mz-Compact Spaces

By A. T. Al-Ani
Irbid National University, Jordan

Abstract- In this work we study mz-compact spaces and mz-Lindelof spaces, where m is an infinite cardinal number. Several new properties of them are given. It is proved that every mz-compact space is pseudocompact (a space on which every real valued continuous function is bounded). Characterizations of mz-compact and mz-Lindelof spaces by multifunctions are given.

Keywords: mz-compact space, mz-Lindelof space, m-compact space, pseudocompact space, realcompact space.

GJSFR-F Classification : AMS 2010: 54C, 54D
mz-Compact Spaces

A. T. Al-Ani

Abstract: In this work we study mz-compact spaces and mz-Lindelöf spaces, where m is an infinite cardinal number. Several new properties of them are given. It is proved that every mz-compact space is pseudocompact (a space on which every real valued continuous function is bounded). Characterizations of mz-compact and mz-Lindelöf spaces by multifunctions are given.

Keywords and Phrases: mz-compact space, mz-Lindelöf space, m-compact space, pseudocompact space, realcompact space.

AMS classification 54C, 54D.

I. INTRODUCTION

In this paper we study some properties of mz-compact spaces and mz-Lindelöf spaces. Modifications of results about countably z-compact spaces [1] are proved. We relate mz-compact spaces to pseudocompact spaces (Theorem III b). Then we give some characterizations of mz-compact spaces. The collection of real valued continuous functions on a topological space X forms a ring denoted by C(X) [2]. Characterizations of mz-compact spaces in terms of z-filter bases and z-ideals are given. No separation property is assumed unless otherwise is stated. For definitions and notations not stated here see [2].

II. PRELIMINARIES

a) Definition

A space X is called m-compact if every open cover of X of cardinality at most m, has a finite subcover. Recall that a cozero set in a space X = (X, τ) is an \( f^{-1}[\mathbb{R} \setminus \{0\}] \) with a continuous function \( f: X \rightarrow \mathbb{R} \). Cozero sets constitute a base of a topology \( z\tau \) on X. (X, \( z\tau \)) is said to be mz-compact, (mz-Lindelöf) if \( zX = (X, z\tau) \) is m-compact (m-Lindelöf). Filters and z-ideals here are modifications of their respective definitions [2] by taking z-closed set (closed in \( zX \)) instead of zero-set.

b) Definition

A multifunction \( \alpha \) of a space X into a space Y is a set valued function on X into Y such that \( \alpha(x) = \Phi \) for every \( x \in X \). The class of all multifunctions on X into Y is denoted by \( m(X, Y) \).

c) Definition

A multifunction \( \alpha \) on X into Y is called closed graph if its graph \( G(\alpha) = \{ (x, y) \in X \times Y : y \in \alpha(x) \text{ is closed in } X \times Y \} \).
1. $X$ is mz-compact.
2. Every family of subsets of $X$ of cardinality at most $m$ each is an intersection of zero sets, with the finite intersection property has a non empty intersection.
3. Every mz-filter on $X$ is fixed.
4. Every mz-ideal in $C(X)$ is fixed.
5. $zX$ is m-compact.

b) Theorem

Every mz-compact space is pseudo compact.

c) Example ([3], [4])

Let $N$ be the set of positive integers. Topologize $N$ by taking a subbase the collection $\beta = \{U(b) : b + np \in \mathbb{N}, p \text{ prime }, b \text{ is not divisible by } p\}$ This space is a $T_2$ countably z-compact Lindelöf of space which is not countably compact. For $m = \chi_0$ in this example the space $N$ is mz-compact but not m-compact.

IV. A Characterization of MZ-compactness in Terms of Multifunctions

A space $X$ is said to be of character $m$ if every point of $X$ has a local base of cardinality at most $m$. We give here a characterization of mz-compact space $X$ in terms of multifunctions. Equivalently a characterization of m-compactness of $zX$. It is to be noted that a space is m-compact if every family of closed sets with cardinality at most $m$, satisfying the finite intersection property has a non-empty intersection. We shall use this fact in the proof of the second part of the following theorem.

a) Theorem

A space $X$ is mz-compact if for every space $Y$ with character $m$ and closed graph multifunction $\alpha \in m(zX, Y)$, the image of every closed set in $zX$ is closed in $Y$.

Proof

Let $zX$ be m-compact space, $Y$ be a space of character $m$, $\alpha \in m(zX, Y)$ with closed graph. Let $K$ be closed in $zX$ and $y \in Y - \alpha(K)$.

Let $\{B_\lambda : \lambda \in \Lambda\}$ be a local base of cardinality at most $m$ at $y$.

For each $x \in K$, there exist open set $V_x$ in $zX$ and $B_\lambda$ in $Y$ such that

$$(x, y) \in V_x \times B_\lambda$$

and $(V_x \times B_\lambda) \cap G(\alpha) = \emptyset$.

For each $\lambda \in \Lambda$, let

$W_\lambda = \cup \{V_x \times B_\lambda : x \in K, (x, y) \in V_x \times B_\lambda\}$

Then $\{W_\lambda : \lambda \in \Lambda\}$ is an open cover of $K$ of cardinality at most $m$. So, it has a finite subcover $\{W_{\lambda_i} : i = 1, 2, \ldots, n\}$.

Now, let $W = \bigcup \{W_{\lambda_i} : i = 1, 2, \ldots, n\}$. Then $W$ is open in $Y$ with $y \in W$ and $W \cap \alpha(K) = \emptyset$.

So, $\alpha(K)$ is closed in $Y$.

To prove the converse, let $\{K_\lambda : \lambda \in \Lambda\}$ be a family of closed sets in $zX$ of cardinality at most $m$, with the finite intersection property, let $y_0 \notin zX$. Topologize $\tilde{z}X \cup \{y_0\}$ by taking open sets all subsets of $zX$ and sets containing $y_0 \cup \alpha(K_\lambda)$ for some $\lambda \in \Lambda$. Obviously, $\tilde{z}X \cup \{y_0\}$ has character $m$. Let $\beta$ be the closure of the identity function of $zX$. Then $\beta$ has a closed graph and so, by hypothesis, it maps closed sets in $zX$ onto closed subsets in $Y$.
So, \( \beta(K_\lambda) \) is closed in \( zX \cup \{y_0\} \), for every \( \lambda \in \Lambda \). So, \( y_0 \in \beta(K_\lambda) \) for every \( \lambda \in \Lambda \).

Hence \( \{K_\lambda : \lambda \in \Lambda\} \) has a non-empty intersection.

Therefore, \( zX \) is m-compact.

V. MZ-LINDELOF SPACE

a) Definition

A space \( X \) is a \( P(m) \)-space if every intersection of at most \( m \) open sets in \( X \) is open.

The following result about mz-Lindelof space can be proved by the same technique of Theorems IV b.

b) Theorem

A space \( X \) is mz-Lindelof if for every \( P(m) \)-space \( Y \) and \( z \)-closed graph multifunction \( \alpha \in m(X,Y) \) the image of every \( z \)-closed set in \( X \) is closed in \( Y \).

References Références Referencias

This page is intentionally left blank