Certain Indefinite Integrals Involving Harmonic Number

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I. Introduction and Preliminaries

a) Harmonic Number

The $n^{th}$ harmonic number is the sum of the reciprocals of the first $n$ natural numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \quad (1.1)$$

Harmonic numbers were studied in antiquity and are important in various branches of number theory. They are sometimes loosely termed harmonic series, are closely related to the Riemann zeta function, and appear in various expressions for various special functions.

An integral representation is given by Euler

$$H_n = \int_{0}^{1} \frac{1-x^n}{1-x} \, dx \quad (1.2)$$

The equality above is obvious by the simple algebraic identity below

$$\frac{1-x^n}{1-x} = 1 + x + \ldots + x^n \quad (1.3)$$

An elegant combinatorial expression can be obtained for $H_n$ using the simple integral transform $x = 1 - u$:

$$H_n = \int_{0}^{1} \frac{1-x^n}{1-x} = -\int_{1}^{0} \frac{1-(1-u)^n}{u} \, du = \int_{0}^{1} \frac{1-(1-u)^n}{u} \, du$$

$$= \int_{0}^{1} \left[ \sum_{k=1}^{n} (-1)^{k-1} \frac{n}{k} u^{k-1} \right] \, du$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \frac{n}{k} \int_{0}^{1} u^{k-1} \, du$$

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b) Generalized Hypergeometric Functions

A generalized hypergeometric function \( {}_pF_q(a_1, \ldots a_p; b_1, \ldots b_q; z) \) is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

\[
\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k + a_1)(k + a_2) \cdots (k + a_p)}{(k + b_1)(k + b_2) \cdots (k + b_q)(k + 1)} z.
\]  

(1.5)

Where \( k + 1 \) in the denominator is present for historical reasons of notation, and the resulting generalized hypergeometric function is written

\[
{}_pF_q \left[ \begin{array}{c} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{array} ; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k z^k}{(b_1)_k(b_2)_k \cdots (b_q)_k k!}
\]  

(1.6)

or

\[
{}_pF_q \left[ \begin{array}{c} (a_p) \\ (b_q) \end{array} ; z \right] \equiv {}_pF_q \left[ \begin{array}{c} (a_j)_{j=1}^p \\ (b_j)_{j=1}^q \end{array} ; z \right] = \sum_{k=0}^{\infty} \frac{(a_p)_k z^k}{((b_q)_k) k!}
\]  

(1.7)

where the parameters \( b_1, b_2, \cdots, b_q \) are neither zero nor negative integers and \( p, q \) are non-negative integers.

The \( {}_pF_q \) series converges for all finite \( z \) if \( p \leq q \), converges for \( |z| < 1 \) if \( p \neq q + 1 \), diverges for all \( z, z \neq 0 \) if \( p > q + 1 \).

The \( {}_pF_q \) series absolutely converges for \( |z| = 1 \) if \( R(\zeta) < 0 \), conditionally converges for \( |z| = 1, z \neq 0 \) if \( 0 \leq R(\zeta) < 1 \), diverges for \( |z| = 1 \) if \( 1 \leq R(\zeta) \), \( \zeta = \sum_{i=1}^{p} a_i - \sum_{i=0}^{q} b_i \).

The function \( {}_2F_1(a, b; c; z) \) corresponding to \( p = 2, q = 1 \), is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss’s hypergeometric function (Gauss 1812, Barnes 1908). To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

\[
z(1-z)y'' + [c - (a + b + 1)z]y' - aby = 0
\]  

(1.8)

The solution of this equation is

\[
y = A_0 \left[ 1 + \frac{ab}{1!} z + \frac{a(a+1)b(b+1)}{2! c(c+1)} z^2 + \cdots \right]
\]  

(1.9)

This is the so-called regular solution, denoted

\[
{}_{2}F_{1}(a, b; c; z) = \left[ 1 + \frac{ab}{1!} z + \frac{a(a+1)b(b+1)}{2! c(c+1)} z^2 + \cdots \right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}
\]  

(1.10)

which converges if \( c \) is not a negative integer for all of \( |z| < 1 \) and on the unit circle \( |z| = 1 \) if \( R(c - a - b) > 0 \).

It is known as Gauss hypergeometric function in terms of Pochhammer symbol \((a)_k\) or generalized factorial function.
Many of the common mathematical functions can be expressed in terms of the hypergeometric function, or as limiting cases of it. Some typical examples are

\[(1 - z)^{-a} = z \, _2F_1(1, 1; 2; -z) \tag{1.11}\]

\[\sin^{-1} z = z \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) \tag{1.12}\]

The special case of (1.3.4) when \(a = c\) and \(b = 1\), or \(a = 1\) and \(b = c\), yields the elementary geometric series

\[\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots \tag{1.13}\]

Hence the term “Hypergeometric” is given. The term hypergeometric was first used by Wallis in his work “Arithmetica Infinitorum”. Hypergeometric series or more precisely Gauss series is due to Carl Friedrich Gauss (1777-1855) who in year 1812 introduced and studied this series in his thesis presented at Gottingen and gave the \(F\)-notation for it.

Here \(z\) is a real or complex variable. If \(c\) is zero or negative integer, the series (1.10) does not exist and hence the function \(_2F_1(a, b; c; z)\) is not defined unless one of the parameters \(a\) or \(b\) is also a negative integer such that \(-c < -a\). If either of the parameters \(a\) or \(b\) is a negative integer, say \(-m\) then in this case (1.10) reduce to the hypergeometric polynomial defined as

\[\_2F_1(-m, b; c; z) = \sum_{n=0}^{m} \frac{(-m)_n(b)_n}{(c)_n n!} \tag{1.14}\]

c) Hypergeometric Function of Second Kind

\[G(a, b; c; z) = \frac{\Gamma(1 - c)}{\Gamma(a - c + 1) \Gamma(b - c + 1)} \times _2F_1\left[\begin{array}{c} a, b \\ c \end{array}; z\right] + \frac{\Gamma(c - 1)}{\Gamma(a) \Gamma(b)} \times _2F_1\left[\begin{array}{c} 1 + a - c, 1 + b - c \\ 2 - c \end{array}; z\right] \tag{1.15}\]

where \(c \neq 0, \pm 1, \pm 2, \ldots\)

\[G(a, b; c; z) = z^{(1-c)} G(1 + a - c, 1 + b - c; 2 - c; z) \tag{1.16}\]

Each of the following functions is a solution of differential equation (1.8).

A system of two linearly independent solutions of differential equation (1.8) in the vicinity of the singular point \(z = 0, 1\) and \(\infty\) are given by

\[w_1^{(0)}(z) = _2F_1\left[\begin{array}{c} a, b \\ c \end{array}; z\right] \quad \text{and} \quad w_1^{(1)}(z) = _2F_1\left[\begin{array}{c} a, b \\ 1 + a + b - c \end{array}; 1 - z\right] \tag{1.17}\]
The equation (1.8) is also denoted by

\[ w_2^{(0)}(z) = z^{(1-c)} \binom{a, b}{c, z} \]

\[ w_2^{(1)}(z) = (1-z)^{(c-a-b)} \binom{a, b}{c-a, c-b} \]

\[ w_1^{(\infty)}(z) = (-z)^{-a} \binom{a, b}{1-a-b} \]

where \( c \neq 0, \pm 1, \pm 2, \ldots; (c - a - b) \) and \( (a - b) \) are not integers.

The equation (1.8) is also denoted by

\[ \binom{a, b}{c, z} = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{(c)_m m!} \]

\[ = 1 + \frac{a b z}{c} + \frac{a (a+1) b (b+1) z^2}{c (c+1) 2!} + \]

\[ + \frac{a (a+1) (a+2) b (b+1) (b+2) z^3}{c (c+1) (c+2) 3!} + \cdots + \text{ad inf.} \]

(1.20)

It is convergent for \( |z| < 1 \).

Note:

\[ \binom{a, b}{c, 0} = 2F_1 \left[ \begin{array}{c} 0, b \\ c \end{array} ; z \right] = 1 \]

(1.21)

(1 - z)^{-a} = \sum_{r=0}^{\infty} \frac{(a)_r z^r}{r!} = 1 \binom{a}{c} z^r ; |z| < 1 \]

(1.22)

d) Generalized Ordinary Hypergeometric Function of One Variable

The generalized Gaussian hypergeometric function of one variable is defined as follows

\[ \binom{a_1, a_2, a_3, \ldots, a_A}{b_1, b_2, b_3, \ldots, b_B} z = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n \cdots (a_A)_n z^n}{(b_1)_n (b_2)_n (b_3)_n \cdots (b_B)_n n!} \]

(1.23)

or,

\[ \binom{a_A}{b_B} z = \sum_{n=0}^{\infty} \frac{[(a_A)_n z^n}{[(b_B)_n n!} \]

(1.24)

where for the sake of convenience (in the contracted notation), \( (a_A) \) denotes the array of “A” number of parameters given by \( a_1, a_2, a_3, \ldots, a_A \). The denominator parameters are neither zero nor negative integers. The numerator parameters may be zero and negative integers. \( A \) and
Certain Indefinite Integrals Involving Harmonic Number

$\sum_{n=a}^{b}$ and $\prod_{n=a}^{b}$ are empty if $b < a$.

$[(a_A)]_{-n} = \frac{(-1)^{nA}}{[1 - (a_A)]_n}$

$[(a_A)]_n = (a_1)_n(a_2)_n(a_3)_n \cdots (a_A)_n = \prod_{m=1}^{A} (a_m)_n = \prod_{m=1}^{A} \frac{\Gamma(a_m + n)}{\Gamma(a_m)}$

where $a_1, a_2, a_3, \ldots, a_A; b_1, b_2, b_3, \ldots, b_B$ and $z$ may be real and complex numbers.

$\frac{3F_2}{3} \left[ \begin{array}{c} a, b, 1 \\ c, 2 \end{array} ; z \right] = \frac{(c-1)}{(a-1)(b-1)} z \times \frac{1}{2F_1} \left[ \begin{array}{c} a-1, b-1 \\ c-1 \end{array} ; z \right] - 1$

The convergence conditions of $A F_B$ are given below

Suppose that numerator parameters are neither zero nor negative integers (otherwise question of convergence will not arise).

(i) If $A \leq B$, then series $A F_B$ is always convergent for all finite values of $z$ (real or complex) i.e., $|z| < \infty$.

(ii) If $A = B + 1$ and $|z| < 1$, then series $A F_B$ is convergent.

(iii) If $A = B + 1$ and $|z| > 1$, then series $A F_B$ is divergent.

(iv) If $A = B + 1$ and $|z| = 1$, then series $A F_B$ is absolutely convergent, when

$$\Re\left\{ \sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n \right\} > 0$$

(v) If $A = B + 1$ and $z = 1$, then series $A F_B$ is convergent, when

$$\Re\left\{ \sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n \right\} > 0$$

(vi) If $A = B + 1$ and $z = 1$, then series $A F_B$ is divergent, when

$$\Re\left\{ \sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n \right\} \leq 0$$

(vii) If $A = B + 1$ and $z = -1$, then series $A F_B$ is convergent, when

$$\Re\left\{ \sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n \right\} > -1$$

(viii) If $A = B + 1$ and $|z| = 1$, but $z \neq 1$, then series $A F_B$ is conditionally convergent, when
-1 < \text{Re}\left\{ \sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n \right\} \leq 0

(ix) If \( A > B + 1 \), then series \( A F_B \) is convergent, when \( z = 0 \).

(x) If \( A = B + 1 \) and \( |z| \geq 1 \), then it is defined as an analytic continuation of this series.

(xi) If \( A = B + 1 \) and \( |z| = 1 \), then series \( A F_B \) is divergent, when

\[
\text{Re}\left\{ \sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n \right\} \leq -1
\]

(xii) If \( A > B + 1 \), then a meaningful independent attempts were made to define MacRobert’s \( E \)-function, Meijer’s \( G \)-function, Fox’s \( H \)-function and its related functions.

(xiii) If one or more of the numerator parameters are zero or negative integers, then series \( A F_B \) terminates for all finite values of \( z \) i.e., \( A F_B \) will be a hypergeometric polynomial and the question of convergence does not enter the discussion.

II. **Main Indefinite Integrals**

\[
\int \frac{\cosh x}{\sqrt{1 - \sin x}} \, dx = -\frac{1}{\sqrt{1 - \sin x}} \left( \frac{3}{5} - \frac{\ell}{5} \right) \left( \cos \frac{x}{2} - \sin \frac{x}{2} \right) \left\{ \text{cosh} \left( 1 + \frac{\ell}{2} \right) x - \text{sinh} \left( 1 + \frac{\ell}{2} \right) x \right\} \times
\]

\[
\times \left[ (\sin x - \ell \cos x) {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2} + \ell; \sin x - \ell \cos x \right) - \left( \text{sinh}(2x) + \text{cosh}(2x) \right) \times
\]

\[
\times_2F_1 \left( -\frac{1}{2} - \ell, \frac{1}{2} - \ell; \sin x - \ell \cos x \right) + \left( \text{sinh}(2x) + \text{cosh}(2x) \right) \right] + \text{Constant}
\]  

(2.1)

\[
\int \frac{\sinh x}{\sqrt{1 - \cos x}} \, dx = \frac{1}{\sqrt{1 - \cos x}} \left( \frac{3}{5} - \frac{\ell}{5} \right) \left( \cos \frac{x}{2} - \sin \frac{x}{2} \right) \left\{ \text{cosh} \left( 1 + \frac{\ell}{2} \right) x - \text{sinh} \left( 1 + \frac{\ell}{2} \right) x \right\} \times
\]

\[
\times \left[ (\ell \sin x + \ell \cos x) {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2} + \ell; \sin x - \ell \cos x \right) - \left( \text{sinh}(2x) + \text{cosh}(2x) \right) \times
\]

\[
\times_2F_1 \left( -\frac{1}{2} - \ell, \frac{1}{2} - \ell; \sin x - \ell \cos x \right) + \left( \text{sinh}(2x) + \text{cosh}(2x) \right) \right] + \text{Constant}
\]  

(2.2)

\[
\int \frac{\sinh x}{\sqrt{1 - \cos x}} \, dx = -\frac{1}{5\sqrt{1 - \cos x}} 2 e^{(-1+\frac{\ell}{2})x} \left[ (1 + 2\ell)e^{2x} {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2} - \ell; e^x \right) -
\]

\[-(1 - 2\ell) {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2} + \ell; e^x \right) \right] \sin \frac{x}{2} + \text{Constant}
\]  

(2.3)

\[
\int \frac{\cosh x}{\sqrt{1 - \cos x}} \, dx = -\frac{1}{5\sqrt{1 - \cos x}} 2 e^{(-1+\frac{\ell}{2})x} \left[ (1 + 2\ell)e^{2x} {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2} - \ell; e^x \right) +
\]

\[+(1 - 2\ell) {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2} + \ell; e^x \right) \right] \sin \frac{x}{2} + \text{Constant}
\]  

(2.4)

\[
\int \frac{\cos x}{\sqrt{1 - \cos x}} \, dx = -\frac{1}{5\sqrt{1 - \cos x}} e^{-ix}(e^x - 1) \left[ (1 + 2\ell)e^{2x} {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2} - \ell; e^x \right) +
\]

\[+(1 - 2\ell) {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2} + \ell; e^x \right) \right] + \text{Constant}
\]  

(2.5)
\[
\int \frac{\sin x \, H_1^{(x)}}{\sqrt{1 - \cosh x}} \, dx = \frac{1}{5\sqrt{1 - \cosh x}} \, e^{-ix}(e^x - 1) \left[ (2 - i) \, _2F_1 \left( \frac{1}{2} - i, 1; \frac{3}{2} - i; e^x \right) + (2 + i) \, e^{2ix} \, _2F_1 \left( \frac{1}{2} + i, 1; \frac{3}{2} + i; e^x \right) \right] + \text{Constant} \quad (2.6)
\]

\[
\int \frac{\sin x \, H_2^{(x)}}{\sqrt{1 - \cosh x}} \, dx = \frac{1}{(25 + 5 \log^2 4 - 10 \log 4)\sqrt{1 - \cosh x}} \times \left[ 2^{-x} e^{-ix} (e^x - 1) \left\{ (2 - i) \, 2^x (5 + 4 \log^2 2 - 4 \log 2) \, _2F_1 \left( \frac{1}{2} - i, 1; \frac{3}{2} - i; e^x \right) + (2 + i) \left( 2^x e^{2ix} (5 + 4 \log^2 2 - 4 \log 2) \, _2F_1 \left( \frac{1}{2} + i, 1; \frac{3}{2} + i; e^x \right) - (1 + 2i) e^{2ix} (log 4 - 1 + 2i) \, _2F_1 \left( 1, \frac{1}{2} + i; 2; \frac{3}{2} + i; e^x \right) + (3 - 4i + (2 + 4i) \log 2) \, _2F_1 \left( 1, \frac{1}{2} - i; 2; \frac{3}{2} - i; e^x \right) \right\} \right] + \text{Constant} \quad (2.7)
\]

\[
\int \frac{\sin x \, H_3^{(x)}}{\sqrt{1 - \cosh x}} \, dx = -\frac{1}{5(2^x + 3^x + 6^x)\sqrt{1 - \cosh x}} \left[ i2^x 3^x (2^x + 3^x + 1) \sinh \left( \frac{x}{2} \right) \times \right.
\]

\[
\left( \frac{1}{(\log 9 + (-1 + 2i))} \right) (1 + 2i) e^{\left( \frac{1}{2} - i \right)x} \, _2F_1 \left( \frac{1}{2} - i, 1; \frac{3}{2} - i; e^x \right) - (1 - 2i) e^{\left( \frac{3}{2} + i \right)x} \, _2F_1 \left( \frac{1}{2} + i, 1; \frac{3}{2} + i; e^x \right) - \left( 5e^{\left( \frac{1}{2} - i \right) \log 9 + (-1 + 2i)} \times \right.
\]

\[
\left. \times _2F_1 \left( 1, \frac{1}{2} - i; 2; \frac{3}{2} - i; e^x \right) \right) \bigg] + 5 \times 3^{-x} e^{\left( \frac{1}{2} + i \right)x} \, _2F_1 \left( 1, \left( \frac{1}{2} + i \right); \left( \frac{3}{2} + i \right); e^x \right) + \frac{5 \times 3^{-x} e^{\left( \frac{1}{2} + i \right)x} \, _2F_1 \left( 1, \left( \frac{1}{2} + i \right); \left( \frac{3}{2} + i \right); e^x \right)}{(\log 9 + (-1 - 2i))} \bigg] + \left( \frac{5 \times 2^{-x} e^{\left( \frac{1}{2} + i \right)x} \, _2F_1 \left( 1, \left( \frac{1}{2} + i \right); \left( \frac{3}{2} + i \right); e^x \right)}{\log 4 + (-1 + 2i)} \right) + \text{Constant} \quad (2.8)
\]

\[
\int \frac{\cos x \, H_2^{(x)}}{\sqrt{1 - \cosh x}} \, dx = \frac{1}{(25 + 5 \log^2 4 - 10 \log 4)\sqrt{1 - \cosh x}} \times \left[ 2^{-x} e^{\left( \frac{1}{2} - i \right)x} \sinh \left( \frac{x}{2} \right) \left\{ (1 + 2i) \, 2^x (5 + 4 \log^2 2 - 4 \log 2) \, _2F_1 \left( \frac{1}{2} - i, 1; \frac{3}{2} - i; e^x \right) + (1 - 2i) \, 2^x e^{2ix} (5 + 4 \log^2 2 - 4 \log 2) \, _2F_1 \left( \frac{1}{2} + i, 1; \frac{3}{2} + i; e^x \right) \right\} \right] + \text{Constant} \quad (2.7)
\]


\[-5(\log 4 - 1 - 2t)_{2}F_{1}\left(1, \left(\frac{1}{2} - t\right) - \log 2; \left(\frac{3}{2} - t\right) - \log 2; e^{x}\right) + \\
+ e^{2ix}(\log 4 - 1 + 2t)_{2}F_{1}\left(1, \left(\frac{1}{2} + t\right) - \log 2; \left(\frac{3}{2} + t\right) - \log 2; e^{x}\right)\right] + \text{Constant} \quad (2.9)\]

III. Derivation of the Integrals

Involving the same parallel method of ref[4], one can derive the integrals.

IV. Applications

The integrals which are presented here are very special integrals. These are applied in the field of engineering and other allied sciences.

V. Conclusion

In our work we have established certain indefinite integrals involving Harmonic Number and Hypergeometric function. However, one can establish such type of integrals which are very useful for different field of engineering and sciences by involving these integrals. Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions.

REFERENCES Références Referencias