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Certain New Formulae Involving Modified Bessel Function of First Kind

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1. INTRODUCTION

a) Bessel Function

Bessel functions, first defined by the mathematician Daniel Bernoulli and generalized by Friedrich Bessel, are the canonical solutions $y(x)$ of Bessel's differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2)y = 0 \quad (1.1)$$

for an arbitrary complex number α (the order of the Bessel function). The most important cases are for α an integer or half-integer.

b) Bessel Function of First Kind

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel's differential equation that are finite at the origin ($x = 0$) for integer or positive α , and diverge as x approaches zero for negative non-integer α . It is possible to define the function by its Taylor series expansion around $x = 0$.

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha} \quad (1.2)$$

where Γz is the gamma function, a shifted generalization of the factorial function to non-integer values. The Bessel function of the first kind is an entire function if α is an integer. The graphs of Bessel functions look roughly like oscillating sine or cosine functions that decay proportionally to $\frac{1}{\sqrt{x}}$.

c) Modified Bessel Function of First Kind

The Bessel functions are valid even for complex arguments x , and an important special case is that of a purely imaginary argument. In this case, the solutions to the Bessel equation are called

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the modified Bessel functions (or occasionally the hyperbolic Bessel functions) of the first and second kind. The first kind of Bessel function is defined as

$$I_{\alpha}(x) = \iota^{-\alpha} J_{\alpha}(\iota x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha} \quad (1.3)$$

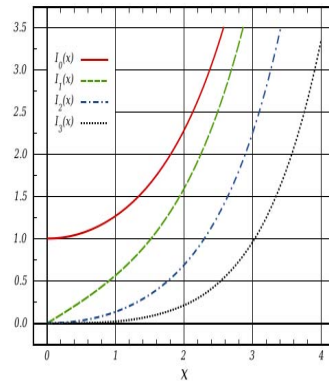


Figure 1 : Modified Bessel Function of First Kind

d) Exponential Function

In mathematics, the exponential function is the function e^x , where e is the number (approximately 2.718281828) such that the function e^x is its own derivative. The exponential function is used to model a relationship in which a constant change in the independent variable gives the same proportional change (i.e. percentage increase or decrease) in the dependent variable. The function is often written as $\exp(x)$, especially when it is impractical to write the independent variable as a superscript. The exponential function is widely used in physics, chemistry, engineering, mathematical biology, economics and mathematics.

In particular the exponential function may be defined as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (1.4)$$

e) Generalized Hypergeometric Functions

A generalized hypergeometric function ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\dots(k+a_p)}{(k+b_1)(k+b_2)\dots(k+b_q)(k+1)} z. \quad (1.5)$$

Where $k+1$ in the denominator is present for historical reasons of notation, and the resulting generalized hypergeometric function is written

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k z^k}{(b_1)_k (b_2)_k \dots (b_q)_k k!} \quad (1.6)$$

or

$${}_pF_q \left[\begin{matrix} (a_p) & ; \\ (b_q) & ; \end{matrix} \quad z \right] = \sum_{k=0}^{\infty} \frac{((a_p))_k z^k}{((b_q))_k k!} \quad (1.7)$$

where the parameters $b_1, b_2, \dots, b_q$ are neither zero nor negative integers and p, q are non-negative integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $|z| < 1$ if $p \neq q + 1$, diverges for all $z, z \neq 0$ if $p > q + 1$.

The ${}_pF_q$ series absolutely converges for $|z| = 1$ if $R(\zeta) < 0$, conditionally converges for $|z| = 1, z \neq 0$ if $0 \leq R(\zeta) < 1$, diverges for $|z| = 1$, if $1 \leq R(\zeta)$, $\zeta = \sum_{i=1}^p a_i - \sum_{i=0}^q b_i$.

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (Gauss 1812, Barnes 1908). To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

$$z(1-z)y'' + [c - (a+b+1)z]y' - aby = 0 \quad (1.8)$$

The solution of this equation is

$$y = A_0 \left[1 + \frac{ab}{1!c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \dots \right] \quad (1.9)$$

This is the so-called regular solution, denoted

$${}_2F_1(a, b; c; z) = \left[1 + \frac{ab}{1!c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \dots \right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!} \quad (1.10)$$

which converges if c is not a negative integer for all of $|z| < 1$ and on the unit circle $|z| = 1$ if $R(c-a-b) > 0$.

It is known as Gauss hypergeometric function in terms of Pochhammer symbol $(a)_k$ or generalized factorial function.

f) Whittaker Function

In mathematics, a Whittaker function is a special solution of Whittaker's equation, a modified form of the confluent hypergeometric equation introduced by Whittaker (1904) to make the formulas involving the solutions more symmetric. More generally, Jacquet (1966, 1967) introduced Whittaker functions of reductive groups over local fields, where the functions studied by Whittaker are essentially the case where the local field is the real numbers and the group is $SL_2(R)$. Whittaker's equation is

$$\frac{d^2 w}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) w = 0 \quad (1.11)$$

It has a regular singular point at 0 and an irregular singular point at ∞ . Two solutions are given by the Whittaker functions $M_{k,\mu}(z), W_{k,\mu}(z)$, defined in terms of Kummer's confluent hypergeometric functions M and U by

$$M_{k,\mu}(z) = \exp\left(-\frac{z}{2}\right) z^{\mu+\frac{1}{2}} M\left(\mu - k + \frac{1}{2}, 1 + 2\mu; z\right) \quad (1.12)$$

$$W_{k,\mu}(z) = \exp\left(-\frac{z}{2}\right) z^{\mu+\frac{1}{2}} U\left(\mu - k + \frac{1}{2}, 1 + 2\mu; z\right) \quad (1.13)$$

Whittaker functions appear as coefficients of certain representations of the group $SL_2(R)$, called Whittaker models.

g) Associated Laguerre Polynomials

The Rodrigues representation for the associated Laguerre polynomials is

$$L_n^k(x) = \frac{e^x x^{-k}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+k}) \quad (1.14)$$

$$= \sum_{m=0}^n (-1)^m \frac{(n+k)!}{(n-m)!(k+m)!m!} x^m \quad (1.15)$$

The first few associated Laguerre polynomials are

$$\begin{cases} L_0^k(x) = 1 \\ L_1^k(x) = -x + k + 1 \\ L_2^k(x) = \frac{1}{2}[x^2 - 2(k+2)x + (k+1)(k+2)] \end{cases} \quad (1.16)$$

II. MAIN RESULTS

$$\begin{aligned} e^{-\frac{x}{2}} {}_1F_1(a; 2a+1; x) &= 2^{2a-1} x^{\frac{1}{2}-a} \Gamma\left(a + \frac{1}{2}\right) I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right) - \\ &\quad - 2^{2a-1} x^{\frac{1}{2}-a} \Gamma\left(a + \frac{1}{2}\right) I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right) \end{aligned} \quad (2.1)$$

$$\begin{aligned} e^{-\frac{x}{2}} {}_1F_1(a; 2a-1; x) &= 2^{2a-3} (4a+x-2) x^{\frac{1}{2}-a} \Gamma\left(a - \frac{1}{2}\right) I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right) + \\ &\quad + 2^{2a-3} x^{\frac{3}{2}-a} \Gamma\left(a - \frac{1}{2}\right) I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right) \end{aligned} \quad (2.2)$$

$$\begin{aligned} e^{-\frac{x}{2}} {}_1F_1(a; 2a+2; x) &= \frac{2^{2a} x^{\frac{1}{2}-a} \Gamma\left(a + \frac{3}{2}\right) I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right)}{a+1} - \\ &\quad - \frac{2^{2a} x^{-\frac{1}{2}-a} (2a+x) \Gamma\left(a + \frac{3}{2}\right) I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right)}{a+1} \end{aligned} \quad (2.3)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a-2; x) = \frac{2^{2a-4}(8a^2 + 4ax - 12a + x^2 - 2x + 4)x^{\frac{1}{2}-a}\Gamma(a - \frac{1}{2})I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right)}{a-1} +$$

$$+ \frac{2^{2a-4}(2a+x-2)x^{-\frac{3}{2}-a}\Gamma(a - \frac{1}{2})I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right)}{a-1} \quad (2.4)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a+3; x) = 2^{2a+1}x^{-\frac{1}{2}-a}(a+x)\Gamma(a + \frac{3}{2})I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right) -$$

$$- \frac{2^{2a+1}x^{-\frac{3}{2}-a}(4a^2 + 3ax + 2a + x^2)\Gamma(a + \frac{3}{2})I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right)}{a+2} \quad (2.5)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a-3; x) +$$

$$= \frac{2^{2a-5}(4a^2 + 3ax - 10a + x^2 - 3x + 6)x^{\frac{3}{2}-a}\Gamma(a - \frac{3}{2})I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right)}{a-1} +$$

$$+ \frac{2^{2a-5}(16a^3 + 12a^2x - 48a^2 + 5ax^2 - 18ax + 44a + x^3 - 3x^2 + 6x - 12)}{a-1} \times$$

$$\times x^{\frac{1}{2}-a}\Gamma(a - \frac{3}{2})I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right) \quad (2.6)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a+4; x) = \frac{2^{2a+2}x^{-\frac{3}{2}-a}(2a^2 + 2ax + 2a + x^2)\Gamma(a + \frac{5}{2})I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right)}{(a+2)(a+3)} -$$

$$- \frac{2^{2a+2}x^{-\frac{5}{2}-a}(8a^3 + 8a^2x + 12a^2 + 4ax^2 + 4ax + 4a + x^3)\Gamma(a + \frac{5}{2})I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right)}{(a+2)(a+3)} \quad (2.7)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a-4; x) =$$

$$= \frac{2^{2a-6}x^{\frac{3}{2}-a}(8a^3 + 8a^2x - 36a^2 + 4ax^2 - 20ax + 52a + x^3 - 4x^2 + 12x - 24)}{(a-2)(a-1)} \times$$

$$\times \Gamma(a - \frac{3}{2})I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right) + 2^{2a-6}x^{\frac{1}{2}-a}\Gamma(a - \frac{3}{2})I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right) \times$$

$$\times \left[\frac{(32a^4 + 32a^3x - 160a^3 + 18a^2x^2 - 96a^2x + 280a^2)}{(a-2)(a-1)} + \right.$$

$$\left. + \frac{(6ax^3 - 30ax^2 + 88ax - 200a + x^4 - 4x^3 + 12x^2 - 24x + 48)}{(a-2)(a-1)} \right] \quad (2.8)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a+5; x) =$$

$$= \frac{2^{2a+3}x^{-\frac{5}{2}-a}(4a^3 + 5a^2x + 10a^2 + 3ax^2 + 5ax + 6a + x^3)\Gamma(a + \frac{5}{2})I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right)}{(a+3)(a+4)} -$$

$$- 2^{2a+3}x^{-\frac{7}{2}-a}\Gamma(a + \frac{5}{2})I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right) \left[\frac{(16a^4 + 20a^3x + 48a^3 + 3a^2x^2 + 30a^2x + 44a^2)}{(a+3)(a+4)} + \right.$$

$$\left. + \frac{(5ax^3 + 7ax^2 + 10ax + 12a + x^4)}{(a+3)(a+4)} \right] \quad (2.9)$$

$$\begin{aligned}
& e^{-\frac{x}{2}} {}_1F_1(a; 2a-5; x) = \\
& = 2^{2a-7} x^{\frac{3}{2}-a} \Gamma\left(a-\frac{5}{2}\right) I_{\frac{1}{2}(2a+1)}\left(\frac{x}{2}\right) \left[\frac{(16a^4 + 20a^3x - 112a^3 + 13a^2x^2 - 90a^2x + 284a^2)}{(a-2)(a-1)} + \right. \\
& \quad \left. + \frac{(5ax^3 - 33ax^2 + 130ax - 308a + x^4 - 5x^3 + 20x^2 - 60x + 120)}{(a-2)(a-1)} \right] + \\
& + 2^{2a-7} x^{\frac{1}{2}-a} \Gamma\left(a-\frac{5}{2}\right) I_{\frac{1}{2}(2a-1)}\left(\frac{x}{2}\right) \left[\frac{(64a^5 + 80a^4x - 480a^4 + 56a^3x^2 - 400a^3x + 1360a^3)}{(a-2)(a-1)} + \right. \\
& \quad \left. + \frac{(25a^2x^3 - 180a^2x^2 + 700a^2x - 1800a^2 + 7ax^4 - 45ax^3 + 184ax^2 - 500ax)}{(a-2)(a-1)} + \right. \\
& \quad \left. + \frac{(1096a + x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 240)}{(a-2)(a-1)} \right] \quad (2.10)
\end{aligned}$$

III. SPECIAL CASES

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a+1; 0) = e^{-\frac{x}{2}} \frac{\Gamma(1-a) \left(\Gamma(1+2a) L_{-a}^{2a}(0) \right)}{\Gamma(1+a)} \quad (3.1)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a+1; 1) = e^{-\frac{x}{2}} \frac{\Gamma(1-a) \left(\Gamma(1+2a) L_{-a}^{2a}(1) \right)}{\Gamma(1+a)} \quad (3.2)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a+1; 2) = e^{-\frac{x}{2}} \frac{e^1 M_{\frac{1}{2},a}^{1,2}(2)}{2^{\frac{1}{2}(1+2a)}} \quad (3.3)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a-1; 1) = e^{-\frac{x}{2}} \frac{\sqrt{e} M_{-\frac{1}{2},\frac{1}{2}}^{1,2}(-2+2a)(1)}{1^{\frac{1}{2}(-1+2a)}} \quad (3.4)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a-1; 2) = e^{-\frac{x}{2}} \frac{e^1 M_{-\frac{1}{2},\frac{1}{2}}^{1,2}(-2+2a)(2)}{2^{\frac{1}{2}(-1+2a)}} \quad (3.5)$$

$$e^{-\frac{x}{2}} {}_1F_1(a; 2a-1; 3) = e^{-\frac{x}{2}} \frac{e^{\frac{3}{2}} M_{-\frac{1}{2},\frac{1}{2}}^{1,2}(-2+2a)(3)}{3^{\frac{1}{2}(-1+2a)}} \quad (3.6)$$

$$e^{-\frac{x}{2}} {}_1F_1(1; 1; -1) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-1)^{k_1} (-x)^{k_2}}{2^{k_2} k_1! k_2!} \quad (3.7)$$

$$e^{-\frac{x}{2}} {}_1F_1(1; 3; -1) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-1)^{k_2} (-x)^{k_1} (1)_{k_2}}{2^{k_1} k_1! k_2! (3)_{k_2}} \quad (3.8)$$

IV. APPLICATIONS

The results are applied in all branches of applied sciences. These are used for solving scientific problems such as diffusion problems on a lattice, solving for patterns of acoustical radiation, Modes of vibration of a thin circular (or annular) artificial membrane (such as a drum or other membranophone), Heat conduction in a cylindrical object.

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