Supersymmetry Quantum Mechanics and Exact Solutions of the Effective Mass of Schrödinger Equations with Rosen-Morse Potential

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Abstract- In quantum mechanics a localized attractive potential typically supports a (possibly infinite) set of bound states, characterized by a discrete spectrum of allowed energies, together with a continuum of scattering states, characterized (in one dimension) by energy-dependent phase shift. The Morse potential named after physicist Philip M. Morse, is a convenient model for the potential energy of a diatomic molecule. It is a better approximation for the vibrational structure of the molecule than the quantum harmonic oscillator because it explicitly includes the effects of bond breaking such as the existence of unbound states. It also accounts for the anharmonicity of real bonds and the non-zero transition probability for overtone and combination bands. The Morse potential can also be used to model other interactions such as the interaction between an atom and a surface. We here make the case that the trigonometric Rosen-Morse potential is exactly soluble in terms of a family of real orthogonal polynomials and present the solutions. And analytical solutions of the Klein-Gordon equation for the Rosen-Morse potential with equal scalar and vector potentials are studied in this paper.

Keywords: supersymmetry, morse-rosen potential, eckart potential, schrödinger equation.

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Abstract: In quantum mechanics, a localized attractive potential typically supports a (possibly infinite) set of bound states, characterized by a discrete spectrum of allowed energies, together with a continuum of scattering states, characterized (in one dimension) by energy-dependent phase shifts. The Morse potential, named after physicist Philip M. Morse, is a convenient model for the potential energy of a diatomic molecule. It is a better approximation for the vibrational structure of the molecule than the quantum harmonic oscillator because it explicitly includes the effects of bond breaking such as the existence of unbound states. It also accounts for the anharmonicity of real bonds and the non-zero transition probability for overtone and combination bands. The Morse potential can also be used to model other interactions such as the interaction between an atom and a surface.

We here make the case that the trigonometric Rosen-Morse potential is exactly soluble in terms of a family of real orthogonal polynomials and present the solutions.

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I. Introduction

The supersymmetric quantum mechanics manages a family of exactly soluble potentials, one of them being the trigonometric Rosen-Morse potential. As long as this potential is obtained from the Eckart potential by complexification of the argument and one of the constants, also its solutions have been concluded from those of the Eckart potential by the same procedure.

Supersymmetric quantum mechanics was originally proposed by Witten as a simple learning ground for the basic concepts of supersymmetric quantum field theories but soon after it evolved to a research field on its own. Supersymmetric quantum mechanics starts with the factorization of one-dimensional Hamiltonians,

\[ H(y) = -\frac{\hbar^2}{2m} \frac{d^2}{dy^2} + V(y) \]

(1)

According to \( H(y) = A^+(y)A^-(y) + \mathcal{E} \) with \( A^\pm(y) \)

\[ (\pm \frac{\hbar}{\sqrt{2m}} \frac{d}{dy} + U(y)) \]

Where \( U(y) \) is the superpotential.

II. The Morse Potential Energy Function

Exact solutions of the effective mass of Schrodinger equations for the mentioned potentials are interesting in the fields of material science and condensed matter physics. There are various methods for exact solutions to energy eigenvalues and corresponding wave function [7,8].

We can quite write the Schrodinger equation in three dimensions for a diatomic molecule with the mass of nuclei \( m_1 \) and \( m_2 \) as the following form:

\[ -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \psi_{nlm}(r, \theta, \varphi) = E\psi_{nlm}(r, \theta, \varphi) \]

(2)

Where \( \hbar, \mu, V(r) \) and \( E \) are Planck’s constant, reduced mass, spherical potential and eigenvalue energy of a quantum system. The radial part of the wave function \( R_{nl}(r) \) is defined by relation of

\[ \psi_{nlm}(r, \theta, \varphi) = \frac{R_{nl}(r)}{r} Y_{lm}(\theta, \varphi) \]

In that case, the radial Schrodinger equation is written by

\[ \frac{d^2R_{nl}(r)}{dr^2} + 2\left(E - V_{eff}\right)R_{nl}(r) = 0 \]

(3)

Where we simply suppose that \( \hbar = \mu = 1 \) and \( V_{eff} \) equal the Rosen-Morse potential plus centrifugal term [8,9].
Exactly solvable 1-dimensional Schrödinger equations play an important role in quantum physics. The best known are the harmonic oscillator and the radial equation for the hydrogen atom, which are covered in every course of quantum mechanics. The Morse potential energy function is of the form

\[
V_{\text{eff}}(r) = D_e \left(1 - e^{-a(r-r_e)}\right)^2
\]  \hspace{1cm} (4)

Here \( r \) is the distance between the atoms, \( r_e \) is the equilibrium bond distance, \( D_e \) is the well depth (defined relative to the dissociated atoms), and \( a \) controls the width of the potential) the smaller \( a \) is, the larger the well). The dissociation energy of the bond can be calculated by subtracting the zero point energy \( E(0) \) from the depth of the well. The force constant of the bond can be found by Taylor expansion of \( V(r) \) around \( r = r_e \) to the second derivative of the potential energy as is.

\[
\frac{\text{d}^2 V_{\text{eff}}}{\text{d}r^2} = -\frac{2m}{\hbar^2} E_{\text{eff}}(r)
\]

(9)

It is convenient to introduce the new variables:

\[
x = ar, \quad x_e = ar_e, \quad \lambda = \frac{\sqrt{2m a^2 D_e}}{\hbar}, \quad E_v = \frac{2m a^2 \hbar^2}{\lambda^2} E(v)
\]  \hspace{1cm} (10)

Then, the schrodinger equation takes the simple form:

\[
\left(-\frac{\text{d}^2}{\text{d}x^2} + V_{\text{eff}}(x)\right) \psi_n(x) = E_n \psi_n(x)
\]

(11)

Its eigenvalues and eigenstates can be written as:

\[
E_n = -\left(\lambda - n - \frac{1}{2}\right)^2
\]

(12)

\[
\psi_n(z) = N_n z^{\lambda-n-1} e^{-z/2} L_n^{2 \lambda - 2 n - 1}(z)
\]

(13)

Where \( z = 2 \lambda e^{-r-x_e} \) whereas the energy spacing between vibrational levels in the quantum harmonic oscillator is constant at \( \hbar \omega \), the energy between adjacent levels decreases with increasing \( v \) in the Rosen-Morse oscillator.

\[
\frac{E_v}{\hbar^2} = \lambda \omega \left(v + \frac{1}{2}\right) - \lambda \omega x_e (v + \frac{1}{2})^2
\]

(15)

In which the constants \( \omega_e \) and \( \omega x_e \) can be directly related to the parameters for the Rosen-Morse potential. As is clear from dimensional analysis, for historical reasons the last equation uses spectroscopic notation in which we represents a wave number obeying \( E = \hbar \omega \), and not an angular frequency given by \( E = \omega \).

**Figure 1**: The Morse potential (dark) and harmonic oscillator potential (light) unlike the energy levels of the harmonic oscillator potential, which are evenly spaced by \( \hbar \omega \), the Morse potential level spacing decreases as the energy approaches the dissociation energy. The dissociation energy \( D_e \) is larger than the true energy required for dissociation \( D_0 \) due to the zero point energy of the lowest (\( v=0 \)) vibrational level. **[12]**

One of the main problems which are involved in many physical processes is the difference of energy state between ground state and first excited state for potential wells. Ground state wave energy \( E^1 \) and first excited state energy \( E^2 \), as a result:

\[
E^1 = \frac{\int_{-\infty}^{+\infty} \varphi(x) (\delta(x) - \delta(x-x_o)) \varphi(x) dx}{\int_{-\infty}^{+\infty} \varphi^2(x) dx} = \frac{\int_{-\infty}^{+\infty} \varphi(x) \left[\frac{\varphi(x)}{l_1}\right]^2 \delta(x-x_o) \varphi(x) dx}{\int_{-\infty}^{+\infty} \varphi^2(x) dx} + \frac{\int_{-\infty}^{+\infty} \varphi(x) \left[\frac{\varphi(x)}{l_2}\right]^2 \delta(x+x_o) \varphi(x) dx}{\int_{-\infty}^{+\infty} \varphi^2(x) dx}
\]

(16)
III. THE SHAPE OF THE TRIGONOMETRIC ROSEN-MORSE POTENTIAL

The study of exactly solvable potentials in quantum mechanics for years has received a lot of attention. For solvable models, the simplest generation technique is supersymmetric quantum mechanics, which is equivalent to the factorization method, the intertwining technique and the Darboux transformation method. It is well known that the spectrum of the generated Hamiltonian differs little from the initial one. This suggests a way to realize, in practice, the spectral design: (i) one starts from a potential having a spectrum close to the desired one. (ii) then, by appropriately moving creating or deleting a certain set of levels, and iterating the method as many times as needed to achieve the required spectrum, one will arrive at a Hamiltonian (or a set of Hamiltonians), which could model the situation under study [13].

We adopt the following form of the trigonometric Rosen-Morse potential

\[ v(z) = -2\beta_n \cot \beta_n z + a(z - 1) \cot \beta_n z \]

(21)

with \( a > 0 \) and \( \beta_n = \frac{\beta}{2^n} \), displayed in Fig. 1. Here, \( z = x \alpha \) and \( \alpha = \sqrt{a(x - 1)} / \sqrt{2}\beta_n \cot \beta_n \) is the dimensionless variable, and \( x \) being the one-dimensional variable of a properly chosen length scale. The potential in ordinary coordinate space and \( \beta_n \), the energy level, are the well-known Jacobi polynomials with \( n \leq \frac{\beta}{2} - 2 \) and \( \beta_n = \frac{\beta}{2^n} \). The exact solutions to the Eckart potential

\[ v(z) = -2\beta_n \cot \beta_n z + a(z - 1) \cot \beta_n z \]

(22)

Next we employ the Eckart potential.

\[ \psi_n(z) = c_n(x - 1)(\beta_n \cot \beta_n z + a(z - 1))^{\frac{1}{2}} \]

(23)

with \( a > 0 \) and \( \beta_n = \frac{\beta}{2^n} \). The exact solutions to the Eckart potential read:

\[ \psi_n(z) = c_n(x - 1)(\beta_n \cot \beta_n z + a(z - 1))^{\frac{1}{2}} \]

(24)

where \( c_n \) are the well-known Jacobi polynomials with \( n \leq \frac{\beta}{2} - 2 \) and \( c_n = \frac{n!}{(\beta_n n)!} \). The allowed energies are not quantized. Perhaps some well-known precursors. Moreover there do exist systems

\[ \Delta E = E_n - E_{n-1} = \frac{\hbar^2}{2m} (\frac{1}{2n}\cot \beta_n z)^2 \]

(25)

for \( n < \frac{\beta}{2} \). We can take Eq. (24) plus a linear like perturbation

\[ \psi_n(x) = \sum_{m=0}^{\infty} \frac{c_m}{\beta_n m!} (\frac{h}{2\beta_n m})^m x^m \psi_n(x) \]

(26)

where \( c_m \) are the well-known Jacobi polynomials with \( m \leq \frac{\beta}{2} - 2 \), and \( c_m = \frac{m!}{(\beta_m m)!} \). The allowed energies are not quantized. Perhaps some well-known precursors. Moreover there do exist systems

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(35)

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where \( c_m \) are the well-known Jacobi polynomials with \( m \leq \frac{\beta}{2} - 2 \), and \( c_m = \frac{m!}{(\beta_m m)!} \). The allowed energies are not quantized. Perhaps some well-known precursors. Moreover there do exist systems
The corresponding energy value spectrum is determined by
\[ \epsilon_n = -(n + a)^2 - \frac{b^2}{(n+a)^2} \]  
\hspace{1cm} (28)

Complexify the argument of the Eckart potential and one of its constants according to
\[ \psi_n(ix) = c_n((ix)^2 - 1)^{-\frac{(n+a)}{2}}e^{-\beta_n} \]  
\hspace{1cm} [19]

V. Spectrum and Wave Functions of the Trigonometric Rosen-Morse Potential

Solving schrodinger equation for the trigonometric Rosen-Morse potential as obtained in Ref[20,21] All in all, the trigonometric Rosen-Morse potential and its real orthogonal polynomial solutions open new venues in the calculation of interesting observables in both supersymmetric quantum mechanics and particle spectroscopy [20,21].

The one-dimensional schrodinger equation with the trigonometric Rosen-Morse potential reads:
\[ \frac{d^2}{dz^2}R_m(z) + (2bcotz - a(a + 1) \csc^2z + \epsilon_n)R_m(z) = 0 \]  
\hspace{1cm} (32)

Our pursued strategy in solving it will be to first reshape it to the particular case of a Sturm-Liouville
\[ s(x)\frac{d^2}{dx^2}F_m(x) + \frac{1}{w(x)}\left(\frac{d}{dx}s(x)w(x)\right)\frac{d}{dx}F_m(x) + \lambda_mF_m(x) = 0 \]  
\hspace{1cm} (33)

Where Km is the normalization constant. And
\[ \lambda_m = -m(K_1\frac{d}{dx}F_1(x) + \frac{1}{2}m - 1)\frac{d^2}{dx^2}s(x) \]  
\hspace{1cm} (35)

VI. Conclusion

In this paper we have discussed on the solution of the schrodinger equation with Rosen-Morse potential. In that case we have written radial part of schrodinger equatin with centrifugal term. Helping the factorization method and comparing it with associated Jacobi differential equation, we have obtained energy eigenvalue and wave function. We also have considered two different cases such as s-wave and Eckart potential for the Rosen-Morse.

All in all the Rosen-Morse potential and its real orthogonal polynomial solutions open new venues in the calculation of interesting observables in both supersymmetric quantum mechanics and particle spectroscopy.

References Références Referencias

10. P.M.Morse, Diatomic molecules according to the wave mechanics II Vibrational levels, Physicals Review, Vol. 34, no. I, pp. 57-64, (1929).
Table 1: Some Potentials and Relations [22-28].

<table>
<thead>
<tr>
<th>Potential</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rosen-Morse</td>
<td>$V(r) = -V_1 \text{sech}^2 ar + V_2 \tanh ar$</td>
</tr>
<tr>
<td>Woods-Saxon</td>
<td>$V(r) = -\frac{V_0}{1 + \exp \left( \frac{r - R}{a} \right)}$</td>
</tr>
<tr>
<td>Eckart</td>
<td>$V(r) = -k \frac{\cos hr}{\sin hr} + \left( \frac{\mu^2}{4} - \frac{1}{4} \right) \frac{1}{\sinh^2 r}$</td>
</tr>
<tr>
<td>Poschl-Teller</td>
<td>$V(r) = \left( \alpha^2 - \frac{1}{4} \right) \frac{1}{4 \sin \alpha^2 r} + \left( \beta^2 - \frac{1}{4} \right) \frac{1}{4 \cos \alpha^2 r}$</td>
</tr>
<tr>
<td>Morse</td>
<td>$V(r) = D_r \left( 1 - e^{-\alpha (r - r_0)} \right)^2$</td>
</tr>
<tr>
<td>Manning-Rosen</td>
<td>$V(r) = -k \frac{\sin hr}{\cosh r} - \left( \frac{\mu^2}{4} - \frac{1}{4} \right) \frac{1}{\cosh^2 r}$</td>
</tr>
<tr>
<td>Hulthen</td>
<td>$V(r) = -\frac{z}{\alpha} \frac{\exp \frac{-r}{\alpha}}{1 - \exp \frac{-r}{\alpha}}$</td>
</tr>
</tbody>
</table>

1. **Rosen-Morse**, In this relation $V_1$ and $V_2$ denotes the depth of the potential and $a$ is the range of the potential.
2. **The Wood-Saxon potential**, is a mean field potential for the nucleons (protons and neutrons) inside the atomic nucleus, which is used to approximately describe the forces applied on each nucleon, in the shell model for the structure of the nucleus. In this potential $V_0$ (having dimension of energy) represents the potential well depth, $a$ is a length representing the surface thickness of the nucleus, and $R = r_0 \frac{A^{1/3}}{4}$ is the nuclear radius where $r_0 = 1.25$ fm and $A$ is the mass number.
3. **Eckart**, This potential was proposed and solved by C.Eckart. A natural real domain for this potential is \( ]0, \infty[ \).

4. **Poschl-Teller**, This potential was proposed and solved by Poschl and Teller. A natural real domain for this potential is \( ]0, \pi[ \).

5. **Morse**, In this potential \( r \) is the distance between the atoms, \( r_e \) is the equilibrium bond distance.

6. **Manning-Rosen**, This potential was proposed and solved by M.F.Manning and N.Rosen. A natural real domain for this potential is \( ]-\infty, \infty[ \).

7. **Hulthen**, The Hulthen potential is a short-range potential which behaves like a Coulomb potential for small values for \( r \) and decreases exponentially for large values of \( r \). In this potential \( a \) is the screening parameter and \( z \) is a constant which is identified with the atomic number when the potential is used for atomic phenomena.

![Figure 1: the Morse potential and harmonic oscillator potential](image)