A Class of Multivalent Harmonic Functions Involving Salagean Operator

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Introduction- A continuous complex valued function $f=u+iv$ defined in a simply connected complex domain $D$ is said to be harmonic in $D$ if both $u$ and $v$ are real harmonic in $D$. Let $F$ and $G$ be analytic in $D$ so that $F(0)=G(0)=0$, $ReF = ReG = u$, $ImG = v$ by writing $(F+iG)/2 = h$, $(F-iG)/2 = g$, the function $f$ admits the representation $f = h + g$, where $h$ and $g$ are analytic in $D$. $h$ is called the analytic part of $f$ and $g$, the co-analytic part of $f$.

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A Class of Multivalent Harmonic Functions Involving Salagean Operator

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1. Introduction

A continuous complex valued function \( f = u + iv \) defined in a simply connected complex domain \( D \) is said to be harmonic in \( D \) if both \( u \) and \( v \) are real harmonic in \( D \). Let \( F \) and \( G \) be analytic in \( D \) so that \( F(0) = G(0) = 0 \), \( \text{Re}F = \text{Ref} = u \), \( \text{Re}G = \text{Im}f = v \) by writing \( (F+IG)/2 = h \), \( (F-IG)/2 = g \). The function \( f \) admits the representation \( f = h + g \) where \( h \) and \( g \) are analytic in \( D \). \( h \) is called the analytic part of \( f \) and \( g \), the co-analytic part of \( f \).

Ahuja and Jahangiri \cite{1}, \cite{2} introduce and studied certain subclasses of the family \( SH(m) \), \( m \geq 1 \) of all multivalent harmonic and orientation preserving functions in \( \Delta = \{ z : |z| < 1 \} \). A function \( f \) in \( SH(m) \) can be expressed as \( f = h + g \), where \( h \) and \( g \) are analytic functions of the form

\[
\begin{align*}
    h(z) &= z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \\
    g(z) &= \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1. 
\end{align*}
\]

For analytic function \( h(z) \in S(m) \), Salagean \cite{3} introduced an operator \( D_m^\nu \) defined as follows:

\[
D_m^0 h(z) = h(z), \quad D_m^1 h(z) = D_m(h(z)) = \frac{z}{m} h'(z) \quad \text{and} \quad D_m^\nu h(z) = \frac{z(D_m^{\nu-1} h(z))'}{m},
\]

\[
= z + \sum_{n=2}^{\infty} \left( \frac{n + m - 1}{m} \right)^\nu a_{n+m-1} z^{n+m-1}, \quad \nu \in \mathbb{N}.
\]

Whereas, Jahangiri et al. \cite{4} defined the Salagean operator \( D_m^\nu f(z) \) for multivalent harmonic function as follows:

\[
D_m^\nu f(z) = D_m^\nu h(z) + (-1)^\nu D_m^\nu g(z)
\]

where,

\[
D_m^\nu h(z) = z^m + \sum_{n=2}^{\infty} \left( \frac{n + m - 1}{m} \right)^\nu a_{n+m-1} z^{n+m-1}
\]

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In this paper we define a subclass $H_m(\lambda, \nu, \alpha)$ of $m$-valent harmonic functions involving Salagean operator $D_m^\nu f(z)$ as follows:

**Definition 1**

Let $f(z) = h(z) + \overline{g(z)}$ be the harmonic multivalent function of the form (1), then $f \in H_m(\lambda, \nu, \alpha)$ if and only if

$$\Re\left\{ (1 - \lambda) \frac{D_m^\nu f(z)}{z^m} + \lambda \frac{\partial}{\partial \theta} D_m^\nu f(z) \right\} > \alpha$$

where $0 \leq \alpha < 1$, $\lambda \geq 0$, $z = re^{i\theta} \in \Delta$ and $D_m^\nu f(z)$ is defined by (3) and

$$\frac{\partial}{\partial \theta} D_m^\nu f(z) = i \left[ z(D_m^\nu h(z))' - (-1)^\nu z(D_m^\nu g(z))' \right], \quad \frac{\partial}{\partial \theta} z^m = imz^m.$$

We denote the subclass $TH_m(\lambda, \nu, \alpha)$ consist of harmonic functions $f_\nu = h + g_\nu$ in $H_m(\lambda, \nu, \alpha)$ so that $h$ and $g_\nu$ are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1},$$

$$g_\nu(z) = (-1)^\nu \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}, \quad |b_m| < 1.$$  

Also note that $TH_m(\lambda, \nu, \alpha) \equiv TH_m(\lambda, \nu).$

The class $H_m(\lambda, \nu, \alpha)$ provides a transition between two classes:

$$\Re\left\{ \frac{D_m^\nu f(z)}{z^m} \right\} > \alpha \quad \text{and} \quad \Re\left\{ \frac{\partial}{\partial \theta} D_m^\nu f(z) \right\} > \alpha$$

as $\lambda$ moves between 0 and 1.

Denote $H_m(0, \nu, \alpha)$ by $P_m(\nu, \alpha)$ and $H_m(1, \nu, \alpha)$ by $Q_m(\nu, \alpha)$.

In this paper first we obtained the sufficient coefficient condition for $f(z) \in H_m(\lambda, \nu, \alpha)$ and then it is shown that this coefficient condition is also necessary for $f(z) \in TH_m(\lambda, \nu, \alpha)$. Also distortion bounds, extreme points, convex combination, integral operator, convolution condition, radius of convexity, radius of starlikeness for the functions $f(z) \in TH_m(\lambda, \nu, \alpha)$ are obtained.

II. **Main Results**

a) **Theorem 1 (Sufficient coefficient condition for $H_m(\lambda, \nu, \alpha)$)**

Assume that $f = h + \overline{g}$, $h$ and $g$ be given by (1) and $\lambda \geq 0$, if

$$\sum_{n=2}^{\infty} \left( \frac{n+m-1}{m} \right)^\nu \left[ \frac{n+m-1}{m} \right] \lambda + (1 - \lambda) |a_{n+m-1}| +$$

$$\sum_{n=1}^{\infty} \left( \frac{n+m-1}{m} \right)^\nu \left[ \frac{n+m-1}{m} \right] \lambda - (1 - \lambda) |b_{n+m-1}| \leq 1 - \alpha, 0 \leq \alpha < 1$$

then, $f(z) \in H_m(\lambda, \nu, \alpha)$. 

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b) **Remark 2**

The coefficient bound (6) in above theorem is sharp for the function

\[
f(z) = z^m + \sum_{n=2}^{\infty} \frac{x_n}{(\frac{n+m-1}{m})^\nu \left(\frac{n+m-1}{m}\right)^\lambda + (1-\lambda)} z^{n+m-1}
\]

\[
+ \sum_{n=1}^{\infty} \frac{y_n}{(\frac{n+m-1}{m})^\nu \left(\frac{n+m-1}{m}\right)^\lambda - (1-\lambda)} z^{n+m-1}
\]

where

\[
\frac{1}{1-\alpha} \left( \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| \right) = 1.
\]

c) **Remark 3**

For \( \lambda \geq 1 \),

\[
1 \leq \left(\frac{n+m-1}{m}\right) \leq \left(\frac{n+m-1}{m}\right) + (1-\lambda) \leq \left(\frac{n+m-1}{m}\right) - (1-\lambda).
\]

d) **Corollary 4**

Let \( f = h + \overline{g} \) be such that \( h \) and \( g \) are given by (1) and let

\[
\sum_{n=2}^{\infty} \left(\frac{n+m-1}{m}\right)^\nu \left(\frac{n+m-1}{m}\right)^\lambda + \frac{a_{n+m-1}}{1-\alpha} + \sum_{n=1}^{\infty} \left(\frac{n+m-1}{m}\right)^\nu \left(\frac{n+m-1}{m}\right)^\lambda - (1-\lambda) \frac{b_{n+m-1}}{1-\alpha}
\]

for \( \lambda \geq 1 \) and \( 0 \leq \alpha < 1 \), then \( f \in H(\lambda, \nu, \alpha) \).

Putting \( \lambda = 0 \) in Theorem 1 the following Corollary is obtained.

e) **Corollary 5**

Let \( f = h + \overline{g} \) be such that \( h \) and \( g \) are given by (1) and let

\[
\sum_{n=2}^{\infty} \left(\frac{n+m-1}{m}\right)^\nu \left(\frac{n+m-1}{m}\right)^\lambda + \frac{a_{n+m-1}}{1-\alpha} + \sum_{n=1}^{\infty} \left(\frac{n+m-1}{m}\right)^\nu \left(\frac{n+m-1}{m}\right)^\lambda - (1-\lambda) \frac{b_{n+m-1}}{1-\alpha}
\]

for \( 0 \leq \alpha < 1 \), then \( f \in P_m(\nu, \alpha) \).

Putting \( \lambda = 1 \) in Theorem 1 the following Corollary is obtained.

f) **Corollary 6**

Let \( f = h + \overline{g} \) be such that \( h \) and \( g \) are given by (1) and let

\[
\sum_{n=2}^{\infty} \left(\frac{n+m-1}{m}\right)^\nu \left(\frac{n+m-1}{m}\right)^\lambda + \frac{a_{n+m-1}}{1-\alpha} + \sum_{n=1}^{\infty} \left(\frac{n+m-1}{m}\right)^\nu \left(\frac{n+m-1}{m}\right)^\lambda - (1-\lambda) \frac{b_{n+m-1}}{1-\alpha}
\]

for \( 0 \leq \alpha < 1 \), then \( f \in Q_m(\nu, \alpha) \).

g) **Remark 7**

\( H_m(\lambda, \nu, \alpha_2) \subseteq H_m(\lambda, \nu, \alpha_1) \) for \( \alpha_1 \leq \alpha_2 \). Also, \( Q_m(\nu, \alpha) \subseteq P_m(\nu, \alpha) \).
h) **Theorem 8 (Coefficient inequality for** $\text{TH}_m(\lambda, v, \alpha)$) 

Let $f_v = h + g_v$ be so that $h$ and $g_v$ are given by (5). Then, 

$$
\begin{align*}
\sum_{n=2}^{\infty} \left( \frac{n+m-1}{m} \right)^{\nu} \left[ \left( \frac{n+m-1}{m} \right) \lambda + (1-\lambda) \right] |a_{n+m-1}| \\
\sum_{n=1}^{\infty} \left( \frac{n+m-1}{m} \right)^{\nu} \left[ \left( \frac{n+m-1}{m} \right) \lambda - (1-\lambda) \right] |b_{n+m-1}| \leq 1 - \alpha
\end{align*}
$$

(10)

where $0 \leq \alpha < 1, \lambda \geq 1$ and $|a_m| = 1$.

i) **Theorem 9 (Distortion Bounds)** 

If $f_v \in \text{TH}_m(\lambda, v, \alpha)$ and $\lambda \geq 1, |z| = r < 1$, then

$$
|f_v(z)| \leq (1 + |b_m|)r^m + \frac{r^{m+1}}{(m+1)^\nu} \left[ \frac{m(1-\alpha)}{(m+\lambda)} - \frac{m(2\lambda-1)}{(m+\lambda)} |b_m| \right]
$$

(11)

and

$$
|f_v(z)| \geq (1 - |b_m|)r^m - \frac{r^{m+1}}{(m+1)^\nu} \left[ \frac{m(1-\alpha)}{(m+\lambda)} - \frac{m(2\lambda-1)}{(m+\lambda)} |b_m| \right].
$$

(12)

j) **Corollary 10** 

Let $f_v \in \text{TH}_m(\lambda, v, \alpha)$ then for $|z| = r < 1$ and $\lambda \geq 1$

$$
[w : |w| < \left\{ \left( \frac{m+1}{m} \right)^{\nu} (m+\lambda) - m^{\nu+1} (1-\alpha) \right\} + \left\{ \frac{(2\lambda-1) - (m+1)^{\nu+1} (m+\lambda)}{(m+1)^{\nu+1} (m+\lambda)} \right\} |b_m| \} \subset f_v(\Delta).
$$

(13)

k) **Theorem 11 (Extreme Points)** 

Let $f_v$ be given by (5) then $f_v \in \text{TH}_m(\lambda, v, \alpha)$ ; $\lambda \geq 1$ if and only if.

$$
f_v(z) = \sum_{n=1}^{\infty} \left[ x_{n+m-1} h_{n+m-1}(z) + y_{n+m-1} g_{n+m-1,v}(z) \right],
$$

(14)

where

$$
h_m(z) = z^m, \quad h_{n+m-1}(z) = z^m - \frac{1}{\left( \frac{n+m-1}{m} \right)^{\nu} \left[ \left( \frac{n+m-1}{m} \right) \lambda + (1-\lambda) \right]} z^{n+m-1}, (n = 2, 3, ...)
$$

and

$$
g_{n+m-1,v}(z) = z^m + (-1)^\nu \frac{1}{\left( \frac{n+m-1}{m} \right)^{\nu} \left[ \left( \frac{n+m-1}{m} \right) \lambda - (1-\lambda) \right]} z^{n+m-1}, (n = 1, 2, 3, ...).
$$

$$
x_{n+m-1} \geq 0, \quad y_{n+m-1} \geq 0, \quad x_m = 1 - \sum_{n=2}^{\infty} x_{n+m-1} = \sum_{n=1}^{\infty} y_{n+m-1}.
$$

In particular, the extreme points of $\text{TH}_m(\lambda, v, \alpha)$ are $\{h_{n+m-1}\}$ and $\{g_{n+m-1,v}\}$. 

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1) **Theorem 12 (Convex Combination)**

If \( f_{i,v} (i = 1, 2, \ldots) \) belongs to \( \text{TH}_m (\lambda, v, \alpha) ; \lambda \geq 1 \) then the function
\[
\sum_{i=1}^{\infty} t_i f_{i,v}(z)
\]
is also in \( \text{TH}_m (\lambda, v, \alpha) \) where \( f_{i,v} \) is defined by
\[
f_{i,v} = z^m - \sum_{n=2}^{\infty} |a_{m+n-1,i}| z^{n+m-1} + (-1)^i \sum_{n=1}^{\infty} |b_{m+n-1,i}| z^{n+m-1} (i = 1, 2, \ldots)
\]
and \( 0 \leq t_i < 1, \sum_{i=1}^{\infty} t_i = 1 \).

m) **Definition 2**

The harmonic generalized Bernardi-Libera-Livingston integral operator \( L_c (f(z)) \) for \( m \)-valent functions is defined by
\[
L_c (f(z)) = \frac{c + m}{z^c} \int_0^z t^{c-1} h(t) dt + \frac{c + m}{z^c} \int_0^z t^{c-1} g(t) dt, \quad c > -1.
\]

n) **Theorem 13 (Integral Operator)**

Let \( f \in \text{TH}_m (\lambda, v, \alpha) ; \lambda \geq 1 \). Thus \( L_c (D_m f(z)) \) belongs to the class \( \text{TH}_m (\lambda, v, \alpha) \).

o) **Theorem 14 (Convolution Condition)**

Let \( f_v \in \text{TH}_m (\lambda, v, \alpha) \) and \( F_v \in \text{TH}_m (\lambda, v, \alpha) ; \lambda \geq 1 \) then the convolution
\[
(f_v \ast F_v) (z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1,i}| z^{n+m-1} + (-1)^i \sum_{n=1}^{\infty} |b_{n+m-1,i}| z^{n+m-1} \in \text{TH}_m (\lambda, v, \alpha).
\]

p) **Theorem 15 (Radius of Convexity)**

The radius of convexity for the function \( f_v \in \text{TH}_m (\lambda, v) \) is given by
\[
r_0 = \frac{(m+1)^{v-2}}{m} \frac{1}{1 - (2\lambda - 1)|b_m|}, \text{ for } \lambda \geq 1.
\]

q) **Theorem 16 (Radius of Starlikeness)**

The radius of starlikeness for the function \( f_v \in \text{TH}_m (\lambda, v) \) is given by
\[
r_0 = \frac{(m+1)^{v-1}}{m} \frac{1}{1 - (2\lambda - 1)|b_m|}, \text{ for } \lambda \geq 1.
\]

**References Références Referencias**