A Generalized Probability Distribution Pertaining To Product of Special Functions with Applications

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Abstract—The aim of the present paper is to study a generalized probability distribution involving M-series and H-function. Here, we first obtain the distribution function for the probability density function and then apply the technique of convolution to obtain the distribution of sum of two independent random variables with p.d.f. involving the generalized hyper geometric function. The results obtained here are unified in nature and capable of yielding a very large number of corresponding results (new and known) involving simpler special functions and polynomials as special cases of our results.

Keywords: Probability density function, distribution function, H-function, M-series.

I. INTRODUCTION

In this paper we consider a general class of statistical probability distribution, having the probability density function

\[ f(x) = \frac{x^{\lambda-1}}{C(1+\beta x)^\mu} \alpha^\gamma \sum_{n=0}^{\infty} \frac{(\alpha x)^n}{(1+\beta x)^n} \]

(1)

for \( 0 < x < \infty \) and \( f(x) = 0 \) for other values of \( x \) and the constant \( C \) is given by

\[ C = \beta^{-\lambda} \sum_{k=0}^{\infty} \frac{(u_k)_k ... (u_r)_k w^k}{(v_k)_k ... (v_s)_k} \Gamma(\alpha k + 1) \beta^{-\gamma k} \]

\[ \times H_{\mu+2,q+1} \left[ \frac{Z}{\beta^\gamma} \frac{(l-\lambda-\gamma k,p)(l-\mu-\eta k,l-s)}{(b_j,B_j)_p(l-\mu-\eta k)} \right]. \]

(2)

The M-series introduced by Sharma (2008) is defined as

\[ M_{\alpha} (u_1,...,u_r; v_1,...,v_s, w) = \sum_{k=0}^{\infty} \frac{(u_j)_k ... (u_r)_k w^k}{(v_1)_k ... (v_s)_k} \Gamma(\alpha k + 1) \cdot \ldots \]

(3)

For convergence conditions and other details of M-series, see Sharma (2008).

The H-function introduced by Fox (1961) is defined as

\[ H_{\mu,n,p,q} \left[ \frac{(a,A)_p}{(b,B)_\xi} \right] = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \prod_{j=1}^{m} \Gamma(b_j - B_j \xi) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j \xi) \prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j \xi) \prod_{j=n+1}^{p} \Gamma(a_j - A_j \xi) \]

\[ \cdot \left( \frac{Z}{\beta^\gamma} \frac{(a,A)_p}{(b,B)_\xi} \right). \]

(4)

For convergence conditions and other details of H-function see Fox (1961).

The following conditions are assumed to be satisfied

(i) \( \mu > \lambda + \gamma k > 0, \beta > 0, \sigma \geq \rho > 0, \)

(ii) \( \lambda + \gamma k + \rho \min_{1 \leq i \leq m} \left( \frac{b_j}{B_j} \right) > 0, \)

(iii) \( (\lambda + \gamma k - \mu - \eta k)(\rho - \sigma) \min_{1 \leq i \leq n} \left( \frac{a_j - \Delta}{A_j} \right) < 0, \)

(iv) \( \Delta = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j > 0, \)

(v) \( r \leq s, |w| < 1. \)
(vi) The parameters involved in (1) are real and so restricted that \( f(x) \) remains positive for \( 0 < x < \infty \). \( \ldots (5) \)

The p.d.f. \( f(x) \) given by (1) is generalization of the generalized F-distribution defined by Malik (1967). It is also shown that the p.d.f. defined by Mathai and Saxena (1971) is particular case of the p.d.f. \( f(x) \) given by (1).

II. THE DISTRIBUTION FUNCTION

The distribution function \( F(x) \) for p.d.f. \( f(x) \) is given by

\[
F(x) = \frac{1}{C} \beta^\lambda \sum_{k=0}^{\infty} \frac{(u_1)_k \ldots (u_r)_k}{(v_1)_k \ldots (v_s)_k} \frac{w^k}{\Gamma(\alpha k + 1)} \times \frac{1}{\beta \gamma^k} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \left( \frac{z}{\beta^\rho} \right)^{\xi} \phi(\xi) \, d\xi \\
\times \int_0^{\beta/(1+\beta x)} y^{\lambda + \gamma k + \rho \xi - 1} (1 - y)^{\mu + \eta k + (\sigma - \rho) \xi - \lambda - \gamma k - 1} \, dy
\]

where \( C \) is given by (2) and

\[
\phi(\xi) = \prod_{j=1}^{m} \frac{\Gamma(b_j - B_j \xi)}{\Gamma(1 - b_j + B_j \xi)} \prod_{j=1}^{n} \frac{\Gamma(1 - a_j + A_j \xi)}{\Gamma(a_j - A_j \xi)} \prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j \xi) \prod_{j=n+1}^{p} \Gamma(a_j - A_j \xi)
\]

Further, writing the incomplete beta function occurring in (7) in terms of Gauss hyper geometric function using a result (Erdélyi et al. 1954, p.87), applying Euler's transformation formula (Erdélyi et al. 1954, p.64, eq.(23)) and expressing the result thus obtained in terms of \( H \)-function of two variables with the help of (Srivastava, Gupta and Goyal 1982, p.84, eq.(6.2.1)), we arrive at the following result after a little simplification

\[
F(x) = \frac{x^\lambda}{C(1 + \beta x)^\mu} \sum_{k=0}^{\infty} \frac{(u_1)_k \ldots (u_r)_k w^k}{(v_1)_k \ldots (v_s)_k \Gamma(\alpha k + 1)} \frac{x^\gamma k}{(1 + \beta x)^\eta k}
\]
\begin{align*}
   &
   \binom{\beta, \gamma}{\mu, \nu} \left[ (1-\mu-\eta k ; \sigma, 1) ; (1-\lambda - \gamma k, \rho) (a_j, A_j)_1, p : (0, 1) \\
   &- (1+\beta x)^\sigma
   \right]
   \\
   &\times H_{0,1;m,n+1 : 1,1}^{1,1+p+1,q+1:1,1}
   \left[ \frac{z x^\rho}{1+\beta x} \right]
   \\
   &\times \left[ (1-\lambda - \gamma k, \rho, 1) ; (b_j, B_j)_1, q : (0, 1) \right]
   \end{align*}

\textbf{III. PARTICULAR CASES}

(1) If we set \( k = 0, \rho = \sigma = 0, \) we get the result obtained by Mathai and Saxena (1977, p.201, eq. (132)) and also by Srivastava and Singhal (1972, p.6, eq. (14)) after a little simplification.

\[ f(x) = \begin{cases}
   \frac{x^{\lambda-1}}{C_1 (1+\beta x)^\mu} p F_q \left[ \binom{a_p}{b_q} ; \frac{\beta x}{1+\beta x} \right] & \text{for } 0, \text{ otherwise}
   \end{cases} \]

(2) On taking \( k = 0, \rho = \sigma = 1, z = -\beta \) in (1) and reducing the Fox's H-function to generalized hypergeometric function \( F_p \) with the help of known result (Srivastava and Daoust (1982, p.18, eq. (2.6.3))) the p.d.f. (1) reduces to the following form

\[ F(x) = \frac{x^\lambda}{C_1 \lambda (1+\beta x)^\lambda} F_{1:p:1;1:q:0} \left[ \binom{a_1, \ldots, a_p ; \lambda - \mu}{b_1, \ldots, b_q ; \eta} ; \frac{\beta x}{1+\beta x} \right], \]

provided that \( |\beta x| < 1 \) and the conditions easily obtainable from those stated in (5) are satisfied.

(3) On taking \( \alpha = 1 \) in (1), the M-series reduces to generalized hypergeometric function \( F_s \) (Sharma (2008, p.189, eq. (5))), we get

\[ f(x) = \begin{cases}
   \frac{x^{\lambda-1}}{C_2 (1+\beta x)^\mu} \left[ \frac{w x^\gamma}{(1+\beta x)^\eta} \right] \left[ \frac{z x^\rho}{(1+\beta x)^\sigma} \right] (a_j, A_j)_1, p ; (b_j, B_j)_1, q \end{cases}, \]

where

\[ C_1 = B [\lambda, \mu - \lambda] \beta^{-\lambda} p F_{q+1} [\lambda, (a_p) ; \mu, (b_q) ; 1], \]

and the corresponding distribution function as obtained from (9) is given by

\[ F(x) = \frac{x^\lambda}{C_1 \lambda (1+\beta x)^\lambda} \left[ \frac{w x^\gamma}{(1+\beta x)^\eta} \right] \left[ \frac{z x^\rho}{(1+\beta x)^\sigma} \right] (a_j, A_j)_1, p ; (b_j, B_j)_1, q \]

where

\[ C_1 = B [\lambda, \mu - \lambda] \beta^{-\lambda} p F_{q+1} [\lambda, (a_p) ; \mu, (b_q) ; 1], \]

and the corresponding distribution function as obtained from (9) is given by

\[ F(x) = \frac{x^\lambda}{C_1 \lambda (1+\beta x)^\lambda} \left[ \frac{w x^\gamma}{(1+\beta x)^\eta} \right] \left[ \frac{z x^\rho}{(1+\beta x)^\sigma} \right] (a_j, A_j)_1, p ; (b_j, B_j)_1, q \]

provided that \( |\beta x| < 1 \) and the conditions easily obtainable from those stated in (5) are satisfied.
where

\[ C_2 = \beta^{-\lambda} \sum_{k=0}^{\infty} \frac{(u_1)_k \ldots (u_r)_k}{(v_1)_k \ldots (v_s)_k} \frac{w^k}{k!} \beta^{-\gamma k} \]

\times H_{m.m+2 \mid p+2.q+1} \left[ \frac{Z}{\beta^p} \left| \begin{array}{ccc} (1-\lambda-\gamma k, \rho), (1-\mu-\eta k + \lambda + \gamma k, \sigma - \rho), (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q}, (1-\mu-\eta k, \sigma) \end{array} \right| \right],

and the corresponding distribution function as obtained from (9).

\[ F(x) = \frac{x^\lambda}{C_2 (1 + \beta x)^\mu} \sum_{k=0}^{\infty} \frac{(u_1)_k \ldots (u_r)_k}{(v_1)_k \ldots (v_s)_k} \frac{w^k}{k!} \frac{x^{\gamma k}}{(1 + \beta x)^{\eta k}} \]

\times H_{0,1 : m,n+1 \mid 1,1} \left[ \begin{array}{c} z x^\rho \\ (1 + \beta x)^\sigma \\ 1 + \beta x \end{array} \right] \left| \begin{array}{ccc} (1-\mu-\eta k : \sigma, 1), (1-\lambda-\gamma k, \rho), (a_j, A_j)_{1,p} : (0,1) \\ (1-\lambda-\gamma k : \rho, 1), (b_j, B_j)_{1,q}, (1-\mu-\eta k, \sigma) : (0,1) \end{array} \right|.

If we set \( r = s = 0 \) in (1), the M-series reduces to Mittag-Leffler function (Sharma (2008, p.188, eq.(4))), we get

\[ f(x) = \begin{cases} \frac{x^{\lambda-1}}{C_3 (1 + \beta x)^\mu} E_{\alpha} \left[ \frac{w x^{\gamma}}{(1 + \beta x)^{\eta}} \right] \times H_{m,n \mid p,q} \left[ \frac{z x^\rho}{(1 + \beta x)^\sigma} \right] \left| \begin{array}{c} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q} \end{array} \right], & x > 0 \\ 0, & \text{otherwise} \end{cases} \]

where

\[ C_3 = \beta^{-\lambda} \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\alpha_k + 1)} \beta^{-\gamma k} \]
\[ \times H_{m,n+2}^{p+2, q+1} \left[ \frac{Z}{\beta^p} \begin{bmatrix} (1-\lambda-\gamma k, \rho), (1-\mu-\eta k+\lambda+\gamma k, \sigma-\rho), (a_j, A_j)_{1, p} \\ (b_j, B_j)_{1, q}, (1-\mu-\eta k, \sigma) \end{bmatrix} \right] \]

and the corresponding distribution function is given by

\[ F(x) = \frac{x^\lambda}{C_3 (1+\beta x)^\mu} \sum_{k=0}^{\infty} \frac{W^k}{\Gamma(\alpha k + 1)} \frac{x^{\gamma k}}{(1+\beta x)^{\eta k}} \]

\[ \times H_{0,1; m,n+1}^{1,1} :1,1 \times H_{1,1; p+1, q+1:1,1} \left[ \frac{Zx^\rho}{(1+\beta x)^\sigma} \right] \]

\[ \frac{1}{1+\beta x} \left[ (1-\mu-\eta k : \sigma, 1) : (1-\lambda-\gamma k, \rho), (a_j, A_j)_{1, p} : (0,1) \right] \]

\[ \frac{-(\lambda-\gamma k : \rho, 1) : (b_j, B_j)_{1, q}, (1-\mu-\eta k, \sigma) : (0,1)}{1+\beta x} \]

\[ \cdots (17) \]

\[ \cdots (18) \]

IV. THE DISTRIBUTION OF THE SUM OF TWO INDEPENDENT RANDOM VARIABLES

In this section, we shall obtain the distribution of the sum of two independent random variables with the p.d.f. as given by (10). By convolution formula, the distribution function \( G(y) \) of \( Y = X_1 + X_2 \) is given by

\[ G(y) = \int_0^y F_1(y-x_2) f_2(x_2) dx_2 \]

\[ \cdots (19) \]

where \( f_2(x_2) \) is the p.d.f. of the variable \( X_2 \) and \( F_1(y-x_2) \) is the distribution function of the variate \( X_1 \). The result can be expressed in the form of following theorem:

**Theorem 1.** Let \( X_i (i = 1, 2) \) be two independent random variables with p.d.f. defined by

\[ f_i(x_i) = \begin{cases} \frac{x_i^{\lambda_i-1}}{L_i (1+\beta_i x_i)^{\mu_i}} & F_i \left[ \left( a^{(i)}_{p_i} ; b^{(i)}_{q_i} ; \beta_{x_i} \right)_{1+\beta_i x_i} \right], x > 0 \\ 0, \text{ otherwise} \end{cases} \]

\[ \text{Where for } i = 1,2, \]

\[ L_i = B\left[ \lambda_i, \mu_i, -\lambda_i \right] \beta_i^{-\lambda_i} p_i, q_i F_i \left[ \left( \lambda_i, \left( a^{(i)}_{p_i} ; b^{(i)}_{q_i} ; \beta_{x_i} \right)_{1+\beta_i x_i} \right) \right] + 1 \]

\[ \cdots (21) \]

and the following conditions are assumed to be satisfied, for \( i = 1,2 \)

(i) \( \mu_i > \lambda_i > 0, \beta_i > 0 \)

(ii) \( p_i \leq q_i \) or \( p_i = q_i + 1 \) and

\[ (\mu_i - \lambda_i) + \sum_{j=1}^{q_i} (b^{(i)}_j) - \sum_{j=1}^{p_i} (a^{(i)}_j) > 0, \]

(iii) the parameters involved in (20) are so restricted that \( f_i(x_i) \) remain positive for \( 0 < x_i < \infty \). Then the p.d.f. \( g(y) \) of \( Y = X_1 + X_2 \) is given by

\[ \cdots (20) \]
\[
g(y) = \frac{y^{\lambda_1 + \lambda_2} - 1}{L_1 L_2} B(\lambda_1, \lambda_2) \sum_{1}^{4: \ p_1:1; \ p_2:0;0} \sum_{1: \ q_1:0; \ q_2:1;0;0} \int_{0}^{\infty} \frac{(y - x_2)^{\lambda_1} x_2^{\lambda_2 - 1}}{(1 + \beta_1 (y - x_2))^{\lambda_1} (1 + \beta_2 x_2)^{\lambda_2}} \cdot \frac{\beta_2 x_2}{1 + \beta_2 x_2} \ dx_2
\]

\[
|\beta_i| < 1 \ (i = 1, 2).
\]

**Proof.** On substituting the values of \( F_1 (y - x_2) \) and \( f_2(x_2) \) as obtained from the equations (18) and (20) respectively in equation (19), we get

\[
G(y) = \frac{1}{\lambda_1 L_1 L_2} \int_{0}^{y} \frac{(y - x_2)^{\lambda_1} x_2^{\lambda_2 - 1}}{(1 + \beta_1 (y - x_2))^{\lambda_1} (1 + \beta_2 x_2)^{\lambda_2}} \cdot \frac{\beta_2 x_2}{1 + \beta_2 x_2} \ dx_2
\]

\[
\sum_{1}^{4: \ p_1:1; \ p_2:0;0} \sum_{1: \ q_1:0; \ q_2:1;0;0} \frac{\beta_2 x_2}{1 + \beta_2 x_2}
\]

Now, expressing the generalized Kampé de Fériet function and the generalized hypergeometric function in series form using the results (Bryson (1974, p.27, eq. (28) and p.19, eq. (23)) and interchanging the order of integration and summations, then substituting \( x_2 = yz \) in the resulting integral, and after a little simplification, we get

\[
G(y) = \frac{y^{\lambda_1 + \lambda_2}}{\lambda_1 L_1 L_2} \sum_{r_1, r_2, r=0}^{\infty} r_1 r_2 r
\]
\[
(\lambda_1)_{r_1+r_2} \prod_{j=1}^{p_1} (a_j^{(1)} \lambda_j - \mu_j)_{r_1} \prod_{j=1}^{p_2} (a_j^{(2)} \lambda_j + r_2)_{r_1} \beta_1^{r_1+r_2} \beta_2^{r_1+r_2+r} \\
\times r_1! r_2! r!(1 + \lambda_1)_{r_1+r_2} \prod_{j=1}^{q_1} (b_j^{(1)} \lambda_j)_{r_1} \prod_{j=1}^{q_2} (b_j^{(2)} \lambda_j)_{r}
\]

\[
\times \int_0^1 z^{\lambda_2+r-1} (1-z)^{\lambda_1+r_1+r_2} \left[1 + \beta_1 y (1-z)^{-1}(\lambda_1+r_1+r_2) (1 + \beta_2 y z)^{-1}(\mu_2+r) \right] dz
\]

...\text{(24)}

Further, writing the integral occurring in right hand side of \text{(24)} in terms of Appell's function \(F_3\) using a result (Srivastava and Karlsson (1985, p.279, eq. 18)), expressing the Appell's function \(F_3\) in series form (Exton (1976, p.24, eq.(1.4.3))) and after a little simplification, the distribution function \(G(y)\) can be expressed in terms of generalized Lauricella function (Srivastava and Daoust (1969, p.454)) as follows

\[
G(y) = \frac{B(\lambda_1, \lambda_2)}{L_1 L_2 (\lambda_1 + \lambda_2)} \text{F}_{4:4:0:0:0}^{1:q_1:0:q_2+1:0:0} \left[ \begin{array}{cccc}
\beta_1 y & (\lambda_1 ; 1,1,0,0,1) & (\lambda_2 ; 0,0,1,1,0) & (\mu_2 ; 0,0,1,1,0) \\
\beta_1 y & (1 + \lambda_1 + \lambda_2 ; 1,1,1,1,1) & \ldots & \ldots \\
\beta_1 y & \ldots & \ldots & \ldots \\
-\beta_2 y & \ldots & \ldots & \ldots \\
-\beta_1 y & \ldots & \ldots & \ldots \\
\end{array} \right] \\
\]

\[
(1 + \lambda_1 ; 1,1,0,0,1) \right) , ((a_{p_1}^{(1)}, 1), (\lambda_1 - \mu_1, 1), ((a_{p_2}^{(2)}, 1), \ldots , \ldots \\
\]

\[
\ldots \right) , ((b_{q_1}^{(1)}, 1), \ldots , ((b_{q_2}^{(2)}, 1), (\mu_2, 1), \ldots , \ldots \\
\]

...\text{(25)}
Since $G(y)$ is the distribution function of the random variable $Y$, so the p.d.f. $g(y)$ of random variable $Y$ is obtained by differentiating the expression $G(y)$ with respect to $y$ and we get the desired result (22).

The result obtained here is quite general in nature and is capable of yielding a large number of corresponding results merely by specializing the parameters involved in it. To illustrate we give the following known special case of our result.

If we take the variables $X_1$ and $X_2$ have generalized $F$-distribution obtained by Malik (1967) then the density function of random variable $Y$ can be easily obtained from equation (22). Further on integrating the result thus obtained from 0 to $y$ with respect to $y$, we get the distribution function for the random variable $Y$, which is recently obtained by Dyer (1982, p.185, eq.(8.7)) in a slightly different form.

V. REFERENCES


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