



GLOBAL JOURNAL OF SCIENCE FRONTIER RESEARCH
MATHEMATICS AND DECISION SCIENCES

Volume 13 Issue 8 Version 1.0 Year 2013

Type : Double Blind Peer Reviewed International Research Journal

Publisher: Global Journals Inc. (USA)

Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On the Maximal Ideals in the Banach Space of ${}^+Q$ uasicontinuous Functions

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GJSFR-F Classification : MSC 2010: 13A15, 26A15, 26A48, 46J10, 46J20



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On the Maximal Ideals in the Banach Space of $^+$ Quasicontinuous Functions

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Abstract : In this paper some interesting properties of $^+$ Quasicontinuous functions are presented. The maximal ideals in the Banach space of bounded real valued $^+$ Quasicontinuous functions defined on $[0,1]$ are investigated.

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I. INTRODUCTION

In this paper, it is shown that the set of all bounded real $^+$ Quasicontinuous functions defined on $[0,1]$ forms a commutative Banach algebra with identity under the supremum norm. The maximal ideals in this Banach algebra are identified to be of the form $M_x = \{f / f(x) = 0\}$ or $M_x^+ = \{f / f(x+) = 0\}$ for $x \in [0,1]$.

In what follows $\mathbb{R}$, I and J stand for the real line, the unit closed interval $[0,1]$ and any closed and bounded interval $[a,b]$ respectively.

a) Preliminaries

1.1 Definition: Let $f : J \rightarrow \mathbb{R}$. We define $f(a-) = f(a)$ and $f(b+) = f(b)$.

We say that $f(p+)$ exists at $p \in [a,b)$ and we write $f(p+) = L$, where $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon \quad \forall x \in (p, p + \delta) \subset J$

Similarly for $p \in (a,b]$ we write $f(p-) = l \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - l| < \varepsilon \quad \forall x \in (p - \delta, p) \subset J$

1.2 Definition: A function $f : J \rightarrow \mathbb{R}$ is said to be $^+$ Quasicontinuous on J if

- (i) $f(p+)$ exists at every $p \in [a,b)$

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$$(ii) \quad f(b-) = f(b)$$

1.3 Definition: A function $f: J \rightarrow \mathbb{R}$ is said to be cliquish at a point $p \in J$ if for every $\varepsilon > 0$ and every neighborhood U of p in J there exists a non-empty open set $W \subset U$ such that $|f(x) - f(y)| < \varepsilon \quad \forall x, y \in W$. We say that f is cliquish on J if it is cliquish at every point of J .

1.4 Definition: A mapping T from a linear space $\mathcal{V}$ into a linear space $\mathcal{W}$ is said to be linear if $T(cx + dy) = cT(x) + dT(y)$ for all x and y in $\mathcal{V}$ and constants c and d .

1.5 Definition: Let $\mathcal{V}$ and $\mathcal{W}$ be normed linear spaces. A linear map $T: \mathcal{V} \rightarrow \mathcal{W}$ is said to be bounded if there exists a real number $K \geq 0$ such that $\|T(x)\| \leq K\|x\| \quad \forall x \in \mathcal{V}$.

1.6 Definition: A linear functional on a vector space $\mathcal{V}$ over a field $\mathcal{K}$ is a linear mapping from $\mathcal{V}$ to $\mathcal{K}$.

b) *Properties of ${}^+Q$ uasicontinuous functions*

2.1 Proposition: Let $c \in \mathbb{R}$. If $f: J \rightarrow \mathbb{R}$ and $g: J \rightarrow \mathbb{R}$ are ${}^+Q$ uasicontinuous on J then $f + g, cf, fg, f \vee g$ and $f \wedge g$ are ${}^+Q$ uasicontinuous on J , where $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$.

Proof: Let $p \in [a, b)$. (i) Let $\varepsilon > 0$ be given. Then there exist $\delta_1 > 0$ and

$\delta_2 > 0$ such that $|f(x) - f(p+)| < \frac{\varepsilon}{2} \quad \forall x \in (p, p + \delta_1) \subset J$ and

$|g(x) - g(p+)| < \frac{\varepsilon}{2} \quad \forall x \in (p, p + \delta_2) \subset J$. Put $\delta = \min\{\delta_1, \delta_2\}$.

Then $x \in (p, p + \delta) \Rightarrow |(f + g)(x) - (f(p+) + g(p+))| \leq |f(x) - f(p+)| +$

$$|g(x) - g(p+)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|(f + g)(x) - (f(p+) + g(p+))| < \varepsilon \quad \forall x \in (p, p + \delta).$$

Hence $(f+g)(p+)$ exists and $(f+g)(p+) = f(p+) + g(p+)$. Since f and g are continuous at b , $f+g$ is continuous at b .

Hence $f+g$ is ${}^+Q$ uasicontinuous on J .

(ii) If $c=0$ then $cf=O$, where $O: J \rightarrow \mathbb{R}$ is defined by $O(x)=0$.

Then cf is ${}^+Q$ uasicontinuous on J . Now suppose that $c \neq 0$.

Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$\begin{aligned} |f(x) - f(p+)| &< \frac{\varepsilon}{|c|} \quad \forall x \in (p, p+\delta) \subset J \\ \Rightarrow |(cf)(x) - (cf)(p+)| &< \varepsilon \quad \forall x \in (p, p+\delta) \end{aligned}$$

Hence $(cf)(p+)$ exists and $(cf)(p+) = c f(p+)$. Since f is continuous at b , cf is continuous at b . Hence cf is ${}^+Q$ uasicontinuous on J .

(iii) Since f and g are ${}^+Q$ uasicontinuous at p , for every $\varepsilon > 0$ there exists a

$$\begin{aligned} \delta > 0 \text{ such that } |f(x) - f(p+)| &< \varepsilon \text{ and } |g(x) - g(p+)| < \varepsilon \quad \forall x \in (p, p+\delta) \subset J \\ \Rightarrow |(fg)(x) - f(p+)g(p+)| &= |f(x)g(x) - f(x)g(p+) + f(x)g(p+) - f(p+)g(p+)| \\ &\leq |f(x)||g(x) - g(p+)| + |g(p+)||f(x) - f(p+)| \\ &< |f(x)|\varepsilon + |g(p+)|\varepsilon \quad \forall x \in (p, p+\delta) \\ &= |f(x) - f(p+) + f(p+)|\varepsilon + |g(p+)|\varepsilon \\ &< \varepsilon(\varepsilon + |f(p+)| + |g(p+)|) \quad \forall x \in (p, p+\delta). \end{aligned}$$

Hence $(fg)(p+)$ exists and $(fg)(p+) = f(p+)g(p+)$. Since f and g are continuous at b , fg is continuous at b .

Hence fg is ${}^+Q$ uasicontinuous on J .

It is easy to verify that $f \vee g$ and $f \wedge g$ are ${}^+Q$ uasicontinuous on J and we have the following.

$$(f \vee g)(p+) = \max \{f(p+), g(p+)\} \text{ and } (f \wedge g)(p+) = \min \{f(p+), g(p+)\}.$$

2.2 Proposition: Let $f_n: J \rightarrow \mathbb{R}$, $n=1,2,3,\dots$, be ${}^+Q$ uasicontinuous on J and

$f_n \rightarrow f$ uniformly on J . Then f is ${}^+Q$ uasicontinuous on J .

Proof: Let $p \in [a, b)$. Let $\varepsilon > 0$ be given. Then there exists an integer N such that $n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall x \in J$.

Since f_N is ${}^+Q$ uasicontinuous at p , there exists a $\delta > 0$ such that

$$\begin{aligned} |f_N(x) - f_N(p+)| &< \varepsilon \quad \forall x \in (p, p+\delta) \subset J \\ x \in (p, p+\delta) &\Rightarrow |f(x) - f_N(p+)| = |f(x) - f_N(x) + f_N(x) - f_N(p+)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(p+)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f_N(p+)| < \varepsilon \quad \forall x \in (p, p+\delta) \subset J.$$

Hence $f(p+)$ exists for every $p \in [a, b)$.

Since each f_n is continuous at b and $f_n \rightarrow f$ uniformly on J , f is continuous at b . Hence f is ${}^+Q$ uasicontinuous on J .

2.3 Remark: It is not necessary that a ${}^+Q$ uasicontinuous function defined on a compact domain is bounded. It can be seen from the following example.

2.4 Example: Define $f : [-1, 1] \rightarrow \mathbb{R}$ by $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ \frac{1}{x} & \text{if } -1 \leq x < 0 \end{cases}$

This function f is ${}^+Q$ uasicontinuous on $[-1, 1]$ but it is not bounded.

2.5 Remark: We denote the set of all bounded real valued ${}^+Q$ uasicontinuous functions defined on I by the symbol $\mathcal{C}^+(I)$. By the propositions 2.1 and 2.2 it follows that $\mathcal{C}^+(I)$ forms a commutative Banach algebra with identity under the supremum norm, where the identity $e : I \rightarrow \mathbb{R}$ is defined by $e(x) = 1 \quad \forall x \in I$.

2.6 Proposition: Let $f : J \rightarrow \mathbb{R}$ and $p \in J$. If $f(p+)$ exists then f is cliquish at p .

Proof: Let $\varepsilon > 0$ be given and let U be a neighborhood of p in J . Then there exists a $\delta_1 > 0$ such that $(p - \delta_1, p + \delta_1) \cap J \subset U$.

Given $f(p+)$ exists. So there exists $\delta_2 > 0$ such that

$$|f(x) - f(p+)| < \frac{\varepsilon}{2} \quad \forall x \in (p, p + \delta_2) \subset J.$$

Put $\delta = \min\{\delta_1, \delta_2\}$ and $W = (p, p + \delta)$.

Then for $x, y \in W$, we have $|f(x) - f(y)| = |f(x) - f(p+) + f(p+) - f(y)|$
 $\leq |f(x) - f(p+)| + |f(y) - f(p+)|$
 $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Thus for every $\varepsilon > 0$ and every neighborhood U of p , there exists a non-empty open set $W \subset U$ such that $|f(x) - f(y)| < \varepsilon \quad \forall x, y \in W$

$\Rightarrow f$ is cliquish at p .

2.7 Remark: From the above proposition it is clear that every ${}^+ \text{Quasicontinuous}$ function is cliquish. The converse is not true as is evident from the following example.

2.8 Example: Define $f : [-1, 1] \rightarrow \mathbb{R}$ as follows.

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } -1 \leq x \leq 0 \end{cases}$$

Clearly f is cliquish at $x = 0$ but it is not ${}^+ \text{Quasicontinuous}$.

2.9 Theorem [2]: If $f : J \rightarrow \mathbb{R}$ is ${}^+ \text{Quasicontinuous}$ then the set of points of discontinuity of f is at most countable.

c) *Maximal Ideals in $\mathcal{C}^+(I)$*

3.1 Definition: For each $x \in I$, we define the following.

(a) $M_x = \{f \in \mathcal{C}^+(I) / f(x) = 0\}$ (b) $M_x^+ = \{f \in \mathcal{C}^+(I) / f(x+) = 0\}.$

3.2 Proposition: For each $x \in I$, the sets M_x and M_x^+ are maximal ideals in the commutative Banach algebra $\mathcal{C}^+(I)$.

Proof: For $x \in I$, define F_x and F_x^+ on $\mathcal{C}^+(I)$ by $F_x(f) = f(x)$ and

$$F_x^+(f) = f(x+) \quad \text{for } f \in \mathcal{C}^+(I).$$

Clearly F_x and F_x^+ are multiplicative linear functionals in the dual space $\mathcal{B}_+$ with kernels M_x and M_x^+ respectively. Hence M_x and M_x^+ are ideals. Moreover M_x and M_x^+ are maximal ideals in $\mathcal{C}^+(I)$.

3.3 Proposition: If M is a maximal ideal in $\mathcal{C}^+(I)$ then either $M = M_x$ or $M = M_x^+$ for some $x \in I$.

Proof: For $x \in I$, define F_x and F_x^+ on $\mathcal{C}^+(I)$ by $F_x(f) = f(x)$ and $F_x^+(f) = f(x+)$ for $f \in \mathcal{C}^+(I)$.

Clearly F_x and F_x^+ are multiplicative linear functionals in the dual space $\mathcal{B}_+$ with kernels M_x and M_x^+ respectively. Hence M_x and M_x^+ are ideals. Moreover M_x and M_x^+ are maximal ideals in $\mathcal{C}^+(I)$.

3.3 Proposition: If M is a maximal ideal in $\mathcal{C}^+(I)$ then either $M = M_x$ or $M = M_x^+$ for some $x \in I$.

Proof: Assume that $M \neq M_x$ and $M \neq M_x^+$ for any $x \in I$.

Then there exist f_x and g_x in M such that $f_x \notin M_x$ and $g_x \notin M_x^+$.

Define $\varphi_x : I \rightarrow \mathbb{R}$ by $\varphi_x(t) = f_x^2(t) + g_x^2(t+)$ $\forall t \in I$.

Clearly $\varphi_x \in \mathcal{C}^+(I)$. Since φ_x is ${}^+Q$ uasicontinuous at x and $\varphi_x(x) > 0$, there exists $\delta_x > 0$ such that $\varphi_x(t) > 0 \quad \forall t \in [0, \delta_x)$ and for $x \neq 1$

We have $\varphi_1(t) = f_1^2(t) + g_1^2(t+) \quad \forall t \in I$.

Since φ_1 is continuous at 1 and $\varphi_1(1) > 0$ there exists a $\delta > 0$ such that

$\varphi_1(t) > 0 \quad \forall t \in (1-\delta, 1]$. Then $[0, 1] = \left(\bigcup_{x \neq 1} [0, \delta_x) \right) \cup (1-\delta, 1]$. Since I is compact,

there exists $x_1 \neq 1$ in I such that $[0, 1] = [0, \delta_{x_1}) \cup (1-\delta, 1]$.

Put $\varphi = \varphi_{x_1}^2 + \varphi_1^2$. Then $\varphi \in M$ and $\varphi(t) > 0 \quad \forall t \in I \Rightarrow \frac{1}{\varphi} \in M$

Then $e = \varphi \cdot \frac{1}{\varphi} \in M$. This is a contradiction.

Hence it follows that $M = M_x$ or $M = M_x^+$ for some $x \in I$.

3.4 Remark: Let $\mathcal{M}_+$ be the space of all maximal ideals in $\mathcal{C}^+(I)$. Then $\mathcal{M}_+$ is a compact Hausdorff space with the weak* topology on $\mathcal{C}^+(I)$. Hence $\mathcal{M}_+^2 = \mathcal{M}_+ \times \mathcal{M}_+$ is a compact Hausdorff space with the product topology on $\mathcal{C}^+(I) \times \mathcal{C}^+(I)$.

3.5 Proposition: Let $\mathcal{A}^+ = \{(M_x, M_x^+) / x \in I\}$. Then there exists a one-to-one correspondence between I and $\mathcal{A}^+$.

Proof: Define $\Psi^+ : I \rightarrow \mathcal{A}^+$ by $\Psi^+(x) = (M_x, M_x^+)$.

Clearly Ψ^+ is surjective. If $0 \leq s < t \leq 1$, the function

$$\Psi_0^+(p) = \begin{cases} 0 & \text{if } t \leq p \leq 1 \\ \frac{1}{x-t} & \text{if } 0 \leq p < t \end{cases}$$

satisfies $\Psi_0^+ \in M_t$ and $\Psi_0^+ \notin M_s$.

$$\Rightarrow M_s \neq M_t$$

$$\Rightarrow (M_s, M_s^+) \neq (M_t, M_t^+)$$

$$\Rightarrow \Psi^+(s) \neq \Psi^+(t)$$

Hence Ψ^+ is 1-1.

Hence Ψ^+ is a one-to-one correspondence between I and $\mathcal{A}^+$.

3.6 Remark: Each maximal ideal in $\mathcal{C}^+(I)$ is the kernel of some multiplicative linear functional on $\mathcal{C}^+(I)$, hence can be identified with a multiplicative linear functional on $\mathcal{C}^+(I)$. Let M_x and M_x^+ be identified with the multiplicative linear functional F_x and F_x^+ respectively. So we can write

$$\mathcal{A}^+ = \{(F_x, F_x^+) / x \in I\}.$$

3.7 Proposition: $\mathcal{A}^+$ is closed in $\mathcal{B}_+^2 = \mathcal{B}_+ \times \mathcal{B}_+$ and hence compact.

Proof: We prove that $\mathcal{A}^+$ is closed. Compactness is an immediate consequence of the Banach – Alaoglu theorem [5]. If $F = (F_1, F_2) \in \mathcal{B}_+^2$, we define

$$\|F\| = \max\{\|F_1\|, \|F_2\|\}. \text{ Then } \mathcal{B}_+^2 \text{ is a Banach space under the above norm.}$$

Let $S = \{F / \|F\| \leq 1\} \subset \mathcal{B}_+^2$. Put $\mathcal{A} = \mathcal{A}^+ \cup \{O\}$.

The $\mathcal{A}^+ \subset \mathcal{M}_+^2 \subset \mathcal{A} \subset S \subset \mathcal{B}_+^2$.

Define $\mathcal{P}^+ : \mathcal{A} \rightarrow \mathbb{R}$ by

$$\mathcal{P}^+(F) = \begin{cases} 1 & \text{if } F \in \mathcal{A} \text{ and } F \neq 0 \\ 0 & \text{if } F = 0 \end{cases}$$

Since $\mathcal{P}^+$ is continuous, $\mathcal{A}^+$ and $\mathcal{A}$ are closed in $\mathcal{B}_+^2$.

d) *Further Properties*

4.1 Proposition: Fix $f \in \mathcal{C}^+(I)$. Define $\psi_f : I \rightarrow \mathbb{R}^2$ by

$\psi_f(x) = (f(x), f(x+))$, where $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is considered with the norm

$\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$. Then ψ_f is continuous on I if and only if f is

continuous on I .

Proof: Assume that ψ_f is continuous on I . Let $p \in I$ and let $\varepsilon > 0$ be given.

Since ψ_f is continuous at p , there exists a $\delta > 0$ such that

$$\|\psi_f(x) - \psi_f(p)\| < \varepsilon \quad \forall x \in (p - \delta, p + \delta) \cap I.$$

$$\Rightarrow \|(f(x), f(x+)) - (f(p), f(p+))\| < \varepsilon \quad \forall x \in (p - \delta, p + \delta) \cap I$$

$$\Rightarrow \|(f(x) - f(p), f(x+) - f(p+))\| < \varepsilon \quad \forall x \in (p - \delta, p + \delta) \cap I$$

$$\Rightarrow \max\{|f(x) - f(p)|, |f(x+) - f(p+)|\} < \varepsilon \quad \forall x \in (p - \delta, p + \delta) \cap I$$

$$\Rightarrow |f(x) - f(p)| < \varepsilon \quad \forall x \in (p - \delta, p + \delta)$$

$$\Rightarrow f \text{ is continuous at } p.$$

Thus if ψ_f is continuous at p then f is continuous at p .

Conversely suppose that f is continuous on I .

Then $\psi_f(x) = (f(x), f(x)) \quad \forall x \in I$.

Hence ψ_f continuous on I .

4.2 Proposition: Let $\mathbf{B} = \{\psi_f / f \in \mathcal{C}^+(I)\}$. Define $F : \mathcal{C}^+(I) \rightarrow \mathbf{B}$ by

$F(f) = \psi_f$. Then F is a one-to-one continuous multiplicative linear mapping

from $\mathcal{C}^+(I)$ onto $\mathbf{B}$.

Proof: Clearly $F: \mathcal{C}^+(I) \rightarrow \mathbf{B}$ is surjective.

$$\begin{aligned} \text{For } f, g \in \mathcal{C}^+(I), \psi_{f+g}(x) &= ((f+g)(x), (f+g)(x+)) \\ &= (f(x), f(x+)) + (g(x), g(x+)) \\ &= \psi_f(x) + \psi_g(x) \quad \forall x \in I \end{aligned}$$

$$\text{Hence } \psi_{f+g} = \psi_f + \psi_g \quad \forall f, g \in \mathcal{C}^+(I)$$

$$\Rightarrow F(f+g) = F(f) + F(g) \quad \forall f, g \in \mathcal{C}^+(I).$$

Let $c \in \mathbb{R}$.

$$\text{Then it is easy to see that } F(cf) = \psi_{cf} = c\psi_f = cF(f) \quad \forall f \in \mathcal{C}^+(I).$$

Hence F is linear.

$$\begin{aligned} \text{Also we have } \psi_{fg}(x) &= ((fg)(x), (fg)(x+)) \\ &= (f(x), f(x+)) (g(x), g(x+)) \\ &= \psi_f(x) \psi_g(x) \quad \forall x \in I. \end{aligned}$$

$$\text{Hence } F(fg) = \psi_{fg} = \psi_f \psi_g = F(f)F(g).$$

$\Rightarrow F$ is multiplicative now we prove that F is 1-1. For this, suppose that $F(f) = F(g)$

$$\Rightarrow \psi_f = \psi_g$$

$$\Rightarrow \psi_f(x) = \psi_g(x) \quad \forall x \in I$$

$$\Rightarrow (f(x), f(x+)) = (g(x), g(x+)) \quad \forall x \in I$$

$$\Rightarrow f(x) = g(x) \quad \forall x \in I$$

$$\Rightarrow f = g.$$

Hence F is 1-1.

Suppose that $f_n \in \mathcal{C}^+(I)$, $n=1, 2, 3, \dots$, and $f \in \mathcal{C}^+(I)$.

Let $f_n \rightarrow f$ uniformly on I . Then for a given $\varepsilon > 0$ there exists an integer

$$N > 0 \text{ such that } |f_n(x) - f(x)| < \frac{\varepsilon}{3} \text{ for all } n \geq N \text{ and all } x \in I.$$

Fix $x \in I$ and $n \geq N$. Since f_n is ⁺Quasicontinuous there exists a $\delta_1 > 0$ such

$$\text{that } |f_n(t) - f_n(x+)| < \frac{\varepsilon}{3} \quad \forall t \in (x, x + \delta_1).$$

Since f is also ${}^+Q$ uasicontinuous at x , there exists a $\delta_2 > 0$ such that

$$|f(t) - f(x+)| < \frac{\varepsilon}{3} \quad \forall t \in (x, x + \delta_2).$$

Put $\delta = \min\{\delta_1, \delta_2\}$. Then for $t \in (x, x + \delta)$ and $n \geq N$,

$$\begin{aligned} |f_n(x+) - f(x+)| &= |f_n(x+) - f_n(t) + f_n(t) - f(t) + f(t) - f(x+)| \\ &\leq |f_n(x+) - f_n(t)| + |f_n(t) - f(t)| + |f(t) - f(x+)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $|f_n(x+) - f(x+)| < \varepsilon$ for all $n \geq N$ and all $x \in I$.

$$\begin{aligned} n \geq N \Rightarrow \|F(f_n) - F(f)\| &= \|\psi_{f_n} - \psi_f\| \\ &= \sup\{\|\psi_{f_n}(x) - \psi_f(x)\| : x \in I\} < \varepsilon. \end{aligned}$$

$\Rightarrow F(f_n) \rightarrow F(f)$ uniformly on I .

Hence F is continuous on $\mathcal{C}^+(I)$.

4.3 Proposition: The set $\mathbf{B} = \{\psi_f : f \in \mathcal{C}^+(I)\}$ is a commutative Banach

algebra with identity ψ_e under the norm defined by $\|\psi_f\| = \sup\{\|\psi_f(x)\| : x \in I\}$,

where $\psi_e(x) = (1, 1) \quad \forall x \in I$.

REFERENCES RÉFÉRENCES REFERENCIAS

1. Borsik, J., *Points of continuity, Quasicontinuity and Cliquishness*, Pervento in redazoine, pp. 5 – 20, 1993.
2. Gal, I. S., *on the Continuity and Limiting Values of functions*, Proc. American Math. Soc, 86, pp. 321 – 334, 1957.
3. Goffman, C. And Pedrick, G., *First Course in Functional Analysis*, Prentice Hall of India Pvt., Ltd., New Delhi, 1974.
4. Ramabhadrasarma, I. and Srinivasa kumar, V., *on the Maximal Ideals in the Banach space of Quasicontinuous functions*, International Review of Pure and Applied Mathematics, Vol. 6, No. 1, pp. 41 – 45, 2010.
5. Rudin, W., *Functional Analysis*, Tata McGraw – Hill, New York, 1974.
6. Van Rooij, A. C. M. and Schikhof, W. H., *A second Course on Real functions*, Cambridge University Press, Cambridge, 1982.