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The Metric Characterization of the Generalized Fermat Points

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The Metric Characterization of the Generalized Fermat Points

Sándor N. Kiss

Abstract- We build on the sides of a triangle ABC , outwards and inwards, similar triangles with one another. We prove some relations referring to the generalized Fermat points. We deduce these results based on trigonometrical identities.

I. INTRODUCTION

If we build outwards on the sides BC , CA , AB of an arbitrary triangle ABC three equilateral triangles BCX_1 , CAY_1 , ABZ_1 then

$$AX_1 = BY_1 = CZ_1 = \pm AF_1 \pm BF_1 \pm CF_1, \quad (1)$$

where $F_1 = AX_1 \cap BY_1 \cap CZ_1$ is the first (outward) Fermat point of triangle ABC (Figure 1). A minus sign being taken if the angle of triangle ABC at that vertex exceeds 120° .

If we build inwards on the sides BC , CA , AB of an arbitrary triangle ABC three equilateral triangles BCX_2 , CAY_2 , ABZ_2 then

$$AX_2 = BY_2 = CZ_2 = \pm AF_2 \pm BF_2 \pm CF_2, \quad (2)$$

where $F_2 = AX_2 \cap BY_2 \cap CZ_2$ is the second (inward) Fermat point of triangle ABC (Figure 2). A minus sign being taken at each vertex where the angle of triangle ABC is larger than 60° .

In this paper we generalize these results: on the sides of triangle ABC we build, outwards resp. inwards, three similar triangles.

We mention but in this article we don't discuss the Fermat-Steiner problem:

Given a triangle ABC , how can we find a point P in its plane for which $PA+PB+PC$ is minimal?

The solution and some generalization of this problem was a deal for many mathematicians (see [2], [4], [5]). The historical survey of the Fermat-Steiner problem could be found in [3].

Let be given a triangle ABC with the usual notations: A , B , C the angles, a , b , c the sides, R the circumradius, Δ the area of the triangle ABC , S the twice of the area of the triangle ($S = 2\Delta$).

II. THE FIRST GENERALIZED FERMAT POINT OF THE TRIANGLE

We build on the sides of the triangle ABC outwards the similar triangles BCD , CAE and ABF in that way, that

$BAF\angle = \alpha = CAE\angle$, $CBD\angle = \beta = ABF\angle$, $ACE\angle = \gamma = BCD\angle$ (Figure 3). Since $A + B + C = \pi = \alpha + \beta + \gamma$, therefore $BDC\angle = \alpha$, $CEA\angle = \beta$, $AFB\angle = \gamma$.

We work with barycentric coordinates with reference to the triangle ABC . According to the Conway's formulas, the barycentric coordinates of the points D , E , F are: $D = (-a^2 : S_\gamma + S_C : S_\beta + S_B)$, $E = (S_\gamma + S_C : -b^2 : S_\alpha + S_A)$, $F = (S_\beta + S_B : S_\alpha + S_A : -c^2)$, where $S_\theta = S \cdot \text{ctg}\theta$. Consequently $S_A = bc \cos A$, $S_B = ca \cos B$, $S_C = ab \cos C$.

The equations of the lines AD , BE , CF are: $-(S_\beta + S_B)y + (S_\gamma + S_C)z = 0$,

$$(S_\alpha + S_A)x - (S_\gamma + S_C)z = 0,$$

$$-(S_\alpha + S_A)x + (S_\beta + S_B)y = 0.$$

Swimming random by two of the above equations, we get the third equation, the lines AD , BE and CF meet each other in a point $K = AD \cap BE \cap CF$. This point K will be called **the first generalized Fermat point of the triangle**.

The barycentric coordinates of the point K are:

$$K = \left((S_\beta + S_B)(S_\gamma + S_C) : (S_\gamma + S_C)(S_\alpha + S_A) : (S_\alpha + S_A)(S_\beta + S_B) \right) \\ = \left(\frac{1}{S_\alpha + S_A} : \frac{1}{S_\beta + S_B} : \frac{1}{S_\gamma + S_C} \right).$$

We introduce the following notations:

$$\begin{aligned} \lambda &= \lambda(A, B, C, \alpha, \beta, \gamma) = \\ &= \sin A \sin B \sin C \left(\text{ctg} A \sin^2 \alpha + \text{ctg} B \sin^2 \beta + \text{ctg} C \sin^2 \gamma + 2 \sin \alpha \sin \beta \sin \gamma \right) \\ &= \frac{1}{4R^2} \left(S_A \sin^2 \alpha + S_B \sin^2 \beta + S_C \sin^2 \gamma + 2S \sin \alpha \sin \beta \sin \gamma \right). \\ \lambda_0 &= \lambda(\alpha, \beta, \gamma, A, B, C) = \\ &= \sin \alpha \sin \beta \sin \gamma \left(\text{ctg} \alpha \sin^2 A + \text{ctg} \beta \sin^2 B + \text{ctg} \gamma \sin^2 C + 2 \sin A \sin B \sin C \right) \\ &= \frac{\sin \alpha \sin \beta \sin \gamma}{4R^2} \left(a^2 \text{ctg} \alpha + b^2 \text{ctg} \beta + c^2 \text{ctg} \gamma + 2S \right) \\ &= \frac{\sin \alpha \sin \beta \sin \gamma}{4R^2 S} \left(a^2 S_\alpha + b^2 S_\beta + c^2 S_\gamma + 2S^2 \right). \end{aligned}$$

We will prove that $\lambda_0 = \lambda$:

$$\begin{aligned} &\sin \alpha \sin \beta \sin \gamma \left(\text{ctg} \alpha \sin^2 A + \text{ctg} \beta \sin^2 B + \text{ctg} \gamma \sin^2 C + 2 \sin A \sin B \sin C \right) = \\ &= \sin A \sin B \sin C \left(\text{ctg} A \sin^2 \alpha + \text{ctg} B \sin^2 \beta + \text{ctg} C \sin^2 \gamma + 2 \sin \alpha \sin \beta \sin \gamma \right) \Leftrightarrow \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \cos \alpha \sin \beta \sin \gamma \sin^2 A + \sin \alpha \cos \beta \sin \gamma \sin^2 B + \sin \alpha \sin \beta \cos \gamma \sin^2 C = \\
&= \cos A \sin B \sin C \sin^2 \alpha + \sin A \cos B \sin C \sin^2 \beta + \sin A \sin B \cos C \sin^2 \gamma \Leftrightarrow \\
&\Leftrightarrow \cos \alpha (\cos \alpha + \cos \beta \cos \gamma) \sin^2 A + \cos \beta (\cos \beta + \cos \alpha \cos \gamma) \sin^2 B + \\
&+ \cos \gamma (\cos \gamma + \cos \alpha \cos \beta) \sin^2 C = \cos A (\cos A + \cos B \cos C) \sin^2 \alpha + \\
&+ \cos B (\cos B + \cos A \cos C) \sin^2 \beta + \cos C (\cos C + \cos A \cos B) \sin^2 \gamma \Leftrightarrow \\
&\Leftrightarrow \sin^2 A \cos^2 \alpha + \sin^2 B \cos^2 \beta + \sin^2 C \cos^2 \gamma + \cos \alpha \cos \beta \cos \gamma (\sin^2 A + \sin^2 B + \sin^2 C) = \\
&= \sin^2 \alpha \cos^2 A + \sin^2 \beta \cos^2 B + \sin^2 \gamma \cos^2 C + \cos A \cos B \cos C (\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma) \Leftrightarrow \\
&\Leftrightarrow \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma - (\cos^2 A + \cos^2 B + \cos^2 C) + \\
&+ 2 \cos \alpha \cos \beta \cos \gamma (1 + \cos A \cos B \cos C) - 2 \cos A \cos B \cos C (1 + \cos \alpha \cos \beta \cos \gamma) = 0 \Leftrightarrow \\
&\Leftrightarrow 1 - 2 \cos \alpha \cos \beta \cos \gamma - 1 + 2 \cos A \cos B \cos C + 2 \cos \alpha \cos \beta \cos \gamma - 2 \cos A \cos B \cos C = 0, \\
&\text{and this is true.}
\end{aligned}$$

a) *The length of the segments AD, BE and CF*

In absolute barycentric coordinates the distance between two points $P = (x, y, z)$ and $P' = (x', y', z')$ is given by $|PP'| = \sqrt{(x-x')^2 S_A + (y-y')^2 S_B + (z-z')^2 S_C}$.

In order to calculate the length of the segments AD , BE and CF we introduce the following notations:

$$\begin{aligned}
t_\alpha &= S_\beta + S_\gamma = S \operatorname{ctg} \beta + S \operatorname{ctg} \gamma = S \left(\frac{\cos \beta}{\sin \beta} + \frac{\cos \gamma}{\sin \gamma} \right) = \frac{S}{\sin \beta \sin \gamma} \sin(\beta + \gamma) = \frac{S \sin \alpha}{\sin \beta \sin \gamma}, \\
t_\beta &= S_\gamma + S_\alpha = \frac{S \sin \beta}{\sin \gamma \sin \alpha}, \quad t_\gamma = S_\alpha + S_\beta = \frac{S \sin \gamma}{\sin \alpha \sin \beta}.
\end{aligned}$$

The sum of the coordinates of the point D , E resp. F is t_α , t_β resp. t_γ . The length of the segment AD is:

$$\begin{aligned}
AD &= \frac{1}{t_\alpha} \sqrt{(a^2 + t_\alpha)^2 S_A + (S_\gamma + S_C)^2 S_B + (S_\beta + S_B)^2 S_C} \\
&= \frac{1}{t_\alpha} \sqrt{(a^2 + 2t_\alpha)(a^2 S_A + S_{BC}) + S_A t_\alpha^2 + S_B S_\gamma^2 + S_C S_\beta^2} \\
&= \frac{1}{t_\alpha} \sqrt{(a^2 + 2t_\alpha) S^2 + S_A t_\alpha^2 + S_B S_\gamma^2 + S_C S_\beta^2} \\
&= \frac{1}{t_\alpha} \sqrt{(a^2 + 2t_\alpha) S^2 + S_A S^2 \frac{\sin^2 \alpha}{\sin^2 \beta \sin^2 \gamma} + S_B S^2 \frac{\cos^2 \gamma}{\sin^2 \gamma} + S_C S^2 \frac{\cos^2 \beta}{\sin^2 \beta}} \\
&= \frac{1}{\sin \alpha} \sqrt{a^2 \sin^2 \beta \sin^2 \gamma + 2S \sin \alpha \sin \beta \sin \gamma + S_A \sin^2 \alpha + S_B \sin^2 \beta \cos^2 \gamma + S_C \cos^2 \beta \sin^2 \gamma} \\
&= \frac{1}{\sin \alpha} \sqrt{S_A \sin^2 \alpha + S_B \sin^2 \beta + S_C \sin^2 \gamma + 2S \sin \alpha \sin \beta \sin \gamma}
\end{aligned}$$

$$= \frac{2R\sqrt{\lambda}}{\sin \alpha} = \frac{2R\sqrt{\lambda_0}}{\sin \alpha} = \frac{\sqrt{\sin \alpha \sin \beta \sin \gamma (a^2 \operatorname{ctg} \alpha + b^2 \operatorname{ctg} \beta + c^2 \operatorname{ctg} \gamma + 2S)}}{\sin \alpha}.$$

For this reason:

$$AD = \sqrt{\frac{\sin \beta \sin \gamma}{\sin \alpha} (a^2 \operatorname{ctg} \alpha + b^2 \operatorname{ctg} \beta + c^2 \operatorname{ctg} \gamma + 2S)}, \quad (3)$$

$$BE = \sqrt{\frac{\sin \gamma \sin \alpha}{\sin \beta} (a^2 \operatorname{ctg} \alpha + b^2 \operatorname{ctg} \beta + c^2 \operatorname{ctg} \gamma + 2S)}, \quad (4)$$

$$CF = \sqrt{\frac{\sin \alpha \sin \beta}{\sin \gamma} (a^2 \operatorname{ctg} \alpha + b^2 \operatorname{ctg} \beta + c^2 \operatorname{ctg} \gamma + 2S)}. \quad (5)$$

Based on the formulas (3), (4) and (5):

$$AD \cdot \sin \alpha = BE \cdot \sin \beta = CF \cdot \sin \gamma = \sqrt{\sin \alpha \sin \beta \sin \gamma (a^2 \operatorname{ctg} \alpha + b^2 \operatorname{ctg} \beta + c^2 \operatorname{ctg} \gamma + 2S)}. \quad (6)$$

b) *The distance of the first generalized Fermat point from the vertices of the reference triangle*

In what follows we calculate the lengths of the segments AK , BK and CK . The sum of coordinates of the point K is:

$$\begin{aligned} \Phi &= (S_\beta + S_B)(S_\gamma + S_C) + (S_\gamma + S_C)(S_\alpha + S_A) + (S_\alpha + S_A)(S_\beta + S_B) \\ &= S_{BC} + S_{CA} + S_{AB} + S_{\beta\gamma} + S_{\gamma\alpha} + S_{\alpha\beta} + (S_B + S_C)S_\alpha + (S_C + S_A)S_\beta + (S_A + S_B)S_\gamma \\ &= a^2 S_\alpha + b^2 S_\beta + c^2 S_\gamma + 2S^2. \end{aligned}$$

Therefore

$$\lambda = \frac{\sin \alpha \sin \beta \sin \gamma}{4R^2 S} (a^2 S_\alpha + b^2 S_\beta + c^2 S_\gamma + 2S^2) = \frac{\sin \alpha \sin \beta \sin \gamma}{4R^2 S} \Phi.$$

The length of the segment AK is:

$$\begin{aligned} AK &= \frac{1}{|\Phi|} \sqrt{[\Phi - (S_\beta + S_B)(S_\gamma + S_C)]^2 S_A + (S_\gamma + S_C)^2 (S_\alpha + S_A)^2 S_B + (S_\alpha + S_A)^2 (S_\beta + S_B)^2 S_C} \\ &= \frac{|S_\alpha + S_A|}{|\Phi|} \sqrt{(S_B + S_C + S_\beta + S_\gamma)^2 S_A + (S_\gamma + S_C)^2 S_B + (S_\beta + S_B)^2 S_C} \\ &= \frac{|S_\alpha + S_A|}{|\Phi|} \sqrt{(a^2 + t_\alpha)^2 S_A + (S_\gamma + S_C)^2 S_B + (S_\beta + S_B)^2 S_C} \\ &= \frac{|S_\alpha + S_A|}{|\Phi|} AD \cdot t_\alpha = \frac{|S_\alpha + S_A|}{|\Phi|} \frac{2R \cdot t_\alpha \sqrt{\lambda}}{\sin \alpha} = \frac{\sin \alpha}{2R\sqrt{\lambda}} |S_\alpha + S_A| = \frac{S}{2R\sqrt{\lambda} \sin A} |\sin(A + \alpha)| \\ &= \frac{S}{a\sqrt{\lambda}} |\sin(A + \alpha)| = \frac{bc}{2R\sqrt{\lambda}} |\sin(A + \alpha)|. \end{aligned}$$

Therefore:

$$AK = \frac{bc}{2R\sqrt{\lambda}} |\sin(A + \alpha)|, \quad (7) \quad BK = \frac{ca}{2R\sqrt{\lambda}} |\sin(B + \beta)|, \quad (8)$$

$$CK = \frac{ab}{2R\sqrt{\lambda}} |\sin(C + \gamma)|. \quad (9)$$

c) *Relations referring to the segments AK, BK and CK*

Now we will demonstrate the following conditional identity:

$$bc \sin \alpha \sin(A + \alpha) + ca \sin \beta \sin(B + \beta) + ab \sin \gamma \sin(C + \gamma) = 4R^2 \lambda. \quad (10)$$

Indeed

$$\begin{aligned} & bc \sin \alpha \sin(A + \alpha) + ca \sin \beta \sin(B + \beta) + ab \sin \gamma \sin(C + \gamma) = \\ &= 4R^2 \sin B \sin C \sin \alpha (\sin A \cos \alpha + \sin \alpha \cos A) + 4R^2 \sin C \sin A \sin \beta (\sin B \cos \beta + \sin \beta \cos B) + \\ &+ 4R^2 \sin A \sin B \sin \gamma (\sin C \cos \gamma + \sin \gamma \cos C) = \\ &= 4R^2 \sin A \sin B \sin C \left[\operatorname{ctg} A \sin^2 \alpha + \operatorname{ctg} B \sin^2 \beta + \operatorname{ctg} C \sin^2 \gamma + (\sin 2\alpha + \sin 2\beta + \sin 2\gamma)/2 \right] = \\ &= 4R^2 \sin A \sin B \sin C (\operatorname{ctg} A \sin^2 \alpha + \operatorname{ctg} B \sin^2 \beta + \operatorname{ctg} C \sin^2 \gamma + 2 \sin \alpha \sin \beta \sin \gamma) = 4R^2 \lambda. \end{aligned}$$

Since $A + \alpha + B + \beta + C + \gamma = 2\pi$, from inequalities $A + \alpha > \pi$, $B + \beta > \pi$, $C + \gamma > \pi$ only one can be true. If $A + \alpha < \pi$, $B + \beta < \pi$, $C + \gamma < \pi$, then the K is an inside point of the triangle ABC and based on the (10) we have

$$AD \sin \alpha = BE \sin \beta = CF \sin \gamma = 2R\sqrt{\lambda} = AK \sin \alpha + BK \sin \beta + CK \sin \gamma. \quad (11)$$

If for instance $A + \alpha > \pi$, then

$$AD \sin \alpha = BE \sin \beta = CF \sin \gamma = 2R\sqrt{\lambda} = -AK \sin \alpha + BK \sin \beta + CK \sin \gamma. \quad (12)$$

III. THE SECOND GENERALIZED FERMAT POINT OF THE TRIANGLE

We build on the sides of the triangle ABC inwards the similar triangles BCL , CAM and ABN in that way, that

$BAN \angle = \alpha = CAM \angle$, $CBL \angle = \beta = ABN \angle$, $ACM \angle = \gamma = BCL \angle$ (Figure 4). Since $A + B + C = \pi = \alpha + \beta + \gamma$, therefore $BLC \angle = \alpha$, $CMA \angle = \beta$, $ANB \angle = \gamma$.

According to the Conway's formulas, the barycentric coordinates of the points L , M , N are:

$$L = (a^2 : S_\gamma - S_C : S_\beta - S_B), \quad M = (S_\gamma - S_C : b^2 : S_\alpha - S_A), \quad N = (S_\beta - S_B : S_\alpha - S_A : c^2).$$

The equations of the lines AL , BM , CN are: $-(S_\beta - S_B)y + (S_\gamma - S_C)z = 0$,

$$(S_\alpha - S_A)x - (S_\gamma - S_C)z = 0,$$

$$-(S_\alpha - S_A)x + (S_\beta - S_B)y = 0.$$

Since adding whatever two equation, we get the third equation, the lines AL , BM and CN meets each other in a point $T = AL \cap BM \cap CN$. This point T will be called **the second generalized Fermat point of the triangle**.

The barycentric coordinates of the point T are:

$$\begin{aligned} T &= ((S_\beta - S_B)(S_\gamma - S_C) : (S_\gamma - S_C)(S_\alpha - S_A) : (S_\alpha - S_A)(S_\beta - S_B)) \\ &= \left(\frac{1}{S_\alpha - S_A} : \frac{1}{S_\beta - S_B} : \frac{1}{S_\gamma - S_C} \right). \end{aligned}$$

We introduce the following notations:

$$\begin{aligned}
 \mu &= \mu(A, B, C, \alpha, \beta, \gamma) = \\
 &= \sin A \sin B \sin C (ctg A \sin^2 \alpha + ctg B \sin^2 \beta + ctg C \sin^2 \gamma - 2 \sin \alpha \sin \beta \sin \gamma) \\
 &= \frac{1}{4R^2} (S_A \sin^2 \alpha + S_B \sin^2 \beta + S_C \sin^2 \gamma - 2S \sin \alpha \sin \beta \sin \gamma). \\
 \mu_0 &= \mu(\alpha, \beta, \gamma, A, B, C) = \\
 &= \sin \alpha \sin \beta \sin \gamma (ctg \alpha \sin^2 A + ctg \beta \sin^2 B + ctg \gamma \sin^2 C - 2 \sin A \sin B \sin C) \\
 &= \frac{\sin \alpha \sin \beta \sin \gamma}{4R^2} (a^2 ctg \alpha + b^2 ctg \beta + c^2 ctg \gamma - 2S) \\
 &= \frac{\sin \alpha \sin \beta \sin \gamma}{4R^2 S} (a^2 S_\alpha + b^2 S_\beta + c^2 S_\gamma - 2S^2).
 \end{aligned}$$

In this case the equality $\mu_0 = \mu$ is valid too.

a) *The length of the segments AL , BM and CN*

The sum of the coordinates of the point L , M resp. N are t_α , t_β resp. t_γ . The length of the segment AL is:

$$\begin{aligned}
 AL &= \frac{1}{t_\alpha} \sqrt{(a^2 - t_\alpha)^2 S_A + (S_\gamma + S_C)^2 S_B + (S_\beta + S_B)^2 S_C} \\
 &= \frac{1}{t_\alpha} \sqrt{(a^2 - 2t_\alpha)(a^2 S_A + S_{BC}) + S_A t_\alpha^2 + S_B S_\gamma^2 + S_C S_\beta^2} \\
 &= \frac{1}{t_\alpha} \sqrt{(a^2 - 2t_\alpha) S^2 + S_A t_\alpha^2 + S_B S_\gamma^2 + S_C S_\beta^2} \\
 &= \frac{1}{t_\alpha} \sqrt{(a^2 - 2t_\alpha) S^2 + S_A S^2 \frac{\sin^2 \alpha}{\sin^2 \beta \sin^2 \gamma} + S_B S^2 \frac{\cos^2 \gamma}{\sin^2 \gamma} + S_C S^2 \frac{\cos^2 \beta}{\sin^2 \beta}} \\
 &= \frac{1}{\sin \alpha} \sqrt{a^2 \sin^2 \beta \sin^2 \gamma - 2S \sin \alpha \sin \beta \sin \gamma + S_A \sin^2 \alpha + S_B \sin^2 \beta \cos^2 \gamma + S_C \cos^2 \beta \sin^2 \gamma} \\
 &= \frac{1}{\sin \alpha} \sqrt{S_A \sin^2 \alpha + S_B \sin^2 \beta + S_C \sin^2 \gamma - 2S \sin \alpha \sin \beta \sin \gamma} \\
 &= \frac{2R\sqrt{\mu}}{\sin \alpha} = \frac{2R\sqrt{\mu_0}}{\sin \alpha} = \frac{\sqrt{\sin \alpha \sin \beta \sin \gamma (a^2 ctg \alpha + b^2 ctg \beta + c^2 ctg \gamma - 2S)}}{\sin \alpha}.
 \end{aligned}$$

For this reason:

$$AL = \sqrt{\frac{\sin \beta \sin \gamma}{\sin \alpha} (a^2 ctg \alpha + b^2 ctg \beta + c^2 ctg \gamma - 2S)}, \quad (13)$$

$$BM = \sqrt{\frac{\sin \gamma \sin \alpha}{\sin \beta} (a^2 ctg \alpha + b^2 ctg \beta + c^2 ctg \gamma - 2S)}, \quad (14)$$

$$CN = \sqrt{\frac{\sin \alpha \sin \beta}{\sin \gamma} (a^2 ctg \alpha + b^2 ctg \beta + c^2 ctg \gamma - 2S)}. \quad (15)$$

Based on the (13), (14) and (15) we obtain:

$$AL \cdot \sin \alpha = BM \cdot \sin \beta = CN \cdot \sin \gamma = \sqrt{\sin \alpha \sin \beta \sin \gamma (a^2 \cot \alpha + b^2 \cot \beta + c^2 \cot \gamma - 2S)}. \quad (16)$$

b) *The distance of the second generalized Fermat point from the vertices of the reference triangle*

The sum of coordinates of the point T is:

$$\begin{aligned} \Gamma &= (S_\beta - S_B)(S_\gamma - S_C) + (S_\gamma - S_C)(S_\alpha - S_A) + (S_\alpha - S_A)(S_\beta - S_B) \\ &= S_{BC} + S_{CA} + S_{AB} + S_{\beta\gamma} + S_{\gamma\alpha} + S_{\alpha\beta} - (S_B + S_C)S_\alpha - (S_C + S_A)S_\beta - (S_A + S_B)S_\gamma \\ &= -a^2 S_\alpha - b^2 S_\beta - c^2 S_\gamma + 2S^2 = -(a^2 S_\alpha + b^2 S_\beta + c^2 S_\gamma - 2S^2). \end{aligned}$$

Therefore:

$$\mu = \frac{\sin \alpha \sin \beta \sin \gamma}{4R^2 S} (a^2 S_\alpha + b^2 S_\beta + c^2 S_\gamma - 2S^2) = -\frac{\sin \alpha \sin \beta \sin \gamma}{4R^2 S} \Gamma.$$

The length of the segment AT is:

$$\begin{aligned} AT &= \frac{1}{|\Gamma|} \sqrt{[\Gamma - (S_\beta - S_B)(S_\gamma - S_C)]^2 S_A + (S_\gamma - S_C)^2 (S_\alpha - S_A)^2 S_B + (S_\alpha - S_A)^2 (S_\beta - S_B)^2 S_C} \\ &= \frac{|S_\alpha - S_A|}{|\Gamma|} \sqrt{(S_\beta + S_\gamma - S_B - S_C)^2 S_A + (S_\gamma - S_C)^2 S_B + (S_\beta - S_B)^2 S_C} \\ &= \frac{|S_\alpha - S_A|}{|\Gamma|} \sqrt{(-a^2 + t_\alpha)^2 S_A + (S_\gamma - S_C)^2 S_B + (S_\beta - S_B)^2 S_C} \\ &= \frac{|S_\alpha - S_A|}{|\Gamma|} AL \cdot t_\alpha = \frac{|S_\alpha - S_A|}{|\Gamma|} \frac{2R \cdot t_\alpha \sqrt{\mu}}{\sin \alpha} = \frac{\sin \alpha}{2R\sqrt{\mu}} |S_\alpha - S_A| = \frac{S}{2R\sqrt{\mu} \sin A} |\sin(A - \alpha)| \\ &= \frac{S}{a\sqrt{\mu}} |\sin(A - \alpha)| = \frac{bc}{2R\sqrt{\mu}} |\sin(A - \alpha)| = \frac{bc}{2R\sqrt{\mu}} \sin|A - \alpha|. \end{aligned}$$

Therefore:

$$AT = \frac{bc}{2R\sqrt{\mu}} \sin|A - \alpha|, \quad (17) \quad BT = \frac{ca}{2R\sqrt{\mu}} \sin|B - \beta|, \quad (18)$$

$$CT = \frac{ab}{2R\sqrt{\mu}} \sin|C - \gamma|. \quad (19)$$

c) *Relations referring to the segments AT , BT and CT*

Here is valid too the following conditional identity:

$$bc \sin \alpha \sin(A - \alpha) + ca \sin \beta \sin(B - \beta) + ab \sin \gamma \sin(C - \gamma) = -4R^2 \mu. \quad (20)$$

Since $A - \alpha + B - \beta + C - \gamma = 0$, only two of the $A > \alpha$, $B > \beta$, $C > \gamma$ inequalities

can be true at the same time. For instance if $A < \alpha$, $B > \beta$, $C > \gamma$, then based on the formula (20)

$$AL \sin \alpha = BM \sin \beta = CN \sin \gamma = 2R\sqrt{\mu} = AT \sin \alpha - BT \sin \beta - CT \sin \gamma. \quad (21)$$

Or if $A < \alpha$, $B < \beta$, $C > \gamma$, then

$$AL \sin \alpha = BM \sin \beta = CN \sin \gamma = 2R\sqrt{\mu} = AT \sin \alpha + BT \sin \beta - CT \sin \gamma. \quad (22)$$

IV. SPECIALE CASE

In this section we use the following relations:

$$abc(ctgA + ctgB + ctgC) = R(a^2 + b^2 + c^2), \quad (23)$$

$$\begin{aligned} \sin A \sin B \sin C (ctgA + ctgB + ctgC + \sqrt{3}) &= \\ &= 1 + \sqrt{3} \sin A \sin B \sin C + \cos A \cos B \cos C, \end{aligned} \quad (24)$$

$$\begin{aligned} \sin A \sin B \sin C (ctgA + ctgB + ctgC - \sqrt{3}) &= \\ &= 1 - \sqrt{3} \sin A \sin B \sin C + \cos A \cos B \cos C. \end{aligned} \quad (25)$$

The proof of (23):

$$\begin{aligned} abc(ctgA + ctgB + ctgC) &= abc \left(\frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C} \right) = 2Rabc \left(\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} \right) = \\ &= R(2bc \cos A + 2ca \cos B + 2ab \cos C) = R(b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2) = \\ &= R(a^2 + b^2 + c^2). \end{aligned}$$

The proof of (24) and (25):

$$\begin{aligned} \sin A \sin B \sin C (ctgA + ctgB + ctgC \pm \sqrt{3}) &= \sin A \sin B \sin C \left(\frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C} \pm \sqrt{3} \right) = \\ &= \cos A \sin B \sin C + \cos B \sin C \sin A + \cos C \sin A \sin B \pm \sqrt{3} \sin A \sin B \sin C = \\ &= 1 \pm \sqrt{3} \sin A \sin B \sin C + \cos A \cos B \cos C. \end{aligned}$$

If $\alpha = \beta = \gamma = \pi/3$, then $K \equiv F_1$ and $T \equiv F_2$, where F_1 resp. F_2 is the first resp. the second Fermat's point of the triangle ABC (X_{13} resp. X_{14} in [1]). In this case

$$\begin{aligned} \lambda^* &= \lambda \left(A, B, C, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right) = \frac{3}{4} \sin A \sin B \sin C (ctgA + ctgB + ctgC + \sqrt{3}) = \\ &= \frac{3}{32R^2} (a^2 + b^2 + c^2 + 2\sqrt{3}S) = \frac{3}{4} (1 + \sqrt{3} \sin A \sin B \sin C + \cos A \cos B \cos C), \\ \mu^* &= \mu \left(A, B, C, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right) = \frac{3}{4} \sin A \sin B \sin C (ctgA + ctgB + ctgC - \sqrt{3}) = \\ &= \frac{3}{32R^2} (a^2 + b^2 + c^2 - 2\sqrt{3}S) = \frac{3}{4} (1 - \sqrt{3} \sin A \sin B \sin C + \cos A \cos B \cos C). \end{aligned}$$

Based on (6) we have

$$AX_1 = BY_1 = CZ_1 = \frac{4R}{\sqrt{3}}\sqrt{\lambda^*} = \frac{\sqrt{a^2 + b^2 + c^2 + 2\sqrt{3}S}}{\sqrt{2}}. \quad (26)$$

According to (7), (8) and (9)

$$AF_1 = \frac{bc}{2R\sqrt{\lambda^*}} \left| \sin\left(A + \frac{\pi}{3}\right) \right|, \quad (27) \quad BF_1 = \frac{ca}{2R\sqrt{\lambda^*}} \left| \sin\left(B + \frac{\pi}{3}\right) \right|, \quad (28)$$

$$CF_1 = \frac{ab}{2R\sqrt{\lambda^*}} \left| \sin\left(C + \frac{\pi}{3}\right) \right|. \quad (29)$$

If $A < 2\pi/3$, $B < 2\pi/3$, $C < 2\pi/3$, then

$$AX_1 = BY_1 = CZ_1 = \frac{4R}{\sqrt{3}}\sqrt{\lambda^*} = AF_1 + BF_1 + CF_1. \quad (30)$$

If for instance $A > 2\pi/3$, then

$$AX_1 = BY_1 = CZ_1 = \frac{4R}{\sqrt{3}}\sqrt{\lambda^*} = -AF_1 + BF_1 + CF_1. \quad (31)$$

Based on (16) we have

$$AX_2 = BY_2 = CZ_2 = \frac{4R}{\sqrt{3}}\sqrt{\mu^*} = \frac{\sqrt{a^2 + b^2 + c^2 - 2\sqrt{3}S}}{\sqrt{2}}. \quad (32)$$

According to (17), (18) and (19)

$$AF_2 = \frac{bc}{2R\sqrt{\mu^*}} \sin\left|A - \frac{\pi}{3}\right|, \quad (33) \quad BF_2 = \frac{ca}{2R\sqrt{\mu^*}} \sin\left|B - \frac{\pi}{3}\right|, \quad (34)$$

$$CF_2 = \frac{ab}{2R\sqrt{\mu^*}} \sin\left|C - \frac{\pi}{3}\right|. \quad (35)$$

If for instance $A < \pi/3$, $B > \pi/3$, $C > \pi/3$, then

$$AX_2 = BY_2 = CZ_2 = \frac{4R}{\sqrt{3}}\sqrt{\mu^*} = AF_2 - BF_2 - CF_2. \quad (36)$$

Or if $A < \pi/3$, $B < \pi/3$, $C > \pi/3$, then

$$AX_2 = BY_2 = CZ_2 = \frac{4R}{\sqrt{3}}\sqrt{\mu^*} = AF_2 + BF_2 - CF_2. \quad (37)$$

Other arrangements of the triangles BCD , CAE and ABF are possible. To investigate the valabilities of the above metric relations in this cases is a possible subject for further researches.

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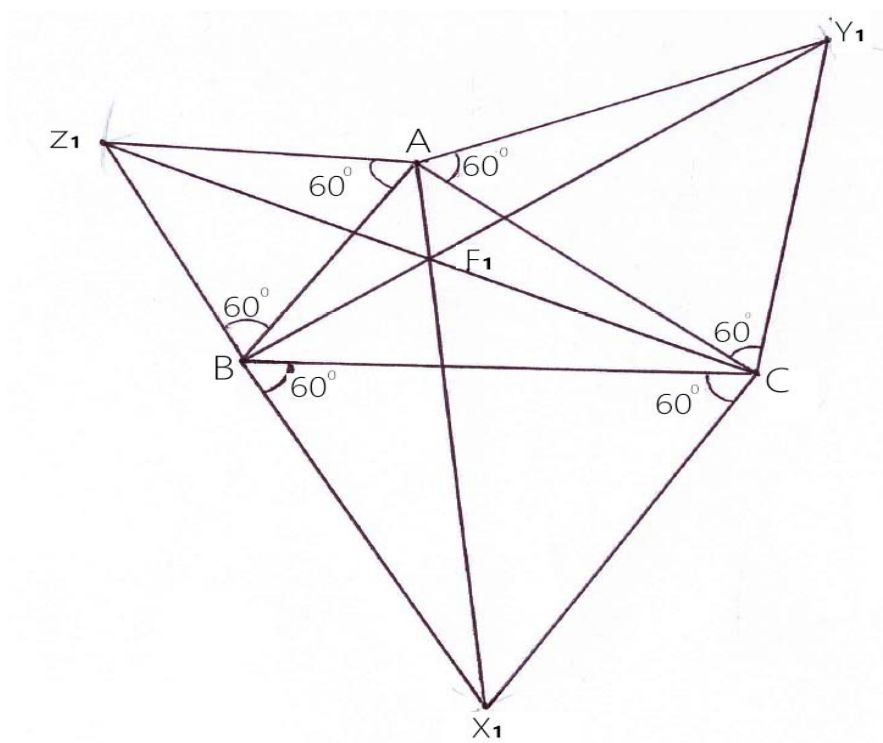


Figure 1

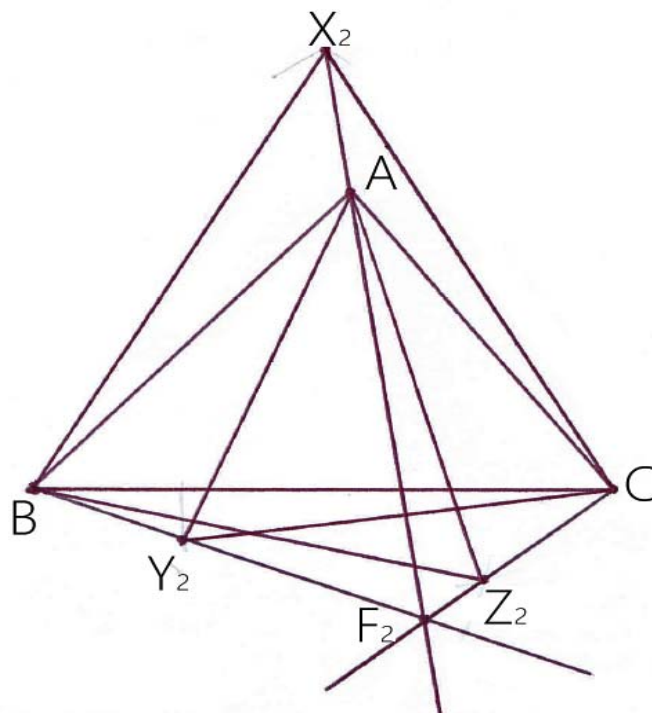


Figure 2

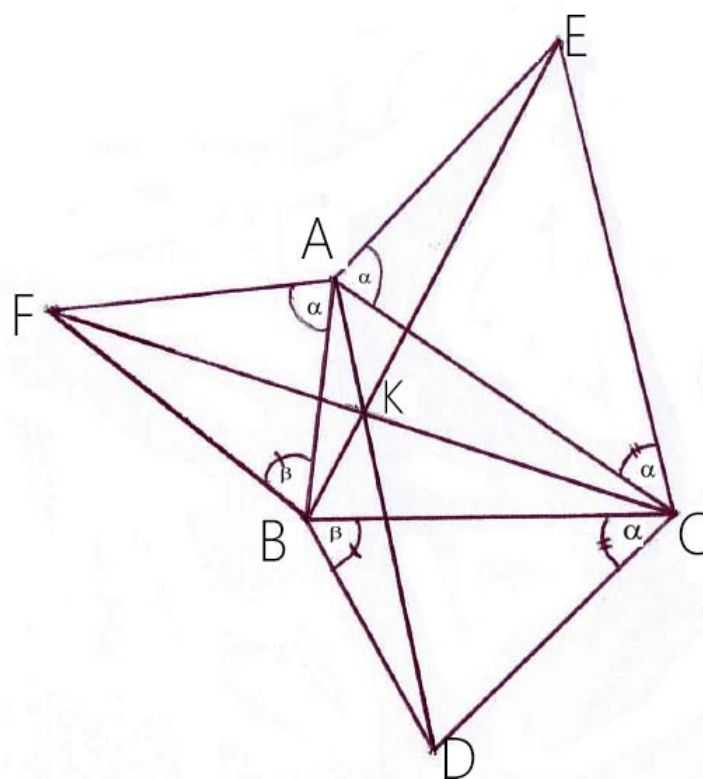


Figure 3

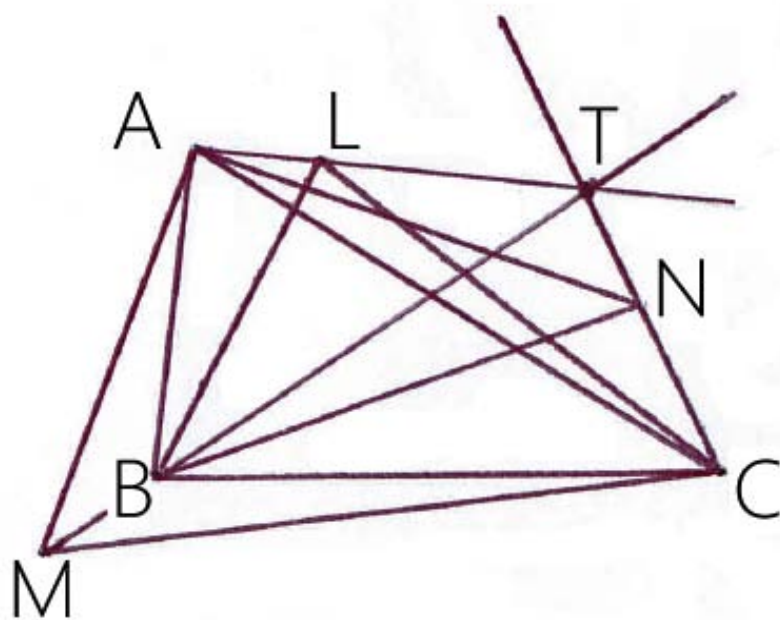


Figure 4

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