcal{P}$. Then the image $g_p(Q_p)$ is identical to the compact subset $\mathbb{Z}_p$, the unit disk of $Q_p$, for all $p \in \mathcal{P}$. Therefore, the Adele ring

$$A_Q = \prod_{p \in \mathcal{P}}' Q_p$$

is identified with

$$A_Q = \prod_{p \in \mathcal{P}} Q_p,$$

by (2.1.10) and (5.1.2), where $g = \{g_p\}_{p \in \mathcal{P}}$ is the system of functions $g_p$ of (5.1.3).

Remark here that, for example, if we have real number $r$ in $R = Q_\infty$, with its decimal notation...
\[ |r| = \sum_{k \in \mathbb{Z}} t_k \cdot 10^{-k} = \cdots t_{-2} t_{-1} t_0 \cdot t_1 t_2 t_3 \cdots \]

with \( 0 \leq t_k < 10 \) in \( \mathbb{N} \), then

\[ g_\infty (r) = 0 \cdot t_1 t_2 t_3 \cdots, \quad (5.1.4) \]

with identification \( g_\infty (\pm 1) = 1 \).

Traditionally, we simply write \( A_Q = \prod'_{p \in P} Q_p \) as before, if there is no confusion.

Remark also that, \( X_i \)'s of (5.1.1) and (5.1.2) may / can be algebraic structures (e.g., semigroups, or groups, or monoids, or groupoids, or vector spaces, etc), or topological spaces (e.g., Hilbert spaces, or Banach spaces, etc). One may put product topology on the weak tensor product, with continuity on \( \{ g_i \}_{i \in \Lambda} \). Similarly, if \( X_i \)'s are topological algebras (e.g., Banach algebras, or \( C^* \)-algebras, or von Neumann algebras, etc), then we may have suitable product topology, with bounded (or continuous) linearity on the system \( \{ g_i \}_{i \in \Lambda} \).

**Notation** In topological-algebraic case, to distinguish with other situations, we use the notation \( \otimes_\Phi \), instead of using \( \prod_\Phi \), for a system \( \Phi \) of functions. \( \Box \)

In this section, we establish a von Neumann algebra \( \mathcal{M} \) generated by the Adele ring \( A_Q \). Recall that the Adele ring \( A_Q \) is a unbounded-measured product topological ring induced by \( \{ Q_p \}_{p \in P} \).

In particular, it is a weak tensor product of \( \{ Q_p \}_{p \in P} \), i.e.,

\[ A_Q = \prod'_{p \in P} Q_p = \prod_{p \in P} g_p, \]

where \( g = \{ g_p \}_{p \in P} \) is the system of functions (5.1.3) satisfying (5.1.4).

By understanding \( A_Q \) as a measure space \( (A_Q, \sigma (A_Q), \rho) \) (e.g., see Section 2.1), we have the \( L^2 \)-Hilbert space \( H_Q \), defined by

\[ H_Q \overset{def}{=} L^2 (A_Q, \rho). \]

It has its inner product \( \langle \cdot, \cdot \rangle \), defined by (5.2.1)

\[ \langle F_1, F_2 \rangle \overset{def}{=} \int_{A_Q} F_1 \overline{F_2} \, d\rho, \]

for all \( F_1, F_2 \in H_Q \). And similar to Section 3.1, the von Neumann algebra \( \mathcal{M} \) is defined by

\[ \mathcal{M} \overset{def}{=} L^\infty (A_Q, \rho). \quad (5.2.2) \]

**Definition 5.2.** We call the Hilbert space \( H_Q \) of (5.2.1) the Adele-ring Hilbert space. The von Neumann algebra \( \mathcal{M} \) of (5.2.2) is said to be the Adele-ring von Neumann algebra.

Let \( F = \sum_{Y \in \sigma (A_Q)} t_Y \chi_Y \) be an element of the Adele-ring von Neumann algebra \( \mathcal{M} \). Then

\[ \int_{A_Q} F \, d\rho = \int_{A_Q} \left( \sum_{Y \in \sigma (A_Q)} t_Y \chi_Y \right) d\rho = \sum_{Y \in \sigma (A_Q)} t_Y \rho (Y). \quad (5.2.3) \]

By construction, if \( Y \) is a subset of the Adele ring \( A_Q \), then

\[ Y = \prod_{p \in P} Y_p, \quad (5.2.4) \]
where $Y_p$'s are subsets of $\mathbb{Q}_p$, for $p \in \mathcal{P}$. So, by (2.1.10), (5.2.3) and (5.2.4), one has
\[
\rho(Y) = \rho \left( \prod_{p \in \mathcal{P}} Y_p \right) = \prod_{p \in \mathcal{P}} \rho_p(Y_p),
\]
(5.2.5)
by identifying $\rho_\infty$ with the usual distance measure on $\mathbb{Q}_\infty = \mathbb{R}$.

As we discussed in (3.1.3) and (3.1.4), if $S \in \sigma(\mathbb{Q}_p)$, for a prime $p$, then the element $\chi_S$ is identically distributed with
\[
\sum_{j=1}^N r_j \chi_{p^j U_p}.
\]
for some $N \in \mathbb{N} \cup \{\infty\}$, $r_j \in [0, 1]$ in $\mathbb{R}$, and $k_j \in \mathbb{Z}$ for $j = 1, ..., N$. Therefore, we obtain the following theorem.

**Theorem 5.1.** Let $Y \in \sigma(\mathbb{A}_Q)$ and let $\chi_Y$ be a generating element of the Adele-ring von Neumann algebra $\mathfrak{M}$. Then there exist $N_p \in \mathbb{N} \cup \{\infty\}$, $r_{p;j} \in [0, 1]$ in $\mathbb{R}$, and $k_{p;j} \in \mathbb{Z}$, for $j = 1, ..., N_p$, for $p \in \mathcal{P}$, such that
\[
\int_{\mathbb{A}_Q} \chi_Y \, d\rho = \prod_{p \in \mathcal{P}} \left( \sum_{j=1}^{N_p} r_{p;j} \left( \frac{1}{p^{k_{p;j}}} - \frac{1}{p^{k_{p;j}+r}} \right) \right). \tag{5.2.6}
\]

**Proof.** Let $Y \in \sigma(\mathbb{A}_Q)$. Then, by (5.2.4), there exist $Y_p \in \sigma(\mathbb{Q}_p)$, for all $p \in \mathcal{P}$, such that $Y = \prod_{p \in \mathcal{P}} Y_p$. By Section 3.1, for each $p \in \mathcal{P}$, the $\rho_p$-measurable subsets $Y_p$ has $N_p \in \mathbb{N} \cup \{\infty\}$, and $k_{p;1}, ..., k_{p;N_p} \in \mathbb{Z}$, and $r_{p;1}, ..., r_{p;N_p} \in [0, 1]$, such that:
\[
\rho_p(Y_p) = \sum_{j=1}^{N_p} r_{p;j} \left( \frac{1}{p^{k_{p;j}}} - \frac{1}{p^{k_{p;j}+r}} \right) = \int_{\mathbb{Q}_p} \chi_{Y_p} \, d\rho_p.
\]
Therefore, by the product measure $\rho = \times_{p \in \mathcal{P}} \rho_p$ on the Adele ring $\mathbb{A}_Q$, we have that:
\[
\int_{\mathbb{A}_Q} \chi_Y \, d\rho = \rho(Y) = \left( \prod_{p \in \mathcal{P}} \rho_p(Y_p) \right) = \prod_{p \in \mathcal{P}} \rho_p(Y_p)
\]
\[
= \prod_{p \in \mathcal{P}} \left( \sum_{j=1}^{N_p} r_{p;j} \left( \frac{1}{p^{k_{p;j}}} - \frac{1}{p^{k_{p;j}+r}} \right) \right). \tag{5.2.6}
\]
Therefore, the formula (5.2.6) holds.$ \blacksquare$

The formula (5.2.6) characterizes the identically-distributedness on elements of the Adele-ring von Neumann algebra $\mathfrak{M}$.

The following theorem provides a structure theorem of the Adele-ring von Neumann algebra $\mathfrak{M}$ in terms of the $p$-prime von Neumann algebras $\{\mathfrak{M}_p\}_{p \in \mathcal{P}}$.

**Theorem 5.2.** Let $\mathfrak{M} = L^\infty(\mathbb{A}_Q, \mu)$ be the Adele-ring von Neumann algebra, and let $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \mu_p)$ be the $p$-prime von Neumann algebras, for $p \in \mathcal{P}$. Then $\mathfrak{M}$ is $*$-isomorphic to the weak tensor product von Neumann algebra $\bigotimes_{p \in \mathcal{P}} \mathfrak{M}_p$ of $\{\mathfrak{M}_p\}_{p \in \mathcal{P}}$, induced by the system of functions $\varphi = \{\varphi_p\}_{p \in \mathcal{P}}$, i.e.,
\[
\mathfrak{M} \overset{\text{iso}}{=} \bigotimes_{p \in \mathcal{P}} \mathfrak{M}_p, \text{ with } \mathfrak{M}_\infty = L^\infty(\mathbb{R}), \tag{5.2.8}
\]
where the weak tensor product $\bigotimes_{p \in \mathcal{P}}$ is not only algebraic, but also topological, satisfying
\[
\varphi_p \left( \sum_{X \in \sigma(\mathbb{Q}_p)} t_X \chi_X \right) \overset{\text{def}}{=} \sum_{X \in \sigma(\mathbb{Q}_p)} t_X \chi_X \chi_{\mathbb{A}_p}, \tag{5.2.9}
\]
for all $p \in \mathcal{P}$.

Proof. By the construction of the Adele ring $\mathbb{A}_Q$, it is the weak tensor product $\bigotimes_{p \in \mathcal{P}} \mathbb{Q}_p$ of $\{\mathbb{Q}_p\}_{p \in \mathcal{P}}$ as we discussed in Sections 2.1 and 5.1. Therefore,

$$\mathcal{M} \overset{\text{def}}{=} L^\infty(\mathbb{A}_Q, \rho) \overset{\ast\text{-iso}}{=} L^\infty \left( \bigotimes_{p \in \mathcal{P}} \mathbb{Q}_p, \times_{p \in \mathcal{P}} \rho_p \right)$$

where $\varphi = \{\varphi_p\}_{p \in \mathcal{P}}$ is the family of *-homomorphisms $\varphi_p : \mathbb{Q}_p \to \mathbb{Z}_p$ of (5.2.9). Now, it suffices to show that $\varphi_p$ are *-homomorphisms, for all $p \in \mathcal{P}$. Trivially $\varphi_p$ are linear and bounded, by the very definition. Also, it satisfies that

$$\varphi_p \left( x_{S_1} x_{S_2} \right) = \varphi_p \left( x_{S_1 \cap S_2} \right) = x_{S_1 \cap S_2} \in \mathbb{Z}_p$$

for all $S_1, S_2 \in \sigma(\mathbb{Q}_p)$. So, for any $g_1, g_2 \in \mathcal{M}_p$,

$$\varphi_p(g_1 g_2) = \varphi_p(g_1) \varphi_p(g_2).$$

Now, observe that

$$\varphi_p \left( (t x_S)^* \right) = \varphi_p \left( \overline{t} x_S^* \right) = \overline{t} \varphi_p(x_S)$$

for all $S \in \sigma(\mathbb{Q}_p)$, and $t \in \mathbb{C}$. Therefore, for $g \in \mathcal{M}_p$,

$$\varphi_p(g^*) = \left( \varphi_p(g) \right)^*. \tag{5.3.1}$$

Therefore, $\varphi_p$ are well-defined *-homomorphisms, for all $p \in \mathcal{P}$. So, the family $\{\varphi_p\}_{p \in \mathcal{P}}$ is a system of *-homomorphisms.

Therefore, indeed, $\mathcal{M}$ is *-isomorphic to the weak tensor product $\bigotimes_{p \in \mathcal{P}} \mathcal{M}_p$, as a well-defined $W^\ast$-subalgebra of the usual tensor product $W^\ast$-algebra $\bigotimes_{p \in \mathcal{P}} \mathcal{M}_p$.

The above theorem shows that, to study the Adele-ring von Neumann algebra $\mathcal{M}$, we can investigate the system of conditional summands $\mathcal{M}_p$, the p-prime von Neumann algebras, for $p \in \mathcal{P}$.

c) Adele $W^\ast$-Dynamical Systems Let’s fix an arbitrary von Neumann algebra $M$ in an operator algebra $B(H)$, and let $\mathcal{M} = L^\infty(\mathbb{A}_Q, \rho)$ be the Adele-ring von Neumann algebra, which is *-isomorphic to the weak tensor product $W^\ast$-algebra $\bigotimes_{p \in \mathcal{P}} \mathcal{M}_p$ of p-prime von Neumann algebras $\mathcal{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$, where $\varphi = \{\varphi_p\}_{p \in \mathcal{P}}$ is in the sense of (5.2.9). Thus, in this section, we understand $\mathcal{M}$ and $\bigotimes_{p \in \mathcal{P}} \mathcal{M}_p$, alternatively.

Consider the $\sigma$-algebra $\sigma(\mathbb{A}_Q)$ of the Adele ring $\mathbb{A}_Q$ as a monoid ($\sigma(\mathbb{A}_Q), \cap$), with its identity $\mathbb{A}_Q$. Define a monoidal action $\alpha$ of $\sigma(\mathbb{A}_Q)$ acting on $M$ in $B(H_Q)$ by

$$\alpha_S(m) \overset{\text{def}}{=} x_S m x_S^* = x_S m x_S, \quad \text{for all } S \in \sigma(\mathbb{A}_Q), \text{ and } m \in M,$$

where

$$H_Q = H \otimes H_Q.$$
Then the action $\alpha$ of $\sigma(A_Q)$ is indeed a well-defined monoidal action, since
\[
\alpha_{S_1 \cap S_2} (m) = \chi_{S_1 \cap S_2} m \chi_{S_1 \cap S_2} = \chi_{S_1 \cap S_2} m \chi_{S_1} \chi_{S_2} = \chi_{S_1} (\alpha_{S_2} (m)) \chi_{S_1} = \alpha_{S_1} (\alpha_{S_2} (m)) = (\alpha_{S_1} \circ \alpha_{S_2}) (m),
\]
and
\[
(\alpha_{S_1} (m))^* = (\chi_{S_1} m \chi_{S_1})^* = \chi_{S_1} m^* \chi_{S_1} = \alpha_{S_1} (m^*),
\]
for all $S_1, S_2 \in \sigma(A_Q)$, and $m \in M$. Thus,
\[
(\alpha_{S_1 \cap S_2} (m)) = (\alpha_{S_1} \circ \alpha_{S_2}) (m), \text{ and } (\alpha_{S_1} (m))^* = \alpha_{S_1} (m^*), \quad (5.3.2)
\]
for all $S_1, S_2 \in \sigma(A_Q)$, for all $m \in M$.

The action $\alpha$ compresses operators of $M$ in $B(H_Q)$, and hence it is bounded. So, $\alpha$ is a well-defined monoidal action of $\sigma(A_Q)$ acting on $M$ in $B(H_Q)$, by (5.3.2).

**Notes** Similar to Section 3.2, we denote $\alpha_S (m)$ simply by $m^S$, for all $S \in \sigma(A_Q)$ and $m \in M$. \(\Box\)

Notice that there exists an action $\chi$ of the monoid $\sigma(A_Q)$ acting on $H_Q = L^2(A_Q, \rho)$, such that
\[
\chi(S) = \chi_S, \text{ the characteristic function of } S, \quad (5.3.3)
\]
for all $S \in \sigma(A_Q)$. By construction,
\[
H_Q = \text{linear span of } \chi(\sigma(A_Q)) <,>,
\]
under the Hilbert space topology induced by $<,>$ of (5.2.1)'.

**Definition 5.3.** The triple $A_M = (M, \sigma(A_Q), \alpha)$ of a fixed von Neumann algebra $M$ in $B(H)$, the $\sigma$-algebra $\sigma(A_Q)$ of the Adele ring $A_Q$, understood as a monoid equipped with $(\cap)$, and the monoid action $\alpha$ of $\sigma(A_Q)$ in the sense of (5.3.1), is called an Adele $W^*$-dynamical system in $B(H_Q)$. For an Adele $W^*$-dynamical system $A_M$, define the corresponding crossed product $W^*$-algebra
\[
M_Q = M \times_{\alpha} \sigma(A_Q)
\]
by the $W^*$-subalgebra of $B(H_Q)$ generated by $M$ and $\alpha(\chi(\sigma(A_Q)))$, where $\chi$ is in the sense of (5.3.3), consisting of all elements
\[
\sum_{S \in \sigma(A_Q)} m_S \chi_S \text{ with } m_S \in M.
\]

This $W^*$-subalgebra $M_Q$ of $B(H_Q)$ is said to be the Adele dynamical $W^*$-algebra induced by $A_M$.

Let $A_M = (M, \sigma(A_Q), \alpha)$ be an Adele $W^*$-dynamical system, and let $M_Q = M \times_{\alpha} \sigma(A_Q)$ be the Adele dynamical $W^*$-algebra induced by $A_M$. Let $m_j \chi_{S_j}$ be elements of $M_Q$, with $m_j \in M$, and $S_j \in \sigma(A_Q)$, for $j = 1, ..., n$, for $n \in \mathbb{N}$. Then one can obtain that
\[
\prod_{j=1}^{n} (m_j \chi_{S_j}) = \left( m \left( \prod_{j=2}^{n} m_{j-1} \chi_{S_{j-1}} \right) \right) \left( \chi_{S_1 \cap S_2} \right), \quad (5.3.4)
\]
since
\[
(m_1 \chi_{S_1})(m_2 \chi_{S_2}) = m_1 \chi_{S_1} m_2 \chi_{S_1} \chi_{S_2} = m_1 \chi_{S_1} m_2 \chi_{S_1} \chi_{S_1 \cap S_2} = m_1 m_2 \chi_{S_1 \cap S_2}.
\]

Also, we have
\[
(m \chi_S)^* = \chi_S m^* = \chi_S m^*, \quad (5.3.5)
\]
\[
= \chi_S m^* \chi_S = \chi_S m^* \chi_S \chi_S = (m^*)^2 \chi_S = (m^*)^2 \chi_S^*,
\]
for all $m \chi_S \in M$, with $m \in M$, and $S \in \sigma(\mathcal{A}_Q)$.

So, the Adele dynamical $W^*$-algebra $M$ is a $W^*$-subalgebra of $B(\mathcal{H}_Q)$ generated by $M$ and $\chi(\mathcal{A}_Q))$, satisfying the conditions (5.3.4) and (5.3.5).

Let $\mathfrak{M} = L^\infty(\mathcal{A}_Q, \rho)$ be the Adele-ring von Neumann algebra. For a fixed von Neumann algebra $M$, construct the tensor product $W^*$-algebra

$$M_0 = M \otimes \mathfrak{M},$$

which is a $W^*$-subalgebra of $B(\mathcal{H}_Q)$. Define now a $W^*$-subalgebra $M$ of $M_0$ by the “conditional” tensor product $W^*$-algebra

$$M_0 \otimes M \otimes \mathfrak{M},$$

(5.3.6)

satisfying the following $\alpha$-relations (5.3.7) and (5.3.8); for all $m_1, m_2, m \in M, and S_1, S_2, S \in \sigma(\mathcal{A}_Q)$. Of course, the $\alpha$-relations; (5.3.7) and (5.3.8); are determined under linearity.

Similar to Section 3.3, we obtain the following structure theorem for $M$.

**Theorem 5.3.** Let $M_0 = M \times_{\alpha_0} \sigma(\mathcal{A}_Q)$ be the Adele dynamical $W^*$-algebra in $B(\mathcal{H}_Q)$ induced by an Adele $W^*$-dynamical system $A_M$, and let $\mathfrak{M}$ be the Adele-ring von Neumann algebra. Then $M_0$ and the conditional tensor product $W^*$-algebra $M \otimes \mathfrak{M}$ of (5.3.6) are $\ast$-isomorphic, i.e.,

$$M_0 = M \times_{\alpha_0} \sigma(\mathcal{A}_Q) \ast-isomorphic = M \otimes \mathfrak{M} = M_0.$$  

**Proof.** Let $M_0$ be the Adele dynamical $W^*$-algebra $M \times_{\alpha_0} \sigma(\mathcal{A}_Q)$ induced by an Adele $W^*$-dynamical system $A_M$, and let $M = M \otimes \mathfrak{M}$ be the conditional tensor product $W^*$-algebra (5.3.6) of a fixed von Neumann algebra $M$, and the Adele-ring von Neumann algebra $\mathfrak{M}$ in $B(\mathcal{H}_Q)$, satisfying the $\alpha$-relations (5.3.7) and (5.3.8).

Define now a morphism

$$\Phi : M_0 \to M_0$$

by a linear transformation satisfying

$$\Phi(m \otimes \chi_S) = m \chi_S,$$  

(5.3.10)

for all $m \in M$, and $S \in \sigma(\mathcal{A}_Q)$. Then it is generator-preserving, and hence, it is bijective and bounded. Also, it satisfies that

$$\Phi((m_1 \otimes \chi_S)(m_2 \otimes \chi_{S_2})) = \Phi((m_1 m_2^{S_1}) \otimes \chi_{S_1} \chi_{S_2})$$

$$= (m_1 m_2^{S_1}) \chi_{S_1 \cap S_2} = (m_1 \chi_{S_1})(m_2 \chi_{S_2})$$

$$= \Phi(m_1 \otimes \chi_{S_1}) \Phi(m_2 \otimes \chi_{S_2}),$$

for all $m_1, m_2 \in M$, and $S_1, S_2 \in \sigma(\mathcal{A}_Q)$. Thus, for any $T_1, T_2 \in M_0$, we have

$$\Phi(T_1 T_2) = \Phi(T_1) \Phi(T_2) \text{ in } M_0,$$  

(5.3.11)

by the linearity of $\Phi$. Furthermore,

$$\Phi((m \otimes \chi_S)^*) = \Phi((m^*)^S \otimes \chi_S^S)$$

$$= (m^*)^S \chi_S = (m \chi_S)^* = (\Phi(m \otimes \chi_S))^*,$$

for all $m \in M$, and $S \in \sigma(\mathcal{A}_Q)$. 


So, for any \( T \in \mathcal{M}_Q \),
\[
\Phi(T^*) = \Phi(T)^* \quad \text{in} \quad \mathcal{M}_Q. \tag{5.3.12}
\]

Therefore, by (5.3.11) and (5.3.12), the bijective linear transformation \( \Phi \) of (5.3.10) is a \(*\)-isomorphism from \( \mathcal{M}_Q \) onto \( \mathcal{M}_Q \). □

The above structure theorem (5.3.9) shows that, just like the \( p \)-adic cases, Adelic dynamical \( W^* \)-algebras \( M \times_\alpha \sigma(\mathbb{A}_Q) \) are understood as conditional tensor product \( W^* \)-algebras \( M \otimes \alpha \mathfrak{M} \). As in Section 3, we handle von Neumann algebras \( \mathfrak{M}_Q \) and \( \mathcal{M}_Q \), alternatively.

One of the most interesting results of the above structure theorem (5.3.9) is the following structure theorem.

**Theorem 5.4.** Let \( \mathfrak{M}_Q \) be the Adelic dynamical \( W^* \)-algebra induced by an Adelic \( W^* \)-dynamical system \( A_M \). Then \( \mathfrak{M}_Q \) is \(*\)-isomorphic to the weak tensor product \( \mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p) \) in the sense of (3.2.5), for \( p \in \mathcal{P} \), i.e.,
\[
\mathfrak{M}_Q \ast\text{-iso} = \otimes_{p \in \mathcal{P}} \mathcal{M}_p, \tag{5.3.13}
\]
with the system \( \varphi_M \),
\[
\varphi_M \overset{\text{def}}{=} 1_M \otimes \varphi = \{1_M \otimes \varphi_p\}_{p \in \mathcal{P}}, \tag{5.3.14}
\]
where \( \varphi = \{\varphi_p\}_{p \in \mathcal{P}} \) is in the sense of (5.2.9).

**Proof.** By (5.3.9), the given Adelic dynamical \( W^* \)-algebra \( \mathfrak{M}_Q \) is \(*\)-isomorphic to \( \mathcal{M}_Q = M \otimes \alpha \mathfrak{M} \);
\[
\mathfrak{M}_Q \ast\text{-iso} = \mathcal{M}_Q. \tag{5.3.15}
\]

Also, by (5.2.8), the Adelic-ring von Neumann algebra \( \mathfrak{M} \) is \(*\)-isomorphic to \( \otimes_{p \in \mathcal{P}} \mathfrak{M}_p \), where \( \varphi \) is in the sense of (5.2.9);
\[
\mathfrak{M} \ast\text{-iso} = \otimes_{p \in \mathcal{P}} \mathfrak{M}_p, \tag{5.3.16}
\]
where \( \mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p) \) are \( p \)-prime von Neumann algebras, for \( p \in \mathcal{P} \).

Thus, one can have that
\[
\mathfrak{M}_Q \ast\text{-iso} = M \otimes \alpha \mathfrak{M} \ast\text{-iso} = M \otimes \alpha \otimes_{p \in \mathcal{P}} \mathfrak{M}_p
\]
\[
\ast\text{-iso} \overset{\varphi_M}{=} \otimes_{p \in \mathcal{P}} \mathcal{M}_p
\]
where \( \varphi_M = \{1_M \otimes \varphi_p\}_{p \in \mathcal{P}} \)
\[
\ast\text{-iso} \overset{\varphi_M}{=} \otimes_{p \in \mathcal{P}} \mathcal{M}_p, \tag{5.3.17}
\]
by (3.3.3). □

The structure theorem (5.3.13) provides a useful tool for studying our Adelic dynamical \( W^* \)-algebras \( \mathfrak{M}_Q \) in terms of \( p \)-adic dynamical \( W^* \)-algebras \( \mathcal{M}_p \)’s.

**VI. Adelic Dynamical \( W^* \)-Algebras**

Let \( M \) be a fixed von Neumann algebra in \( B(H) \), and let
\[
\mathfrak{M}_Q = M \times_\alpha \sigma(\mathbb{A}_Q)
\]
be the Adelic dynamical \( W^* \)-algebra induced by an Adelic \( W^* \)-dynamical system.
\[ A_M = (M, \sigma(\mathbb{A}_\mathbb{Q}), \alpha) \text{ in } B(\mathcal{H}_\mathbb{Q}). \]

In Section 5, we showed that \( \mathcal{M}_\mathbb{Q} \) is \( \ast \)-isomorphic to the conditional tensor product \( W^\ast \)-algebra

\[ \mathcal{M}_\mathbb{Q} = M \otimes_\alpha \mathfrak{M} \]

of \( M \) and the Adele von Neumann algebra \( \mathfrak{M} = L^\infty(\mathbb{A}_\mathbb{Q}, \rho) \) by (5.3.9). And hence, it is \( \ast \)-isomorphic to the weak tensor product von Neumann algebra

\[ \mathcal{M}_p = \otimes_{\mathfrak{M}, p \in \mathcal{P}} \mathcal{M}_p \]

of \( p \)-adic dynamical \( W^\ast \)-algebras \( \mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p) \), for \( p \in \mathcal{P} \), by (5.3.13).

We understand these three von Neumann algebras \( \mathcal{M}_\mathbb{Q}, \mathcal{M}_Q \) and \( \mathcal{M}_Q \), as the same von Neumann algebras \( \mathcal{M}_\mathbb{Q} \). Especially, case-by-case, we use a suitable one among \( \{ \mathcal{M}_\mathbb{Q}, \mathcal{M}_Q, \mathcal{M}_Q \} \) as \( \mathcal{M}_\mathbb{Q} \).

First, recall that, if \( Y \in \sigma(\mathbb{A}_\mathbb{Q}) \), then there exist \( Y_p \in \sigma(\mathbb{Q}_p) \), for all \( p \in \mathcal{P} \), such that

\[ Y = \prod_{p \in \mathcal{P}} Y_p, \]

where most of \( Y_p \)'s are identical to \( Y_q \cap \mathbb{Z}_q \) (i.e., \( Y_q \subseteq \mathbb{Z}_q \), for \( q \in \mathcal{P} \).

For instance, the subset \( U \) of \( \mathbb{A}_\mathbb{Q} \),

\[ U = \prod_{p \in \mathcal{P}} U_p \quad (6.0.1) \]

is a well-determined element of \( \sigma(\mathbb{A}_\mathbb{Q}) \), where \( U_p \) are the unit circles of \( \mathbb{Q}_p \), for all \( p \in \mathcal{P} \). We call \( U \), the unit circle of the Adele ring \( \mathbb{A}_\mathbb{Q} \). Indeed, for any element \( (u_p)_{p \in \mathcal{P}} \in U \), we have

\[ |(u_p)_{p \in \mathcal{P}}|_\mathbb{Q} = \prod_{p \in \mathcal{P}} |u_p|_p = 1, \]

where \( |.|_\mathbb{Q} \) is the non-Archimedean norm on \( \mathbb{A}_\mathbb{Q} \) induced by the \( p \)-norms \( \{|.|_p\}_{p \in \mathcal{P}} \) (e.g., see [18]).

Define now a conditional expectation

\[ E : \mathcal{M}_\mathbb{Q} = \mathcal{M}_Q \to M \otimes_\alpha \mathbb{C} [\{ \chi_U \}] \ast-isom \ M \]

by a linear morphism satisfying that:

\[ E(m\chi_Y) = m(r\chi_U), \quad (6.0.2) \]

where \( r \in [0, 1] \) satisfies that:

\[ \int_{\mathbb{A}_\mathbb{Q}} \chi_{Y \cap U} \, d\rho = r \int_{\mathbb{A}_\mathbb{Q}} \chi_U \, d\rho = r \left( \prod_{p \in \mathcal{P}} \left( 1 - \frac{1}{p} \right) \right). \quad (6.0.3) \]

Remark here that the quantity \( \prod_{p \in \mathcal{P}} \left( 1 - \frac{1}{p} \right) \) on the right-hand side of (6.0.2)' satisfies that:

\[ \prod_{p \in \mathcal{P}} \left( 1 - \frac{1}{p} \right) = \left( 1 - \frac{1}{\infty} \right) \prod_{p:\text{primes}} \left( 1 - \frac{1}{p} \right) = \prod_{p:\text{primes}} \left( 1 - \frac{1}{p} \right) = \frac{1}{\zeta(\mathbb{Q})}, \]

where

\[ \zeta(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p:\text{primes}} \frac{1}{1 - p^{-s}} = \frac{1}{\prod_{p:\text{primes}} (1 - p^{-s})} \]

is the Riemann zeta function, satisfying that:

\[ \frac{1}{\zeta(s)} = \prod_{p:\text{prime}} \left( 1 - \frac{1}{p} \right), \text{ for } s \in \mathbb{C}. \]
By definition, it is clear that
\[ \zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty, \]
and hence,
\[ \frac{1}{\zeta(1)} = 0. \]

Thus, one can verify that the formula (6.0.3) becomes 0, for all \( m_{\chi_Y} \in \mathbb{M}_{\mathbb{Q}}, \) with \( m \in (M, \psi) \) and \( Y \in \sigma(\mathbb{A}_{\mathbb{Q}}). \) In other words, we cannot directly mimic the \( p \)-adic dynamical free-probabilistic approaches as in Section 4.

Therefore, we consider a new, but similar approach to establish a suitable free probability model on our Adelic dynamical \( W^* \)-algebra \( \mathbb{M}_{\mathbb{Q}}. \)

a) Adelic Dynamical \( W^* \)-Probability Spaces \( \{ (\mathbb{M}_{\mathbb{Q}}, \varphi_P) \}_{P \in \mathcal{P}}. \) As we have discussed at the beginning of this section, we cannot directly mimic the free-probabilistic settings from the \( p \)-adic dynamical \( W^* \)-probability spaces to our Adelic \( W^* \)-probability settings. So, we construct suitable linear functionals differently from those of Section 4 (and those of \([10]\)).

Take first a “finite” subset \( P \) of \( \mathcal{P}, \) say
\[ P = \{ p_1, \ldots, p_n \}, \]  
(6.1.1)
for some \( n \in \mathbb{N}, \) in particular, suppose all \( p_1, \ldots, p_n \) of \( P \) are primes (not \( \infty \)) in \( \mathcal{P}. \) We call such subsets \( P \) of \( \mathcal{P}, \) finite prime (sub)sets of \( \mathcal{P}. \)

Let \( P \) be a finite prime set (6.1.1) of \( \mathcal{P}. \) Define an element \( U_P \) of \( \sigma(\mathbb{A}_{\mathbb{Q}}) \) by
\[ U_P \overset{def}{=} \left( \prod_{P \in \mathcal{P}} U_P \right) \times \left( \prod_{q \in \mathcal{P} \setminus \mathcal{P}} \mathbb{Z}_q \right), \]  
(6.1.2)
under possible re-arrangement. i.e., for all \( p \) in \( P, \) take the unit circle \( U_p \) of \( \mathbb{Q}_p, \) and for almost all other \( q \) in \( \mathcal{P}, \) take \( \mathbb{Z}_q \) of \( \mathbb{Q}_q, \) and then product them to construct a \( \rho \)-measurable subset \( U_P \) in the Adele ring \( \mathbb{A}_{\mathbb{Q}}. \)

Then define a subalgebra \( M_P \) of \( \mathcal{M}_{\mathbb{Q}} = \mathbb{M}_{\mathbb{Q}} = \mathbb{M}_{\mathbb{Q}} \) by
\[ M_P \overset{def}{=} M \otimes_{\sigma} \mathbb{C} \left[ \{ \chi_S : S \in \sigma(\mathbb{A}_{\mathbb{Q}}), S \subseteq U_P \} \right]. \]  
(6.1.3)

Define now a conditional expectation
\[ E_P : \mathcal{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}} \rightarrow M_P \]
by a linear morphism satisfying that:
\[ E_P (m_{\chi_Y}) = m \chi_Y \cap U_P, \]  
(6.1.4)
Now, let’s check the morphism \( E_P \) of (6.1.4) is indeed a conditional expectation:
\[ E_P (m_{\chi_S}) = m_{\chi_S \cap U_P} = m_{\chi_S}, \]
since \( S \subseteq U_P, \) and hence, for any \( x \in M_P, \) we have
\[ E_P (x) = x, \] under linearity.
\[ E_P \]
(6.1.6) For \( m_{j \chi_{S_j}} \in M_P, \) for \( j = 1, 2, \) and \( m_{\chi_Y} \in \mathcal{M}_{\mathbb{Q}}, \) we have
\[ E_P \left( (m_{1 \chi_{S_1}})(m_{\chi_Y})(m_{2 \chi_{S_2}}) \right) \]
\[ = E_P \left( m_{1 \chi_{S_1} m_{2 \chi_{S_2} \cap Y}} \chi_{S_1 \cap Y \cap S_2} \right) \]
\[ = E_P \left( m_{1 \chi_{S_1} m_{2 \chi_{S_2} \cap Y}} \chi_{S_1 \cap Y \cap S_2} \right) \]
\[ = \left( m_{1 \chi_{S_1} m_{2 \chi_{S_2} \cap Y}} \chi_{S_1 \cap Y \cap S_2} \right), \]
and
\[ (m_{1 \chi_{S_1}})(E_P (m_{\chi_Y}))(m_{2 \chi_{S_2}}) \]
\[ = (m_{1 \chi_{S_1}})(m_{\chi_Y \cap U_P}))(m_{2 \chi_{S_2}}) \]
\[ = \left( m_{1 \chi_{S_1} m_{2 \chi_{S_2} \cap Y \cap U_P}} \chi_{S_1 \cap Y \cap S_2} \right), \]
\[ = \left( m_{1 \chi_{S_1} m_{2 \chi_{S_2} \cap Y}} \chi_{S_1 \cap Y \cap S_2} \right). \]
because \( S_1 \cap U_P = S_1 \), so, \( S_1 \cap Y \cap U_P = S_1 \cap Y \), and hence,
\[
E_P \left( (m_1 \chi_{U_P})(m \chi_Y)(m_2 \chi_{U_P}) \right) \\
= (m_1 \chi_{U_P}) \left( E_P(m \chi_Y) \right) (m_2 \chi_{U_P}) .
\]
Thus, under linearity, we have that:
\[
E_P (x_1 y x_2) = x_1 \ E_P (y) \ x_2 ,
\]
for all \( x_1 , x_2 \in M_P \) and \( y \in \mathbb{M}_Q \).

(6.1.7) Also, one has that:
\[
E_P (m \chi_Y)^* = E_P ((m^*)^\chi_Y) \\
= (m^*)^\chi_Y (\chi_Y \cap U_P) = (E_P(m \chi_Y))^* ,
\]
and hence, for all \( y \in \mathbb{M}_Q \), we have
\[
E_P(y^*) = E_P(y)^* .
\]

**Proposition 6.1.** The morphism \( E_P \) of (6.1.4) is a well-defined conditional expectation from \( \mathbb{M}_Q \) onto \( M_P \), for any finite prime set \( P \) of \( \mathcal{P} \).

**Proof.** By definition, the morphism \( E_P \) of (6.1.4) is bounded and linear. So, it is a conditional expectation because of (6.1.5), (6.1.6) and (6.1.7). 

Define now a morphism \( F_P : M_P \rightarrow M_P \) by a linear morphism satisfying that:
\[
F_P (m \chi_Y) = m \ (r_Y \chi_{U_P}) , \quad \text{for some} \ r_Y \in [0, 1].
\]

(6.1.8) In particular, the quantity \( r_Y \) in (6.1.8) is determined as follows in \([0, 1]\) of \( \mathbb{R} \):
\[
\int_{\mathbb{A}_\mathbb{Q}} \chi_Y \cap U_P \ d\rho = \rho (Y \cap U_P) \\
= \rho \left( \left( \prod_{p \in P} Y_p \right) \cap \left( \prod_{p \in P} V_p \right) \right) \\
= \rho \left( \left( \bigotimes_{p \in P} \rho_p \right) \left( \prod_{p \in P} (Y_p \cap V_p) \right) \right) \\
= \prod_{p \in P} \rho_p (Y_p \cap V_p) \\
= \left( \prod_{p \in P} r_p (1 - \frac{1}{p}) \right) \left( \prod_{q \in P \setminus P} r_q \cdot 1 \right)
\]
by (6.1.2)
\[
\text{for} \ r_w \in [0, 1], \ \text{since} \\
\rho_w (U_w) = 1 - \frac{1}{w}, \ \text{and} \ \rho_w (Z_w) = 1 \\
\text{for all} \ w \in \mathcal{P}, \ \text{and hence, we have} \n\]
\[
\int_{\mathbb{A}_\mathbb{Q}} \chi_Y \cap U_P d\rho = \left( \prod_{q \in \mathcal{P}} r_q \right) \left( \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \right) .
\]

(6.1.9) Define \( r_Y \) in \([0, 1]\) by
\[
r_Y = \prod_{q \in \mathcal{P}} r_q ,
\]
where the quantity of the right-hand side of (6.1.10) is from (6.1.9).

i.e., the morphism \( F_P \) on \( M_P \) satisfies
\[
F_P (m \chi_Y) = m \ (r_Y \chi_{U_P}) ,
\]

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where \( r_Y \in [0, 1] \) satisfy (6.1.10), for all \( m \chi_Y \in M_P \).

By Section 5.2, without loss of generality, one can verify that: if

\[
Y = \prod_{p \in P} Y_p \in \sigma(A_Q), \text{ with } Y_p \in \sigma(Q_p),
\]

then almost all \( Y_q \)'s are identical to \( Z_q \).

**Assumption** If we take \( Y = \prod_{p \in P} Y_p \in \sigma(A_Q) \), with \( Y_p \in \sigma(Q_p) \), then we assume almost all \( Y_q \)'s are identical to \( Z_q \). \( \square \)

Define now a linear functional

\[
\gamma_0 : M_P \rightarrow \mathbb{C}
\]

by

\[
\gamma_0 \overset{\text{def}}{=} \left( \otimes \int_{A_Q} \right) \circ F_P \quad (6.1.11)
\]

i.e., \( \gamma_0 \) is a linear morphism satisfying that:

\[
\gamma_0 (m \chi_Y) \overset{\text{def}}{=} (m) \left( \int_{A_Q} r_Y \chi_{U_P} d\rho \right) \quad (6.1.12)
\]

\[
= r_Y \left( m \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \right),
\]

for all \( m \in (M, \psi) \) and \( Y \in \sigma(A_Q) \), where \( r_Y \in [0, 1] \) is in the sense of (6.1.10). The linear morphism \( \gamma_0 \) of (6.1.12) is indeed a well-defined linear functional on \( M_P \).

Define now a linear functional \( \gamma_P \) on \( M_Q = M_Q = M_Q \) by

\[
\gamma_P \overset{\text{def}}{=} \gamma_0 \circ E_P, \quad (6.1.13)
\]

for any fixed finite prime sets \( P \).

Since \( \gamma_0 \) is a bounded linear functional, and \( E_P \) is a bounded conditional expectation, \( \gamma_P \) of (6.1.13) is indeed a well-defined linear functional on the Adelic dynamical \( W^* \)-algebra \( M_Q \).

**Definition 6.1.** Let \( M_Q = M_Q = M_Q \) be an Adelic dynamical \( W^* \)-algebra over a \( W^* \)-probability space \((M, \psi)\). Let \( P \) be a finite prime set of \( P \), and \( \gamma_P \), the corresponding linear functional in the sense of (6.1.13). Then the pair \((M_Q, \gamma_P)\) is called the Adelic dynamical \( W^* \)-probability space induced by a finite prime set \( P \) of \( P \).

By definition, for any \( m \chi_S \in M_Q \), one has that:

\[
\gamma_P (m \chi_S) = \gamma_0 (E_P (m \chi_S))
\]

\[
= \gamma_0 (m \chi_{S \cap U_P})
\]

\[
= r_{S \cap U_P} \left( m \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \right),
\]

where \( r_{S \cap U_P} \in [0, 1] \) satisfies (6.1.10) and (6.1.12).

Notice now that

\[
U_P = \left( \prod_{p \in P} U_p \right) \times \left( \prod_{q \in P \setminus P} \mathbb{Z}_q \right),
\]

under possible re-arrangement. Like in Section 4, if we replace \( U_P \)'s to \( p^k U_P \), for some \( k \in \mathbb{Z} \), i.e., if we define

\[
U_{P,k} \overset{\text{def}}{=} \left( \prod_{p \in P} U_{p,k} \right) \times \left( \prod_{q \in P \setminus P} \mathbb{Z}_q \right),
\]

where \( U_{p,k} = p^k U_p \), as in (4.8), then we have similar structures, for all \( k \in \mathbb{Z} \), with identity.
Lemma 6.2. Let $m \chi_S$ be a free random variable in the Adelic dynamical $W^*$-probability space $(M_\mathbb{Q}, \gamma)$, with $m \in M$, and $S \in \sigma(\mathcal{A}_\mathbb{Q})$. Then

$$\gamma((m \chi_S)^n) = r_S \left( (m(m^S)^{n-1}) \left( \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \right) \right), \quad (6.2.1)$$

for all $n \in \mathbb{N}$, where $r_S \cup U_P \in [0, 1]$ satisfies (6.1.10) and (6.1.12).

Proof. If $m \chi_S \in M_\mathbb{Q}$, with $m \in M$, and $S \in \sigma(\mathcal{A}_\mathbb{Q})$, then

$$(m \chi_S)^n = mm^S mm^S \ldots mm^S \chi_{S \cap \cdots \cap S}$$

for all $n \in \mathbb{N}$. Therefore, one can have that

$$\gamma((m \chi_S)^n) = \gamma((m(m^S)^{n-1} \chi_S))$$

$$= r_S \cup U_P \left( (m(m^S)^{n-1}) \left( \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \right) \right)$$

for all $n \in \mathbb{N}$, by (6.1.14), where $r_S \cup U_P \in [0, 1]$ satisfies (6.1.10) and (6.1.12).

More general to (6.2.1), we obtain the following lemma.

Lemma 6.3. Let $m_1 \chi_{S_1}, \ldots, m_n \chi_{S_n}$ be free random variables in an Adelic dynamical $W^*$-probability space $(M_\mathbb{Q}, \gamma)$, with $m_k \in M$, $S_k \in \sigma(\mathbb{Q}_p)$, for $k = 1, \ldots, n$, for $n \in \mathbb{N}$. Then

$$\gamma\left( \prod_{j=1}^n m_j \chi_{S_j} \right) = r_{\bigcap_{j=1}^n S_j} \left( \left( \prod_{j=1}^n m_j \cap \chi_{S_j} \right) \right) \left( \prod_{j=1}^n S_j \right), \quad (6.2.2)$$

where $r_{\bigcap_{j=1}^n S_j} \cup U_P \in [0, 1]$, satisfying (6.1.10) and (6.1.12).

Proof. If $m_k \chi_{S_k} \in (M_\mathbb{Q}, \gamma)$ are given as above, for $k = 1, \ldots, n$, then

$$\prod_{j=1}^n (m_j \chi_{S_j}) = m_1 m_2 m_3 \ldots m_n \chi_{S_1 \cap \cdots \cap S_n}$$

$$= \left( \prod_{j=1}^n m_j \cap \chi_{S_j} \right),$$

in $M_\mathbb{Q}$, for all $n \in \mathbb{N}$.
Thus, one has that:
\[
\gamma_P \left( \prod_{j=1}^{n} (m_j \chi_{S_j}) \right) = \gamma_P \left( \left( \prod_{j=1}^{n} m_j^{j-1} S_j \right) \left( \chi_{\prod_{j=1}^{n} S_j} \right) \right)
\]
\[
= r \left( \prod_{j=1}^{n} S_j \right) \cap U_P \left( \psi \left( \prod_{j=1}^{n} m_j^{j-1} S_j \right) \right) \left( \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \right),
\]
by (6.1.14), where \( r \left( \prod_{j=1}^{n} S_j \right) \cap U_P \in [0, 1] \) satisfies (6.1.10) and (6.1.12).

**Notation 6.2** In the following, we denote \( \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \) by \( \zeta_P \), for convenience.

By (6.2.1) and (6.2.2), we obtain the following free-distributional data of free random variables of \( \mathbb{M}_Q, \gamma_P \).

**Theorem 6.4.** Let \( (\mathbb{M}_Q, \gamma_P) \) be an Adelic dynamical W*-probability space determined by a finite prime set \( P \) of \( \mathcal{P} \), and let
\[
T_k = \sum_{S_k \in \text{Supp}(T_k)} m_{S_k} \chi_{S_k}, \text{ for } k = 1, ..., n,
\]
be free random variables, for \( n \in \mathbb{N} \). Then (6.2.3)
\[
\gamma_P \left( \prod_{j=1}^{n} T_j \right) = \zeta_P \left( \sum_{(S_1, ..., S_n) \in \prod_{j=1}^{n} \text{Supp}(T_j)} r \left( \prod_{j=1}^{n} S_j \right) \cap U_P \left( \psi \left( \prod_{j=1}^{n} m_{S_j}^{j-1} \chi_{\prod_{j=1}^{n} S_j} \right) \right) \right),
\]
where \( \zeta_P \) is in the sense of Notation 6.2, and where \( r \left( \prod_{j=1}^{n} S_j \right) \cap U_P \in [0, 1] \) satisfy (6.1.10) and (6.1.12).

**Proof.** Inductively, one can get that
\[
\prod_{j=1}^{n} T_j = \sum_{(S_1, ..., S_n) \in \prod_{j=1}^{n} \text{Supp}(T_j)} \left( \left( \prod_{j=1}^{n} m_{S_j}^{j-1} \chi_{\prod_{j=1}^{n} S_j} \right) \right),
\]
for all \( j = 1, ..., n \). So,
\[
\gamma_P \left( \prod_{j=1}^{n} T_j \right) = \gamma_P \left( T_1 T_2 \ldots T_n \right)
\]
\[
= \gamma_P \left( \sum_{(S_1, ..., S_n) \in \prod_{j=1}^{n} \text{Supp}(T_j)} \left( \left( \prod_{j=1}^{n} m_{S_j}^{j-1} \chi_{\prod_{j=1}^{n} S_j} \right) \right) \right)
\]
\[
= \sum_{(S_1, ..., S_n) \in \prod_{j=1}^{n} \text{Supp}(T_j)} \left( \gamma_P \left( \left( \prod_{j=1}^{n} m_{S_j}^{j-1} \chi_{\prod_{j=1}^{n} S_j} \right) \right) \right)
\]
\[
= \sum_{(S_1, ..., S_n) \in \prod_{j=1}^{n} \text{Supp}(T_j)} \left( r \left( \prod_{j=1}^{n} S_j \right) \cap U_P \left( \psi \left( \prod_{j=1}^{n} m_{S_j}^{j-1} \chi_{\prod_{j=1}^{n} S_j} \right) \right) \left( \zeta_P \right) \right)
\]
by (6.2.2), where \( r \left( \sum_{i=1}^{n} S_i \right) \cap U_p \in [0, 1] \) satisfy (6.1.10) and (6.1.12), and where \( \zeta_p \) is in the sense of Notation 6.2. 

Thanks to (6.2.3), we obtain the following corollary.

**Corollary 6.5.** Let \( T = \sum_{S \in \text{Supp}(T)} m_S \chi_S \) be a free random variable in \((\mathcal{M}_P, \gamma_p)\).

Then \( \gamma_p(T^n) = \zeta_p \left( \prod_{(S_1, \ldots, S_n) \in \text{Supp}(T)^n} \left( r \left( \sum_{i=1}^{n} S_i \right) \cap U_P \left( \psi \left( \prod_{j=1}^{n} (m_{S_j} \chi_{S_j}) \right) \right) \right) \right) \), (6.2.4)

\( \zeta_p \left( \prod_{(S_1, \ldots, S_n) \in \text{Supp}(T)^n} \left( r \left( \sum_{i=1}^{n} S_i \right) \cap U_P \left( \psi \left( \prod_{j=1}^{n} (m_{S_j} \chi_{S_j}) \right) \right) \right) \right) = (6.2.5) 

for all \( n \in \mathbb{N} \), where \( r \left( \sum_{i=1}^{n} S_i \right) \cap U_p \in [0, 1] \) satisfy (6.1.10) and (6.1.12). \( \square \)

Let \( m_1 \chi_{S_1}, \ldots, m_n \chi_{S_n} \) be free random variables in \((\mathcal{M}_Q, \gamma)\), for \( n \in \mathbb{N} \), where \( m_1, \ldots, m_n \in \mathbb{M} \), and \( S_1, \ldots, S_n \in \sigma(\mathcal{A}_Q) \). Then, by (6.2.3), one can obtain that:

\[
\begin{align*}
&= \sum_{\pi \in NC(n)} (\gamma_p(\pi) \left( m_1 \chi_{S_1}, \ldots, m_n \chi_{S_n} \right) \mu(\pi, 1_n) \\
&= \sum_{\pi \in NC(n)} \left( \prod_{V \in \pi} (\gamma_p(V) \left( m_1 \chi_{S_1}, \ldots, m_n \chi_{S_n} \right) \mu(1_{|V|}, 1_{|V|}) \right)
\end{align*}
\]

by the Möbius inversion (See Section 2.3)

\[
\begin{align*}
&= \sum_{\pi \in NC(n)} \left( \prod_{V=(i_1, \ldots, i_k) \in \pi} \gamma_p \left( m_1 \chi_{S_1}, \ldots, m_k \chi_{S_k} \right) \mu(0_k, 1_k) \right) \tag{6.2.7}
\end{align*}
\]

\[
\begin{align*}
&= \sum_{\pi \in NC(n)} \left( \prod_{V=(i_1, \ldots, i_k) \in \pi} \zeta_p \left( r \psi \left( \prod_{i=1}^{n} (m_{i} \chi_{S_i}) \right) \right) \mu(0_k, 1_k) \right),
\end{align*}
\]

where \( k_p(\ldots) \) mean free cumulants induced by \( \gamma_p \) in the sense of Section 2.3.

By (6.2.7), we obtain the following inner free structure of the given Adelic dynamical \( W^* \)-algebra \( \mathcal{M}_Q \), with respect to \( \gamma_p \).

**Theorem 6.6.** Let \( m_1 \chi_{S_1} \) and \( m_2 \chi_{S_2} \) be free random variables in an Adelic dynamical \( W^* \)-probability space \((\mathcal{M}_Q, \gamma_p)\), with \( m_1, m_2 \in \mathbb{M} \), and \( S \in \sigma(\mathcal{A}_Q) \), with \( p(S) \neq 0 \). Moreover, assume that \( S \) contains \( U_p \), i.e., suppose

\[ S = \prod_{q \in \mathbb{P}} S_q \text{ in } \mathcal{A}_Q, \text{ and } S_p \supset U_p, \text{ for all } p \in \mathbb{P}. \tag{6.2.8} \]

Then \( \{ m_1, m_1^S \} \) and \( \{ m_2, m_2^S \} \) are free in the \( W^* \)-probability space \((\mathbb{M}_Q, \psi)\), if and only if \( m_1 \chi_{S_1} \) and \( m_2 \chi_{S_2} \) are free in \((\mathcal{M}_Q, \gamma_p)\). i.e.,

\[
\{ m_1, m_1^S \} \text{ and } \{ m_2, m_2^S \} \text{ are free in } (\mathbb{M}_Q, \psi)
\]

\[
\iff m_1 \chi_{S_1} \text{ and } m_2 \chi_{S_2} \text{ are free in } (\mathcal{M}_Q, \gamma_p),
\]
under the condition (6.2.8).

Proof. (⇒) Assume that \( \{m_1, m_1^S\} \) and \( \{m_2, m_2^S\} \) are free in \((M, \psi)\). Then, by definition, all mixed free \( \ast \)-cumulants of them (with respect to the linear functional \( \psi \)) vanish (See Section 2.3, or [16]), i.e.,

\[
k_n^\psi(u_{i_1}^r, \ldots, u_{i_n}^r) = 0 \quad \text{in } C,
\]

for all \( n \in N \setminus \{1\} \), where \( (u_i, \ldots, u_n) \in \{m_1, m_2, m_1^S, m_2^S\} \) are “mixed,” and \((i_1, \ldots, i_n) \in \{1, 2\}^n \), and \((r_1, \ldots, r_n) \in \{1, *\}^n\), where \( k_n(\ldots) \) mean free cumulants induced by \( \psi \).

Consider mixed free \( \ast \)-cumulants of \( m_1 \chi_S \) and \( m_2 \chi_S \) in \((M_Q, \gamma)\), for a fixed nonzero \( \rho \) measurable set \( S \in \sigma(A_Q) \). By (6.2.7), one has that

\[
k_n^P((m_1 \chi_S)^{r_1}, \ldots, (m_n \chi_S)^{r_n})
\]

\[
= \left( \zeta^P_\ast \right) \sum_{\pi \in NC(n)} \left( \prod_{V=(j_1, \ldots, j_n) \in \pi} r_{V} \psi \left( \prod_{i=1}^{k} \left( [m_{j_i}^{r_1}]^S \right)^{S_{i_1}} \right) \mu(0_k, 1_k) \right)
\]

where all \( S_{j_i} \) are identical to \( S \), and \( r_{V} \) satisfy (6.1.10) and (6.1.12), and where

\[
|m_{j_i}^{r_1}|^S \quad \text{def} = \begin{cases} m_{Y}^S & \text{if } r = 1, \\ (m_{j_i}^S)^{S_{Y}} & \text{if } r = \ast, \end{cases}
\]

for all \( j, r \in \{1, \ast\} \) and \( S, Y \in \sigma(A_Q) \), and hence,

\[
= \left( \zeta^P_\ast \right) \sum_{\pi \in NC(n)} \left( \prod_{V=(j_1, \ldots, j_n) \in \pi} r_{V} \left( \prod_{i=1}^{k} \left( [m_{j_i}^{r_1}]^S \right)^{S_{i_1}} \right) \mu(0_k, 1_k) \right)
\]

by the condition (6.2.8) (Since the assumption (6.2.8) holds, \( r_V = 1 \), for all \( V \in \pi \), for all \( \pi \in NC(n) \), for all \( n \in N)\)

\[
= \left( \zeta^P_\ast \right) (k_n(u_{i_1}^r, \ldots, u_{i_n}^r)) = (\zeta^P_\ast) \cdot 0
\]

\[
= 0,
\]

for all \( n \in N \setminus \{1\} \). It shows that, if \( \{m_1, m_1^S\} \) and \( \{m_2, m_2^S\} \) are free in \((M, \), \( m_1 \chi_S, m_2 \chi_S \) \) are free in \((M_Q, \gamma)\), under the condition (6.2.8).

(⇐) Assume now that two free random variables \( m_1 \chi_S \) and \( m_2 \chi_S \) are free in \((M_Q, \gamma)\), where \( S \) satisfies \( \rho(S) \neq 0 \) and the condition (6.2.8), i.e.,

\[
k_n^P((m_1 \chi_S)^{r_1}, \ldots, (m_n \chi_S)^{r_n})
\]

(6.2.10)

\[
= \left( \zeta^P_\ast \right) \sum_{\pi \in NC(n)} \left( \prod_{V=(j_1, \ldots, j_n) \in \pi} r_{V} \left( \prod_{i=1}^{k} \left( [m_{j_i}^{r_1}]^S \right)^{S_{i_1}} \right) \mu(0_k, 1_k) \right)
\]

\[
= 0,
\]

whenever \((i_1, \ldots, i_n) \) are “mixed” in \( \{1, 2\}^n \), for \((r_1, \ldots, r_n) \in \{1, *\}^n \), for all \( n \in N \setminus \{1\} \).

The formula (6.2.10) is identical to

\[
(k_n(u_{i_1}^r, \ldots, u_{i_n}^r)),
\]

since \( r_V = 1 \), by (6.2.8), for the mixed \( n \)-tuple \((u_{i_1}, \ldots, u_{i_n}) \) of \( \{m_1, m_1^S\} \cup \{m_2, m_2^S\} \).

Since \( \rho(S) \neq 0 \), and since the condition (6.2.8) is assumed, \( \rho(S \cap U_P) \neq 0 \), and hence,
as in (6.2.10), equivalently,

\[ k_n^\psi(u_{i_1}^\psi, ..., u_{i_n}^\psi) = 0, \]

for all mixed n-tuple \( (u_{i_1}, ..., u_{i_n}) \in \{m_1, m_1^S, m_2, m_2^S\}. \) Equivalently, \( \{m_1, m_1^S\} \) and \( \{m_2, m_2^S\} \) are free in \( (M, \psi) \).

The above theorem shows that, the freeness of \( (M, \psi) \) acts like a certain kind of free-filterizations for the inner freeness of \( (M, \gamma_p) \), under the assumption (6.2.8).

The following corollary is a direct consequence of the above theorem.

**Corollary 6.7.** Let \( M_1 \) and \( M_2 \) be \( W^* \)-subalgebras of \( M \) in \( B(H) \), and assume that the subsets \( \{M_1, \alpha_S(M_1)\} \) and \( \{M_2, \alpha_S(M_2)\} \) are free in \( (M, \psi) \), for \( S \in \sigma(A_Q) \), with \( \rho(S \cup U_P) \neq 0 \), satisfying the condition (6.2.8). Then two subsets

\[ M_1 \otimes \alpha \{\chi_S\} \text{ and } M_2 \otimes \alpha \{\chi_S\} \text{ of } M_Q = M_Q, \]

are free in \( (M_Q, \gamma_p) \), for a fixed finite prime set \( P \) of \( \mathcal{P} \).

Conversely, if \( M_1 \otimes \{\chi_S\} \) and \( M_2 \otimes \{\chi_S\} \) are free in \( (M, \gamma_p) \), where \( S \) satisfies (6.2.8), then \( \{M_1, \alpha_S(M_1)\} \) and \( \{M_2, \alpha_S(M_2)\} \) are free in \( (M, \psi) \), too. \( \Box \)

Let \( U_P \) be in the sense of (6.1.2) for a fixed finite prime set \( P \) of \( \mathcal{P} \). Assume now that \( S_1, S_2 \in \sigma(A_Q) \) satisfies

\[ S_1 \cap U_P \neq \emptyset \text{ and } S_2 \cap U_P = \emptyset. \]  

(6.2.11)

For example, "\( S_2 \cap U_P = \emptyset \)" means that, if \( S_2 = \prod_{q \in \mathcal{P}} S_q^2 \), with \( S_q^2 \in \sigma(Q_q) \),

\[ S_q^2 \cap U_P = \emptyset, \text{ for all } q \in \mathcal{P}, \]

and

\[ S_q^2 \cap \mathbb{Z}_q = \emptyset, \text{ for all } q \in \mathcal{P} \setminus P. \]

By (6.2.11), it is clear that

\[ (S_1 \cap U_P) \cap (S_2 \cap U_P) = \emptyset, \]

(6.2.12)

even though \( S_1 \cap S_2 \neq \emptyset \).

**Theorem 6.8.** Let \( m_1 \chi_S, m_2 \chi_{S_2} \) be free random variables in an Adelic dynamical \( W^* \)-probability space \( (M_Q, \gamma_p) \). If \( S_1 \) and \( S_2 \) satisfy the condition (6.2.11) in \( \sigma(A_Q) \), then they are free in \( (M_Q, \gamma_p) \), i.e.,

\[ S_1 \text{ and } S_2 \text{ satisfy (6.2.11)} \]

(6.2.13)

\[ \implies M \otimes \alpha \mathbb{C}[\{\chi_{S_1}\}] \text{ and } M \otimes \alpha \mathbb{C}[\{\chi_{S_2}\}] \text{ are free in } (M_Q, \gamma_p). \]

**Proof.** Suppose \( S_1, S_2 \in \sigma(A_Q) \) satisfy the condition (6.2.11). Then, with respect to \( U_P \) of (6.1.2), they also satisfy the condition (6.2.12). Therefore, one has that:

\[ k_n^\psi((m_1 \chi_S)^{r_1}, ..., (m_1 \chi_S)^{r_n}) \]

(6.1.14)

\[ = (\xi_P) \sum_{\pi \in NC(n)} \left( \prod_{\nu(v) = (i_1, ..., i_\nu) \in \pi} \left( k \cap_{t=1}^{k-1} S_{i_t} \right) \cap U_P \psi^{k-1} \left( \prod_{t=1}^{k-1} m_{i_t} \right) \right) \mu(0_k, 1_k) \]

by (6.2.7), where \( \left( k \cap_{t=1}^{k-1} S_{i_t} \right) \cap U_P \in [0, 1] \) satisfy (6.1.10) and (6.1.12), and the elements \( \{m_{i_t}^n\} \) are in the sense of the proof of the above Theorem.

Assume that a block \( V = (i_1, ..., i_k) \) of \( \pi \) in (6.1.13) is mixed in \( \{1, 2\}^k \). Then the corresponding quantity
\[ r \left( \bigcap_{i \in \mathbb{N}} S_i \right) \cap U_P = 0 \text{ in } [0, 1], \]

by (6.2.12). Therefore, whenever a noncrossing partition \( \pi \) of \( NC(n) \) contains at least one mixed block, then the corresponding summand vanishes. Even though a noncrossing partition \( \theta \) of \( NC(n) \) does not contain a mixed block, since it contains a block corresponding to \( S_2 \), one obtains the quantity

\[ r_{S_2 \cap \ldots \cap S_2} \cap U_P = 0 \text{ in } [0, 1], \]

for at least one block of \( \theta \). Thus, even though \( \theta \) does not contain a mixed block, the corresponding partition-depending free moment vanishes.

i.e., whenever \( (i_1, \ldots, i_n) \in \{1, 2\}^n \) are mixed for \( n \in \mathbb{N} \setminus \{1\} \), then the free cumulants (6.1.13) vanish. Equivalently, \( m_1 \chi_{S_1} \) and \( m_2 \chi_{S_2} \) are free in \((\mathbb{M}_Q, \gamma_P)\).

With a freeness characterization (6.2.9) (under (6.2.8)), the above freeness necessary condition (6.2.13) provide inner free structures of the Adelic \( W^* \)-algebra \( \mathbb{M}_Q \) in terms of linear functionals \( \gamma_P \), for finite prime sets \( P \) of \( P \).

REFERENCES Références Referencias


