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On Some Properties of the Rising Sun Function

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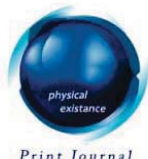
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On Some Properties of the Rising Sun Function

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I. INTRODUCTION

The rising sun function was used as a tool in the proof of the famous Lebesgue's theorem on the differentiability of a real valued monotone function without using the theory of integration [3]. In this paper, some properties of the rising sun function are presented and an operator on the space of all bounded real functions defined on a closed and bounded interval $[a, b]$ is introduced and its properties are investigated.

In what follows X , $\mathbb{R}$ and $\mathbb{N}$ stand for a topological space, the real line and the set of all positive integers respectively. Also $\mathcal{B}$ stands for the Banach space of all bounded real functions defined on a closed and bounded interval $[a, b]$ where $a, b \in \mathbb{R}$ and $a < b$ under the supremum norm.

II. PRELIMINARIES

1.1 Definition [6]: The rising sun function of a function $f \in \mathcal{B}$ is defined by

$$f_{\odot}(x) = \sup \{ f(y) / x \leq y \leq b \}.$$

1.2 Definition: For $f \in \mathcal{B}$ we define the following.

$$(i) \quad \circ f(x) = \sup \{ f(y) / a \leq y \leq x \}$$

$$(ii) \quad f^{\odot}(x) = \inf \{ f(y) / x \leq y \leq b \}$$

$$(iii) \quad {}^{\odot} f(x) = \inf \{ f(y) / a \leq y \leq x \}$$

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1.3 Definition [5]: A function $f : X \rightarrow \mathbb{R}$ is said to be semi-continuous at a point $p \in X$ if for every $\varepsilon > 0$ and every neighborhood U of p in X there exists a non-empty open set $W \subset U$ such that $|f(x) - f(p)| < \varepsilon \quad \forall x \in W$. We say that a function f is semi-continuous on X if it is semi-continuous at every point of X .

1.4 Definition [6]: A function $f : X \rightarrow \mathbb{R}$ is said to be lower semicontinuous (*lsc*) at a point $x \in X$ if for every $\varepsilon > 0$ there exists a neighborhood U of x such that

$$f(y) - f(x) > -\varepsilon \quad \forall y \in U.$$

We say that a function f is *lsc* on X if it is *lsc* at every point of X .

1.5 Definition [6]: A function $f : X \rightarrow \mathbb{R}$ is said to be upper semicontinuous (*usc*) at a point $x \in X$ if for every $\varepsilon > 0$ there exists a neighborhood G of x such that

$$f(y) - f(x) < \varepsilon \quad \forall y \in G.$$

We say that a function f is *usc* on X if it is *usc* at every point of X .

1.6 Definition [4]: A function $f : X \rightarrow \mathbb{R}$ is said to be lower semi-quasicontinuous (*lqsc*) at a point $x \in X$ if for every $\varepsilon > 0$ and every neighborhood U of x there exists a non-empty open set $W \subset U$ such that $f(y) - f(x) > -\varepsilon \quad \forall y \in W$.

We say that a function f is *lqsc* on X if it is *lqsc* at every point of X .

1.7 Definition [4]: A function $f : X \rightarrow \mathbb{R}$ is said to be upper semi-quasicontinuous (*usqc*) at a point $x \in X$ if for every $\varepsilon > 0$ and every neighborhood U of x there exists a non-empty open set $W \subset U$ such that $f(y) - f(x) < \varepsilon \quad \forall y \in W$.

We say that a function f is *usqc* on X if it is *usqc* at every point of X .

1.8 Definition [5]: A function $f : X \rightarrow \mathbb{R}$ is said to be cliquish at a point $x \in X$ if for every $\varepsilon > 0$ and every neighborhood U of x there exists a non-empty open set $W \subset U$ such that $|f(y) - f(z)| < \varepsilon \quad \forall y, z \in W$.

We say that a function f is cliquish on X if it is cliquish at every point of X .

1.9 Definition: Let $f : [a, b] \rightarrow \mathbb{R}$. We define $f(a-) = f(a)$ and $f(b+) = f(b)$.

We say that $f(p+)$ exists at $p \in [a, b)$ and we write $f(p+) = L$, where

$L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$|f(x) - L| < \varepsilon \quad \forall x \in (p, p + \delta) \subset [a, b]$. Similarly for $p \in (a, b]$ we write

$f(p-) = l \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \forall x \in (p - \delta, p) \subset [a, b].$$

1.10 Definition: A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be quasicontinuous at a point $p \in [a, b]$ if $f(p+)$ and $f(p-)$ exist.

1.11 Definition: A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be symmetrically continuous at a point $x \in [a, b]$ if $\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$.

1.12 Definition: A function $f : [a, b] \rightarrow \mathbb{R}$ is Darboux continuous if for all $p, q \in [a, b]$ and for each c between $f(p)$ and $f(q)$ there is an x between p and q such that $f(x) = c$.

1.13 Definition[2]: An operator P on a linear space L is said to be sublinear if
(i) $P(x+y) \leq P(x) + P(y) \quad \forall x, y \in L$ and (ii) $P(\lambda x) = \lambda P(x)$ for any positive real number λ and every $x \in L$.

III. RELATIONS AMONG $f_{\odot}, {}_{\odot}f, f^{\odot}$ AND ${}^{\odot}f$

In this section the relations between the rising sun function and its analogues that are introduced are presented in the following propositions.

2.1 Proposition: For $f \in \mathcal{B}$, (a) $f^{\odot} = -(-f)_{\odot}$ and (b) ${}^{\odot}f = -({}_{\odot}(-f))$.

2.2 Proposition: For $f \in \mathcal{B}$, (a) $(f_{\odot})_{\odot} = f_{\odot}$ (b) ${}_{\odot}({}_{\odot}f) = {}_{\odot}f$

$$(c) (f^{\odot})^{\odot} = f^{\odot} \quad (d) {}^{\odot}({}^{\odot}f) = {}^{\odot}f$$

2.3 Proposition: For $f \in \mathcal{B}$ and $x \in [a, b]$,

$$\begin{aligned} (i) \quad {}_{\odot}(f_{\odot})(x) &= f_{\odot}(a) & (ii) \quad (f_{\odot})^{\odot}(x) &= f_{\odot}(b) & (iii) \quad {}^{\odot}(f_{\odot})(x) &= f_{\odot}(x) \\ (iv) \quad ({}_{\odot}f)_{\odot}(x) &= {}_{\odot}f(b) & (v) \quad ({}_{\odot}f)^{\odot}(x) &= {}_{\odot}f(x) & (vi) \quad {}^{\odot}({}_{\odot}f)(x) &= {}_{\odot}f(a) \\ (vii) \quad (f^{\odot})_{\odot}(x) &= f^{\odot}(b) & (viii) \quad {}_{\odot}(f^{\odot})(x) &= f^{\odot}(x) & (ix) \quad {}^{\odot}(f^{\odot})(x) &= f^{\odot}(a) \\ (x) \quad ({}^{\odot}f)_{\odot}(x) &= {}^{\odot}f(x) & (xi) \quad {}_{\odot}({}^{\odot}f)(x) &= {}^{\odot}f(a) & (xii) \quad ({}^{\odot}f)^{\odot}(x) &= {}^{\odot}f(b) \end{aligned}$$

2.4 Remark: In view of the previous propositions it is enough to investigate the properties of the rising sun function and the properties of ${}_{\odot}f$, $f^{\odot}$ and ${}^{\odot}f$ follow analogously.

IV. CHARACTERISATIONS OF $f_{\odot}, {}_{\odot}f, f^{\odot}$ AND ${}^{\odot}f$

3.1 proposition: For $f \in \mathcal{B}$, $f_{\odot}$ is the smallest decreasing function dominating f

More precisely,

$$(a) f_{\odot}(x) \geq f(x) \quad \forall x \in [a, b]$$

$$(b) f_{\odot} \text{ is decreasing on } [a, b]$$

$$(c) \text{ If } g \text{ satisfies (a) and (b) above, then } f_{\odot}(x) \leq g(x) \quad \forall x \in [a, b].$$

3.2 Proposition: For $f \in \mathcal{B}$, ${}_{\odot}f$ is the smallest increasing function dominating

$$f. \text{ More precisely, } (a) {}_{\odot}f(x) \geq f(x) \quad \forall x \in [a, b]$$

$$(b) {}_{\odot}f \text{ is increasing on } [a, b]$$

$$(c) \text{ If } g \text{ satisfies (a) and (b) above, then } {}_{\odot}f(x) \leq g(x) \quad \forall x \in [a, b].$$

3.3 Proposition: For $f \in \mathcal{B}$,

$$(a) f^{\odot} \text{ is the largest increasing function such that } f^{\odot}(x) \leq f(x) \quad \forall x \in [a, b]$$

$$(b) {}^{\odot}f \text{ is the largest decreasing function such that } {}^{\odot}f(x) \leq f(x) \quad \forall x \in [a, b]$$

V. THE RISING SUN OPERATOR

4.1 Definition: Define $T: \mathcal{B} \rightarrow \mathcal{B}$ by $Tf = f_{\odot}$. We call this operator T , the rising sun operator on $\mathcal{B}$.

4.2 Proposition: The rising sun operator T is sublinear on $\mathcal{B}$. More precisely,

$$(a) T(f + g) \leq Tf + Tg \quad \forall f, g \in \mathcal{B}$$

$$(b) T(\lambda f) = \lambda Tf \text{ for every real number } \lambda > 0 \text{ and every } f \in \mathcal{B}.$$

Proof: Let $f, g \in \mathcal{B}$ and $x \in [a, b]$.

$$(a) \text{ For } y \in [x, b], (f + g)(y) = f(y) + g(y) \leq f_{\odot}(x) + g_{\odot}(x) = (f_{\odot} + g_{\odot})(x)$$

$$\Rightarrow (f + g)(y) \leq (f_{\odot} + g_{\odot})(x) \quad \forall y \in [x, b]$$

$$\Rightarrow (f + g)_{\odot}(x) \leq (f_{\odot} + g_{\odot})(x) \quad \forall x \in [a, b]$$

$$\text{Hence } T(f + g) \leq Tf + Tg \quad \forall f, g \in \mathcal{B}.$$

$$(b) \text{ Suppose that } \lambda \text{ is a positive real number and } f \in \mathcal{B}.$$

$$\text{Then } (\lambda f)_{\odot}(x) = \sup\{(\lambda f)(y) / x \leq y \leq b\}$$

$$= \lambda \sup\{f(y) / x \leq y \leq b\}$$

$$= \lambda f_{\odot}(x)$$

$$T \lambda f = \lambda Tf.$$

4.3 Remark: From the following example, it is clear that $T(f + g) \neq Tf + Tg$.

4.4 Example: Define $f : [0,1] \rightarrow \mathbb{R}$ and $g : [0,1] \rightarrow \mathbb{R}$ by $f(x) = x$ and

$$g(x) = 1 - x \quad \forall x \in [0,1] \quad \Rightarrow (f + g)(x) = 1, \forall x \in [0,1]$$

$$\text{Then} \quad (f + g)_{\odot}(x) = 1 \text{ and } (f_{\odot} + g_{\odot})(x) = 2 - x \quad \forall x \in [0,1] .$$

$$\text{Hence} \quad T(f + g) \neq Tf + Tg .$$

4.5 Proposition: For $f \in \mathcal{B}$ and $k \in \mathbb{R}$, $T(f + k) = Tf + k$.

4.6 Proposition: For $f, g \in \mathcal{B}$, (a) $f \leq g \Rightarrow Tf \leq Tg$ (b) $T(f \vee g) = Tf \vee Tg$

$$\text{where } (f \vee g)(x) = \max\{f(x), g(x)\} .$$

4.7 Proposition: If $\{f_{\alpha} / \alpha \in \Delta\}$ is a collection of functions in $\mathcal{B}$ and if

$$\bigvee_{\alpha \in \Delta} f_{\alpha} = \sup\{f_{\alpha} / \alpha \in \Delta\} \text{ exists in } \mathcal{B} \text{ then } T\left(\bigvee_{\alpha \in \Delta} f_{\alpha}\right) = \bigvee_{\alpha \in \Delta} T(f_{\alpha}) .$$

4.8 Remark: From the following example it can be observed that

$$T(f \wedge g) \neq Tf \wedge Tg , \text{ where } (f \wedge g)(x) = \min\{f(x), g(x)\} .$$

4.9 Example: Define $f : [0,1] \rightarrow \mathbb{R}$ and $g : [0,1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1-2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ -2x+2 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\text{Then } f_{\odot}(x) = 1 \quad \forall x \in [0,1] \quad \text{and} \quad g_{\odot}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ -2x+2 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\Rightarrow (f_{\odot} \wedge g_{\odot})(x) = g_{\odot}(x) \quad \forall x \in [0,1]$$

$$\text{Also } (f \wedge g)(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{4} \\ -2x+1 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ -2x+2 & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$\Rightarrow (f \wedge g)_{\odot}(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq \frac{3}{4} \\ -2x+2 & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}$$

Hence $(f \wedge g)_{\odot}(x) \neq (f_{\odot} \wedge g_{\odot})(x)$.

4.10 Proposition: T is continuous on $\mathcal{B}$.

Proof: Let $f_n \in \mathcal{B}$ for $n \in \mathbb{N}$ and $f_n \rightarrow f$ uniformly on $[a, b]$. Then $f \in \mathcal{B}$.

Let $\varepsilon > 0$ be given. Then there exists a positive integer N such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x \in [a, b]$$

$$\Rightarrow -\varepsilon < f_n(x) - f(x) < \varepsilon \quad \forall x \in [a, b]$$

$$\Rightarrow f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \forall x \in [a, b].$$

Let $x \in [a, b]$ and choose $y \in [x, b]$. Then $y \in [a, b]$.

$$\Rightarrow f(y) - \varepsilon < f_n(y) < f(y) + \varepsilon \quad \forall n \geq N$$

$$n \geq N \Rightarrow f_n(y) < f(y) + \varepsilon \leq f_{\odot}(x) + \varepsilon$$

$$\Rightarrow f_n(y) < f_{\odot}(x) + \varepsilon \quad \forall y \in [x, b]$$

$$\Rightarrow (f_n)_{\odot}(x) \leq f_{\odot}(x) + \varepsilon.$$

Similarly $f_{\odot}(x) - \varepsilon < (f_n)_{\odot}(x)$.

Hence $|(f_n)_{\odot}(x) - f_{\odot}(x)| < \varepsilon \quad \forall n \geq N$ and for every $x \in [a, b]$.

Hence $(f_n)_{\odot} \rightarrow f_{\odot}$ uniformly on $[a, b]$.

Hence $f_n \rightarrow f$ in $\mathcal{B} \Rightarrow Tf_n \rightarrow Tf$ in $\mathcal{B}$.

$\Rightarrow T$ is continuous on $\mathcal{B}$.

4.11 Proposition: (a) $T^n f = f_{\odot} \quad \forall f \in \mathcal{B}$ and for every $n \in \mathbb{N}$

(b) For $f \in \mathcal{B}$, the cycle of T is the set $\{f, f_{\odot}\}$.

4.12 Proposition: If $f \in \mathcal{B}$ and f is monotonically decreasing then f is a fixed point of T .

4.13 Remark: The set of all fixed points of T is the set of all monotonically decreasing functions on $[a, b]$. Let F denote the set of all fixed points of T

Then $F = \{f \in \mathcal{B} / Tf = f\} = \{f \in \mathcal{B} / f \text{ is decreasing}\}$.

4.14 Proposition: The set F of all fixed points of the rising sun operator T is closed in $\mathcal{B}$.

4.15 Proposition: Fix $f \in F$. Let $F^* = \{f \in \mathcal{B} / Tf = f\}$.

Then (a) F^* is closed in $\mathcal{B}$.

(b) $(F^*, \leq)$ is a $\vee$ -semilattice under the relation $\leq$ defined on F^* by

$$f \leq g \Leftrightarrow f(x) \leq g(x) \quad \forall x \in [a, b]$$

4.16 Proposition: The operator T is bounded. More precisely,

$$(i) \|Tf\| \leq \|f\| \quad \forall f \in \mathcal{B} \quad \text{and} \quad (ii) \|T\| = 1$$

VI. INVARIANT PROPERTIES

5.1 Proposition: Let $f \in \mathcal{B}$. If f is *usc* at a point $x \in [a, b]$ then so

is $Tf = f_{\circ}$.

Proof: Let $\varepsilon > 0$ be given and $x \in [a, b]$. Since f is *usc* at $x \in [a, b]$, there exists a $\delta > 0$ such that $f(t) - f(x) < \varepsilon \quad \forall t \in (x - \delta, x + \delta) \cap [a, b] = U$.

$$\Rightarrow f_{\circ}(x) + \varepsilon \geq f(x) + \varepsilon > f(t) \quad \forall t \in U$$

$$\Rightarrow f_{\circ}(x) + \varepsilon > f(t) \quad \forall t \in U$$

Suppose that $x < y$.

$$\Rightarrow f_{\circ}(x) + \varepsilon > f_{\circ}(x) \geq f_{\circ}(y) \geq f(y)$$

$$\Rightarrow f_{\circ}(x) + \varepsilon > f(y).$$

If $x < t$ and $y \in [t, b] \Rightarrow x < y$

$$\Rightarrow f_{\circ}(x) + \varepsilon > f(y)$$

$$\Rightarrow f_{\circ}(x) + \varepsilon \geq f_{\circ}(t)$$

Suppose that $t \leq x$. Then $y \in [t, b] \Rightarrow t \leq y \leq x$ or $x < y < b$.

$$\text{If } t \leq y \leq x \text{ then } y \in U \Rightarrow f_{\circ}(x) + \varepsilon > f(y) \Rightarrow f_{\circ}(x) + \varepsilon > f_{\circ}(t).$$

$$\text{If } x < y \leq b \text{ then } f_{\circ}(x) + \varepsilon > f_{\circ}(t). \text{ Hence } f_{\circ}(x) + \varepsilon > f_{\circ}(t) \quad \forall t \in U.$$

$$\Rightarrow f_{\circ} \text{ is } \textit{usc} \text{ at } x.$$

5.2 Proposition: Let $f \in \mathcal{B}$. If f is *lsc* at a point $x \in [a, b]$ then so

is $Tf = f_{\circ}$.

Proof: Let $\varepsilon > 0$ be given and $x \in [a, b]$. Since f is *lsc* at $x \in [a, b]$, there

exists a $\delta_1 > 0$ such that $f(t) - f(x) > \frac{-\varepsilon}{2} \quad \forall t \in (x - \delta_1, x + \delta_1) \cap [a, b] = U$

$$\Rightarrow f_{\odot}(t) + \frac{\varepsilon}{2} \geq f(t) + \frac{\varepsilon}{2} > f(x) \quad \forall t \in U$$

$$\Rightarrow f_{\odot}(t) + \frac{\varepsilon}{2} > f(x) \quad \forall t \in U$$

Since $f_{\odot}(x) - \frac{\varepsilon}{2}$ is not an upper bound of the set $\{f(y) / x \leq y \leq b\}$, there

exists a point $y \in [x, b]$ such that $f(y) > f_{\odot}(x) - \frac{\varepsilon}{2}$.

If $y = x$ then $f_{\odot}(t) + \frac{\varepsilon}{2} > f(x) > f_{\odot}(x) - \frac{\varepsilon}{2} \quad \forall t \in U$

$$\Rightarrow f_{\odot}(t) + \varepsilon > f_{\odot}(x) \quad \forall t \in U.$$

$y \neq x$. If $x < z < y$ then $f_{\odot}(z) \geq f_{\odot}(y) \geq f(y) > f_{\odot}(x) - \frac{\varepsilon}{2}$

$$\Rightarrow f_{\odot}(z) > f_{\odot}(x) - \frac{\varepsilon}{2} > f_{\odot}(x) - \varepsilon.$$

If $z \leq x$ then $f_{\odot}(z) \geq f_{\odot}(x) > f_{\odot}(x) - \varepsilon$.

Choose $\delta > 0$ such that $a \leq x - \delta < x < x + \delta \leq y$.

Then $f_{\odot}(z) > f_{\odot}(x) - \varepsilon \quad \forall z \in (x - \delta, x + \delta) \cap [a, b]$. Hence $f_{\odot}$ is *lsc* at x .

5.3 Corollary: Let $f \in \mathcal{B}$. If f is continuous at a point $x \in [a, b]$, then

$Tf = f_{\odot}$ is continuous at x .

5.4 Proposition: For any $f \in \mathcal{B}$, $Tf = f_{\odot}$ is *lsqc* at every $x \in (a, b]$.

Proof: Let $f \in \mathcal{B}$ and $x \in (a, b]$. Let $\varepsilon > 0$ be given and let U be a

neighborhood of x in $[a, b]$. Then there exists a $\delta > 0$ such that

$(x - \delta, x + \delta) \cap [a, b] \subset U$. Choose $W = (x - \delta, x) \cap [a, b]$.

W is a non-empty open subset of U .

$$y \in W \Rightarrow x - \delta < y < x$$

$$\Rightarrow f_{\odot}(y) \geq f_{\odot}(x)$$

$$\Rightarrow f_{\odot}(y) + \varepsilon > f_{\odot}(y) \geq f_{\odot}(x)$$

Hence $f_{\odot}(y) + \varepsilon > f_{\odot}(x) \quad \forall y \in W$.

Thus for every $\varepsilon > 0$ and every neighborhood U of x there exists a non-empty open set $W \subset U$ such that $f_{\odot}(y) + \varepsilon > f_{\odot}(x) \quad \forall y \in W$.

5.5 Proposition: Let $f \in \mathcal{B}$. If $f : [a, b] \rightarrow \mathbb{R}$ is *lsqc* at the point a then

$Tf = f_{\odot}$ is *lsqc* at a .

Proof: Let $f \in \mathcal{B}$ and $\varepsilon > 0$ be given. Let U be a neighborhood of a in $[a, b]$. Since f is *lsqc* at a , there exists a non-empty open set $W \subset U$ such that $f(t) - f(a) > -\varepsilon \quad \forall t \in W$.

$$\Rightarrow f_{\odot}(t) \geq f(t) > f(a) - \varepsilon \quad \forall t \in W$$

$$\Rightarrow f_{\odot}(t) > f(a) - \varepsilon \quad \forall t \in W.$$

Since $f_{\odot}(a) - \varepsilon$ is not an upper bound of $\{f(y) / a \leq y \leq b\}$, there exists a point $y \in [a, b]$ such that $f_{\odot}(a) - \varepsilon < f(y)$.

If $y = a$ then $f_{\odot}(t) + \varepsilon > f(a) > f_{\odot}(a) - \varepsilon \quad \forall t \in W$.

Suppose that $y \neq a$.

Since $a \in U$ and U is open in $[a, b]$, there exists a $\delta_1 > 0$ such that

$[a, a + \delta_1) \subset U$. Choose $\delta_2 > 0$ such that $a < a + \delta_2 < y$.

Put $\delta = \min\{\delta_1, \delta_2\}$ and $W_1 = (a, a + \delta)$. Clearly, W_1 is a non-empty open set such that $W_1 \subset U$.

Then $z \in W_1 \Rightarrow z < y$

$$\Rightarrow f_{\odot}(z) \geq f_{\odot}(y) \geq f(y) > f_{\odot}(a) - \varepsilon$$

Hence $f_{\odot}(z) + \varepsilon > f_{\odot}(a) \quad \forall z \in W_1$

$$\Rightarrow f_{\odot} \text{ is } \textit{lsqc} \text{ at } a.$$

5.6 Proposition: Let $f \in \mathcal{B}$. Then

(a) $f_{\odot}$ is *usqc* at every $x \in [a, b)$.

(b) If $f : [a, b] \rightarrow \mathbb{R}$ is *usqc* at b then so is $f_{\odot}$.

5.7 Proposition [5]: Let $f : [a, b] \rightarrow \mathbb{R}$ and $p \in [a, b]$. If $f(p+)$ exists then f is cliquish at p .

5.8 Corollary: For any $f \in \mathcal{B}$, $Tf = f_{\odot}$ is cliquish on $[a, b]$.

5.9 Proposition: For any $f \in \mathcal{B}$, $Tf = f_{\odot}$ is quasicontinuous on $[a, b]$.

VII. VARIANT PROPERTIES

6.1 Symmetric continuity: It is not necessary that the rising sun function of a symmetrically continuous function is symmetrically continuous. For example, define $f : [-1, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then

$$f_{\odot}(x) = \begin{cases} x^2 & \text{if } -1 \leq x \leq 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases}$$

Clearly f is symmetrically continuous on $[-1, 1]$, but $f_{\odot}$ is not.

6.2 Semi-continuity: The semi-continuity of f need not imply the semi-continuity of $f_{\odot}$ as is evident from the following example.

Define $f : [-1, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ (x+1)2^{\frac{-2}{x}} & \text{if } 0 < x \leq 1 \\ x+1 & \text{if } -1 \leq x < 0 \end{cases}$$

Then

$$f_{\odot}(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0 \\ \frac{1}{2} & \text{if } 0 < x \leq 1 \end{cases}$$

Clearly f is semi-continuous on $[-1, 1]$. But $f_{\odot}$ is not semi-continuous at $x = 0$.

6.3 Darboux continuity: It is not necessary that the rising sun function of a Darboux continuous function is Darboux continuous. The function $f : [-1, 1] \rightarrow \mathbb{R}$ defined in the above example is Darboux continuous on $[-1, 1]$ but its rising sun function is not Darboux continuous.

6.4 Differentiability: The rising sun function of a differentiable function is not necessarily differentiable as can be observed from the following example.

Define $f : [0,1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} +\sqrt{\frac{2x}{5} - x^2} & \text{if } 0 \leq x \leq \frac{2}{5} \\ -\sqrt{\frac{6x}{5} - \frac{8}{25} - x^2} & \text{if } \frac{2}{5} \leq x \leq \frac{4}{5} \\ +\sqrt{\frac{9x}{5} - \frac{4}{5} - x^2} & \text{if } \frac{4}{5} \leq x \leq 1 \end{cases}$$

$$\text{Then } f_{\odot}(x) = \begin{cases} \frac{1}{5} & \text{if } 0 \leq x \leq \frac{1}{5} \\ f(x) & \text{if } \frac{1}{5} \leq x \leq \frac{9}{10} \\ \frac{1}{10} & \text{if } \frac{2}{5} \leq x \leq \frac{9}{10} \\ f(x) & \text{if } \frac{9}{10} \leq x \leq 1 \end{cases}$$

Clearly f is differentiable at $a = 0.3732$, but $f_{\odot}$ is not differentiable at this point.

6.5 Pointwise Convergence: If $\{f_n\}$ converges pointwise to f on $[a,b]$, it is not necessary that $\{Tf_n\}$ converges to Tf as can be seen from the following example. Define $f_n : [0,1] \rightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \left\{ \frac{1}{n} / n \in \mathbb{N} \right\} \\ 0 & \text{if } x \notin \left\{ \frac{1}{n} / n \in \mathbb{N} \right\} \end{cases}$$

Then $\{f_n\}$ converges pointwise to 0.

But $(f_n)_{\odot}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1 \end{cases}$ does not converge to 0.

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