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I. Introduction

In Harper [1] it is stated that sphere on which there is no shear stress are found as boundaries in slow viscous fluid flow in two important contexts. The earth's core is, to a good approximation, such a boundary for the convection in its mantle, and the surface of a gas bubble is such a boundary for the flow outside it. In the literature corresponding to Harper's sphere theorem [1] for the axisymmetrical slow viscous flow past a shear stress-free sphere, there are the sphere theorems for axisymmetrical potential flows outside or inside a rigid sphere due to Butler [2] in terms of Stokes stream function [3]. Again there are exterior sphere theorem due to Weiss [4] and the interior sphere theorem of Ludford et al. [5] each for a general irrotational motion of inviscid fluid, both being expressed in terms of the potential function. Furthermore, for axisymmetrical slow viscous fluid motion outside or inside a rigid sphere there are sphere theorems in terms of the Stokes stream function, which are due to Collins [6, 7].

The two dimensional analogue of Harper's theorem [1] referred above is the circle theorem due to Usha et al. [8] for the slow viscous flow past a shear free circular boundary. Relevantly, there is a circle theorem for potential flow past a circular boundary, which is due to Milne - Thomson [3, 9]. Further, in the two-dimensional viscous flow theory similar theorems are found in Avudainayagon et al. [10] and Sen [11] for solving the problems of slow viscous flow past a rigid circular boundary with shear stress.

Following Batchelor [12], we may that when a body of small size moves through fluid, it generates a flow problem which is important in a variety of physical contexts, such as setting of sediment in liquid and fall of mist droplets in air. The matter of great practical interest is the drag force exerted by the fluid on the body. Except in a few simple bodies, such as spherical ones exact solutions for arbitrary body shapes in viscous fluid motions are, in general, not found in the literature.

Our main interest lies in studying the viscous flow about arbitrary rigid bodies which are shear stress-free. With this object in mind first we derive Harper's theorem for a shear stress-free sphere by an analytic technique; and this is done in section 3. For this purpose we need some relevant mathematical results, which are established in the following section.

Ref

5. G.S.S. Ludford, J. Martinek and G.C.K.Yeh (1955), The sphere theorem in rotational theory, Proc. Camb. Phil. Soc. 51, 389-393.

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II. Mathematical Theory

In this section, we first derive Stokes' equation in terms of the Stokes stream function $\psi=\psi(r,\theta)$ for the axisymmetrical motion about axisymmetrical bodies, such as a rigid sphere and then the condition of no shear stress on the sphere, due to an axisymmetrical fluid motion.

When the inertia force in a steady viscous flow field is negligibly small, the Navier-Stokes equations, governing of the flow become

$$\text{grad}\, p = \mu \nabla^2 q\,, \tag{1}$$

and

$$\text{div}\, q = 0, \tag{2}$$

where q is the fluid velocity, p the pressure, and μ the coefficient of viscosity of the fluid.

In the present paper it is convenient to derive the scalar expression of the vector equations (1) and (2) in spherical polar coordinates $(r,\ \theta,\ \phi)$ and then to express them in terms of Stokes stream function $\psi = \psi\ (r,\ \theta)$ as a dependent variable for the differential equations for an axis-symmetrical slow fluid motion in a viscous fluid.

The scalar expressions of equations (1) and (2) can be derived with the help of the relevant results in Batchelor [12, appendix 2] as

$$-\frac{\partial p}{\partial r} = \mu\left\{ \nabla^2 q_r - \frac{2q_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(q_\theta \sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial q_\phi}{\partial\phi} \right\}, \tag{3}$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \mu\left\{ \nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial\theta} - \frac{q_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\phi}{\partial\phi} \right\}, \tag{4}$$

$$\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} = \mu\left\{ \nabla^2 q_\phi + \frac{2}{r^2\sin\theta}\frac{\partial q_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial\phi} - \frac{q_\phi}{r^2\sin^2\theta} \right\}, \tag{5}$$

and

$$\frac{\partial\left(r^2 q_r\right)}{\partial r} + \frac{r}{\sin\theta}\frac{\partial}{\partial\theta}(q_\theta\sin\theta) + \frac{r}{\sin\theta}\frac{\partial q_\phi}{\partial\phi} = 0. \tag{6}$$

If the fluid motion is axisymmetrical about the z-axis, the fluid velocity everywhere in the flow field becomes independent of the azimuthal coordinate ϕ and the azimuthal velocity component $q_\phi = 0$. Thus equations (3) to (5) appear as

$$-\frac{\partial p}{\partial r} = \mu\left\{ \nabla^2 q_r - \frac{2q_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(q_\theta\sin\theta) \right\}, \tag{7}$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \mu\left\{ \nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial\theta} - \frac{q_\theta}{r^2\sin^2\theta} \right\}, \tag{8}$$

And

$$\frac{\partial}{\partial r}\left(r^2 q_r\right) + \frac{r}{\sin\theta}\frac{\partial}{\partial\theta}(q_\theta\sin\theta) = 0, \tag{9}$$

where ∇^2 is the three – dimensional Laplace's operator.

Notes

These equations can be further simplified with the help of the formulae for velocity components q_r and q_θ, defined in terms of Stokes stream function $\psi = \psi(r, \theta)$ for the axisymmetrical fluid motion; and these formulae are

$$q_r = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \text{ and } q_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}, \tag{10}$$

which clearly satisfy the mass conservation equation (9).

Now eliminating q_r and q_θ from equations (7) and (8) by using the relations (10), yields

$$\frac{\partial p}{\partial r} = \mu\frac{1}{r^2\sin\theta}\left[\frac{\partial^2\psi}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(-\cot\theta\frac{\partial\psi}{\partial\theta} + \frac{\partial^2\psi}{\partial\theta^2}\right)\right], \tag{11}$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{1}{r^2\sin\theta}\left[r\frac{\partial^3\psi}{\partial r^3} - \frac{\sin\theta}{r}\left(\frac{\partial^2}{\partial r\partial\theta}\left(\operatorname{cosec}\theta\frac{\partial\psi}{\partial\theta}\right)\frac{2}{r^2}\frac{\partial}{\partial\theta}\left(\operatorname{cosec}\theta\frac{\partial\psi}{\partial\theta}\right)\right]. \tag{12}$$

On using the operator, $E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$ for treating the axisymmetrical fluid motion [3], two concise forms of equations (11) and (12) are easily obtained as

$$\frac{\partial p}{\partial r} = \mu\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi), \tag{13}$$

$$\frac{\partial p}{\partial\theta} = -\mu\frac{1}{\sin\theta}\frac{\partial}{\partial r}(E^2\psi). \tag{14}$$

Next eliminating the pressure p from these equations results in

$$E^4\psi = 0, \tag{15}$$

which is obtained by different methods in Milne-Thomson [3]. The last equation may be called Stokes equation for the stream function of axisymmetrical and slow viscous fluid motion. We note that the stream function $\psi = \psi(r, \theta)$ for a slow viscous fluid motion past a axisymmetrical rigid body must satisfy the differential equation (15).

A general solution for the stream function ψ in spherical polar coordinates is given in [13]. For the convenience of the reference in our present study for a viscous flow past a shear stress-free sphere, we only quote here the relevant part of the general solution and this is

$$\psi(r,\theta) = \sum_{n=2}^{\infty}(a_n r^n + b_n r^{n+1} + c_n r^{n+2} + d_n r^{-n+3})\phi_n(\xi), \tag{16}$$

where a_n, b_n, c_n and d_n are arbitrary real constants, $\xi = \cos\theta$ and $\phi_n(\xi)$ is the Gegenbauer function of the first kind defined by

$$\phi_n(\xi) = \frac{P_{n-2}(\xi) - P_n(\xi)}{2n-1}, n \geq 2, \tag{17}$$

where $P_n(\xi)$ is the Legendre function of the first kind.

Ref

13. J. Happel and H. Brenner ((1986)), Low Reynolds number hydrodynamics, 4th Print, Martinus Nijhoff Publ. Dordrecht.

Now we are interested in deriving the condition for no shear stress on a rigid sphere in axisymmetrical viscous fluid motion. Here on the surface of a sphere $r = a$, out of the six components of the stress tensor [12, appendix 2] only three exist, which are

$$\sigma_{rr} = 2\mu\left(\frac{\partial q_r}{\partial r}\right)_{r=a},$$

$$\sigma_{\phi r} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial \phi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right)_{r=a}$$

and

$$\sigma_{\theta r} = \left(\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{2r}\frac{\partial q_r}{\partial \theta}\right)_{r=a}. \tag{18}$$

We first evaluate the velocity components q_r and q_θ by substituting Stokes' stream function (16) in the formulae (10) for the axisymmetrical flow past the shear stress- free sphere $r = a$.

On using the stress components (18), we then find that, on $r = a$, the normal stress $\sigma_{rr} \neq 0$, the shearing stress $\sigma_{\phi r} = 0$ in the ϕ - direction, since $q_\phi = 0$; and q_r is independent of ϕ and the shear stress in the θ - direction $\sigma_{\theta r} \neq 0$.

Finally, it is easy to calculate that

$$\sigma_{\theta r} = 0 \text{ on } r = a \quad \text{when} \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi}{\partial r}\right) = 0 , \tag{19}$$

and $q_r = 0$ on $r = a$ when $\psi = 0.$ (20)

Therefore, the results (19) and (20) are the required conditions for shear stress free-sphere $r = a$ for axisymmetrical fluid motion past the same sphere.

Our future aim is to solve the problems of axisymmetrical flows past arbitrary symmetrical body shapes which are shear stress free e.g. oblate and prolate spheroids, etc.

With this object in mind, we now present a relatively different analysis to establish the Harper's theorem [1] for the slow axisymmetrical viscous flow exterior to a shear stress-free sphere, and finally we also add an extension of the same theorem for the flow interior to the same sphere.

III. Harper's Theorem

In an unlimited incompressible viscous fluid there is a steady and slow axisymmetrical motion and the motion is characterized by the Stokes stream function $\psi_0 = \psi_0(r, \theta)$, whose singularities are all at a distance greater than 'a' from the origin and $\psi_0(r, \theta) \sim O(r^2)$ near the origin. Then if a shear stress-free sphere is introduced into the flow, Stokes stream function for the new flow outside the same sphere become

$$\psi(r,\theta) = \psi_0(r,\theta) - \frac{r^3}{a^3}\psi_0\left(\frac{a^2}{r},\theta\right). \tag{21}$$

Ref

1. J.F. Harper(1983), Axisymmetric Stokes flow images in Spherical free surfaces with application to rising bubbles, J. Australian. Math. Soc. Ser B 25 , 217-231.

Notes

Proof : Since the singularities of $\psi_0 = \psi_0(r, \theta)$ are at a distance greater than 'a' from the origin, $\psi_0(r, \theta)$ is regular at the origin. Then we suppose ψ_0 in the absence of any boundary, has an expression of the from

$$\psi_0(r,\theta)=\sum_{n=2}^{\infty}(A_n r^n + B_n r^{n+2})\phi(\xi), \tag{22}$$

where A_n, B_n are all known constants and $\phi_n(\xi)$ is the Gegenbauer function of, $\xi=\cos\theta$, of the first kind.

If a shear stress free sphere $r = a$ is now introduced into the viscous flow, the Stokes stream function for a possible new fluid motion must be obtained from the general expression (16), that is,

$$\psi(r,\theta)=\sum_{n=2}^{\infty}(A_n r^n + B_n r^{n+2} + C_n r^{-n+1} + D_n r^{-n+3})\phi_n(\xi), \tag{23}$$

where the last two terms constitute the perturbation stream function of the flow due to the presence of the sphere, and where C_n and D_n are the constants to be determined.

Here the conditions for the flow to be possible are

on $r=a$, $\psi(r, \theta) = 0$,

and on $r=a$, $\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right)=0.$

Thus, by using these conditions, we have

$$A_n a^n + B_n a^{n+2} + C_n a^{-n+1} + D_n a^{-n+3} = 0, \tag{24}$$

$$A_n n(n-3) a^{n-4} + B_n (n-1)(n+2) a^{n-2} + C_n(-n+1)(-n-2) a^{-n-3} + D_n(-n+3)(-n) a^{-n-1} = 0. \tag{25}$$

Solving (24) and (25) for C_n and D_n, we obtain

$$C_n=-B_n a^{2n+1}, \tag{26}$$

$$D_n=-A_n a^{2n-3}. \tag{27}$$

Next, we adopt the following analysis to obtain the result (21). Using the basic stream function (22) in (23) gives

$$\psi(r,\theta) = \psi_0(r,\theta) + \sum_{n=2}^{\infty}(C_n r^{-n+1} + D_n r^{-n+3})\phi_n(\xi). \tag{28}$$

Substituting (26) and (27) in this expression yields

$$\psi(r,\theta) = \psi_0(r,\theta) - \sum_{n=2}^{\infty}(B_n a^{2n+1} r^{-n+1} + A_n a^{2n-3} r^{-n+3})\phi_n(\xi). \tag{29}$$

Now our object is to give the expression (29) a closed form and this is done as follows. From the expression (22) one gets

$$\sum_{n=2}^{\infty}(A_n a^{2n} r^{-n} + B_n a^{2n+4} r^{-n-2})\phi_n(\xi) = \psi_0\left(\frac{a^2}{r},\theta\right)$$

Multiplying both sides by $\frac{r^3}{a^3}$, gives

$$\sum_{n=2}^{\infty} (A_n a^{2n-3} r^{-n+3} + B_n a^{2n+1} r^{-n+1}) \phi_n(\xi) = \frac{a^3}{r^3} \psi_0\left(\frac{a^2}{r}, \theta\right). \quad (30)$$

Finally, substituting (30) in (29) yields the Stokes stream function for the slow viscous fluid motion exterior to the shear stress-free sphere $r = a$ as

$$\psi\ (r,\ \theta) = \psi_0\ (r,\ \theta) - \left(\frac{r^3}{a^3}\right) \psi_0\left(\frac{a^2}{r}, \theta\right),$$

which is in agreement with Harper's result [1]. We then show that the perturbation velocity due to the last term in (3.14) vanishes at infinity. Since $\psi_0(r,\ \theta)$ is $O(r^2)$ near the origin the perturbation stream function $\left(\frac{r^3}{a^3}\right) \psi_0\left(\frac{a^2}{r}, \theta\right)$ is clearly $O(r)$ at infinity which implies a vanishing velocity at infinity. Hence the theorem is established.

IV. Extension of Harper's Theorem

We now extend Harper's sphere theorem for the viscous flow exterior to a shear stress-free sphere, to case of the flow interior to the same sphere. This extension corresponds to the Butler's interior sphere theorem [2] for the axi-symmetric and irrotational inviscid fluid flow within a sphere.

a) An Extension of Harper's Sphere Theorem

Let an axi-symmetric slow flow in an incompressible viscous fluid in the absence of rigid boundaries be characterized by Stokes steam $\psi_0 = \psi_0(r,\ \theta)$, whose singularities are all at a distance less '*a*' from the origin. Let $\psi_0 \sim O\left(\frac{1}{r^k}\right)$, $k \geq 1$ as $r \to \infty$. Now if a shear stress-free rigid sphere be introduced into the flow, the resultant flow interior to the sphere becomes

$$\psi = \psi_0(r,\ \theta) - \frac{r^3}{a^3}\, \psi_0\left(\frac{a^2}{r},\ \theta\right). \quad (31)$$

Proof : Since the singularities of the Stokes stream function $\psi_0(r,\ \theta)$ are all at a distance less than '*a*' from the origin, the function is regular everywhere in the region outside the sphere $r = a$, i.e., the region $r \geq a$.

Therefore a relevant expansion of $\psi_0(r,\theta)$ must be an expansion of the from

$$\psi_0(r,\theta) = \sum_{n=2}^{\infty} \left(A_n \frac{1}{r^{n-1}} + B_n \frac{1}{r^{-n+3}}\right) \phi_n(\xi)\ , \quad (32)$$

where A_n and B_n are all known constants, and $\phi_n\ (\xi)$ is the Gegenbauer function of the first kind already referred above.

If the shear free rigid sphere $r = a$ now is introduced into the basic flow characterized by the stream function (32), the Stokes stream function for the disturbed fluid motion may be given by

$$\psi(r,\theta) = \sum_{n=2}^{\infty} \left(A_n \frac{1}{r^{n-1}} + B_n \frac{1}{r^{n-3}} + \mathrm{C_n\ r^n} + \mathrm{D_n\ r^{n+2}} \right) \phi_n(\xi), \quad (33)$$

Ref

2. Butler. S.F.J. (1954), A note on Stokes stream function for motion with a spherical boundary, Proc. Camb phil. Soc, Vol. 49, 169-174.

Notes

where the last two terms constitute the perturbation stream function with the undetermined constants C_n and D_n.

First we determine the constants C_n and D_n as follows . On the shear stress-free sphere $r = a$, the Stokes stream function (33) must satisfy the boundary conditions on $r = a$, $\psi=0,$
and

$$r=a,\ \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi}{\partial r}\right) = 0\,.$$

Using these boundary conditions one obtains

$$A_n\, a^{-n+1} + B_n\, a^{-n+3} + C_n\, a^n + D_n\, a^{n+2} = 0, \tag{34}$$

$$A_n(n-1)\,(n+2)a^{-n-3} + B_n\, n(n-3)\, a^{-n-1} + C_n\, n(n-3)a^{n-4} + D_n(n-1)(n+2)a^{n-2} = 0. \tag{35}$$

Solving (34) and (35) for the values of C_n and D_n, we get very simple results as

$$C_n = -\, B_n a^{-2n+3} \quad \text{and} \quad D_n = -\, A_n a^{-2n-1}. \tag{36}$$

We now give a closed form of the stream function (33) in the following way. At the outset we note that the first two terms of (33) may be replaced by the stream function $\psi_0(r,\ \theta)$ referred to the expansion (32). Thus we have

$$\psi(r,\ \theta) = \psi_0(r,\ \theta) - \sum_{n=2}^{\infty} (B_n\, a^{-2n+3}\, r^{n} + A_n\, a^{-2n-1}\, r^{n+2})\, \phi_n\,(\xi). \tag{37}$$

By using the relation (32) we at once have

$$\sum_{n=2}^{\infty} (A_n\, a^{-2n-1}\, r^{n+2} + B_n\, a^{-2n+3}\, r^n) = \frac{r^3}{a^3}\psi_0\left(\frac{a^2}{r},\theta\right). \tag{38}$$

Finally, the use of this relation in (37), yields the Stokes stream function in closed form for the flow within a shear stress-free sphere as

$$\psi(r,\theta) = \psi_0\,(r,\ \theta) - \left(\frac{r^3}{a^3}\right)\psi_0\left(\frac{a^2}{r},\theta\right). \tag{39}$$

Next we show that the stream function $\psi(r,\ \theta)$ gives a finite value at the origin.

Since $\psi_0(r,\theta) \sim O\left(\frac{1}{r}\right)$ for large r, the last term on the right hand side of the stream function (39) is $O\left(r^4\right)$ near the origin so that the same term gives the vanishing velocity at the origin. Thus the theorem is established.

Example : A source and sink interior to a shear stress free sphere.

Let us consider, there be a source of strength m at the point $A_1(-c,0,\ 0)$ and a sink of strength $-m$ at the point $A_2(c,\ 0,\ 0)$ on the axis of symmetry, z-axis. The Stokes stream due to their combination is given by

$$\psi_0\left(r,\theta\right) = m\cos\theta_1 - m\cos\theta_2\,. \tag{40}$$

To find out the Stokes stream function for the flow within the sphere $r = a,$ first we are to show that $\psi_0(r,\theta) \approx O\left(\frac{1}{r}\right)$ for large r.

Since the source and the sink lie within the sphere $r = a$, we see that c is less than a, i.e., $c<a$, then we can rewrite the stream function (40) as

$$\psi_0(r,\theta)=\frac{m(r\cos\theta+c)}{\sqrt{r^2+c^2+2rc\cos\theta}}-\frac{m(r\cos\theta-c)}{\sqrt{r^2+c^2-2rc\cos\theta}}, \quad (41)$$

which can be expended as

$$\psi_0(r,\theta)=m(r\cos\theta+c)\ \sum_{n=0}^{\infty}\frac{(-1)^n c^n}{r^{n+1}}P_n(\cos\theta)$$
$$-m(r\cos\theta-c)\sum_{n=0}^{\infty}\ \frac{c^n}{r^{n+1}}P_n(\cos\theta). \quad (42)$$

After the reduction we see that $\psi_0(r,\theta)\approx 0\left(\frac{1}{r}\right)$ for large *r*.

Therefore here the extension of Harper's theorem applies and yields the Stokes stream function for the flow within the sphere as

$$\psi(r,\theta)=m\cos\theta_1-m\cos\theta_2+\frac{m}{ac}\cos\theta_3,-\frac{m}{ac}\cos\theta_4+\frac{m}{ac}r^2\left(R_{01}^2-R_{02}^2\right) \quad (43)$$

where the last four terms constitute the image system outside the sphere *r=a,* and where

$$R_{01}^2=r^2+\frac{a^4}{c^2}+2\left(\frac{a^2}{c}\right)r\cos\theta,\ \text{and}\ \ R_{02}^2=r^2+\frac{a^4}{c^2}-2\left(\frac{a^2}{c}\right)r\cos\theta\ .$$

V. Conclusion

We have shown an alternative way of the proof of Harper's theorem. In addition, we have extended the theorem for the flow interior to the same sphere and illustrate with an example. Numerical solutions of the problem can be useful for bubble rising research. This type of the problem has a great interest in geophysical applications.

References Références Referencias

1. J.F. Harper(1983), Axisymmetric Stokes flow images in Spherical free surfaces with application to rising bubbles, J. Australian. Math. Soc. Ser B 25 , 217-231.
2. Butler. S.F.J. (1954), A note on Stokes stream function for motion with a spherical boundary, Proc. Camb phil. Soc, Vol. 49, 169-174.
3. L.M. Milne-Thomson (1972), Theoretical Hydrodynamics, 5th edition, Macmillan.
4. P. Weiss (1944), On hydradynamical images, abitray irrotational flow disturbed by a sphere, Proc. Camb. Phil. Soc, 40, 259-261.
5. G.S.S. Ludford, J. Martinek and G.C.K.Yeh (1955), The sphere theorem in rotational theory, Proc. Camb. Phil. Soc. 51, 389-393.
6. W.D. Collins (1954), A note on Stokes' stream function for the slow steady motion of viscous fluid before a plane and spherical boundary, Mathematika. 1, 125-130.
7. W.D. Collins (1958), Note on a sphere theorem for the axi-symmetric Stokes' flow of a viscous fluid, Mathematika. 5, 118-121.
8. R. Usha and K. Hemalatha (1993), A note on plane Stokes' flow past a shear free impermeable cylinder, Z. angew. Math. Phys. 44, 73-84
9. L.M. Milne-Thomson (1940) , Hydrodynamical images, Proc. Camb. Phil. Soc. 36, 246-247.
10. A. Avudainayagam and B. Jothiram (1988). A Circle theorem for plane Stokes' flows, Q. J. Mech. Appl. Math, 41, Pt. 3 383-393.
11. S.K. Sen (1989), Circle theorems for steady Stokes flow, Z. Angew. Math. Phys. (ZAMP), 40., 139-146.

Notes

12. G. K. Batchelor (1969), An Introduction to Fluid Dynamics, Cambridge University press.
13. J. Happel and H. Brenner ((1986)), Low Reynolds number hydrodynamics, 4th Print, Martinus Nijhoff Publ. Dordrecht.

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