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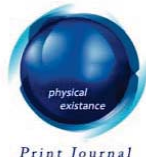
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Ref.

[3] A. A. Dabbour and N. A. Soutan, *Sheaves with many morphisms*, ICTP (Trieste), **291**(1992).

Multi-Presheaves and Multi-Sheaves Constructions

E. M. Solouma ^α & Abdullah Aljouiee ^σ

Abstract - In this paper, we modify the classical theory of sheaves, in such a way that it includes applications where more than one restriction map between sections are needed, we also show how the class of multi-presheaves on a topological space may be made into a category, with suitable notion of morphisms based on a generalization of natural transformation. Several examples are given.

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1. INTRODUCTION

In many occasions, we may be interested in algebraic structures defined over local neighborhoods. For example, a theory of cohomology of a topological space often concerns with sets of maps from a local neighborhood to some abelian groups, which possesses a natural $\mathbb{Z}$ -module structure. Another example is line bundles (either real or complex): since $\mathbb{R}$ or $\mathbb{C}$ are themselves rings, the set of sections over a local neighborhood forms an $\mathbb{R}$ or $\mathbb{C}$ -module. To analyze this local algebraic information, mathematicians came up with the notion of sheaves, which accommodate local and global data in a natural way. However, there are many fashions of introducing sheaves; Tennison [7] and Bredon [1] have done it in two very different styles in their separate books, though both of which bear the name "Sheaf Theory".

In this paper, we introduce some modifications on the set up in [3], putting extra structure on the set of restriction maps to developing a genuine "Multivalued" sheaf theory. One of the main results we will prove is the fact, just as in the classical case, there is a canonical way of associating to any multi-presheaf a "minimal" multi-sheaf. As this construction may be viewed as that of a right adjoint to inclusion, it follows that the category of multi-sheaves is a reflective subcategory of the category of multi-presheaves. This paper is organized as follows: In section 2, we introduce the notion of multi-presheaf over a local monoid defined in [8], and we give some examples. In section 3, we make use of stalks of

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multi-presheaves which are used as a main ingredient of construction and study of multi-sheaves. In successive Section 5, we introduce and study the notion of multi-sheaf, which is just a multi-presheaf satisfying some additional properties.

II. MULTI-PRESHEAVES

Throughout this section, we fix a topological space X with a topology τ and $\mathcal{N}_x$, the family of the open neighborhoods of the point $x \in X$ directed by inclusion, i.e., $U \leq V$ if $U \supseteq V$. In order to introduce the notion of multi-presheaf and multi-sheaf, we will need the notion of *local monoid* defined in [8]. This notion will also be needed later, when we define and study stalks of multi-presheaves and their use in the framework of sheafification.

Let us fix a set I such that, for any $a \leq b$ in a direct index set $\mathcal{A}$, there exists a non-empty set $I_{ab} \subseteq I$. We call $\mathcal{I} = \bigcup \{I_{ab}; I_{ab} \subseteq I\}$ a local monoid (index by $\mathcal{A}$), if the following properties are satisfied:

1. There exists some $e \in I$ such that $e \in \bigcap_{b \geq a} I_{ab}$.
2. It is endowed with some associated composition

$$I_{ab} \times I_{bc} \rightarrow I_{ac} : (i, j) \mapsto ij$$

with the property that for any $a \leq b \leq c$ in $\mathcal{A}$ and any $i \in I_{ab}$ resp. $j \in I_{bc}$, we have $ie = i$ resp. $ej = j$. In particular, this implies $I_{ab} \subseteq I_{ac}$ resp. $I_{bc} \subseteq I_{ac}$ and I_{aa} is a monoid with unit e for every $a \in \mathcal{A}$.

3. Whenever $a \leq b, c$ in $\mathcal{A}$ and $i \in I_{ab}$ resp. $j \in I_{bc}$, then there exists some $d \in \mathcal{A}$ with $d \geq b, c$ and some $k \in I_{bd}$ resp. $\ell \in I_{cd}$ with the property that $ik = j\ell$ in I_{ad} .

Note that once some $d \in \mathcal{A}$ and corresponding elements k and ℓ with the feature required in the last property has been found, This property clearly remains valid for all $d' \geq d$.

Definition 2.1. Consider a topological space X , with topology τ directed by inclusion. Consider a fixed set I , such that for any $U \supseteq V$ in τ , a non-empty set $I_{UV} \subseteq I$. The union $\mathfrak{I} = \bigcup \{I_{UV}; I_{UV} \subseteq I\}$ is called a global monoid if the following properties are satisfied:

G₁: There exists some elements $e \in \bigcap_{U \supseteq V} I_{UV}$.

G₂: It is endowed with some associated composition

$$I_{UV} \times I_{VW} \rightarrow I_{UW} : (i, j) \mapsto ij,$$

R_{ef.}

[8] A. Verschoren and N. A. Soutan, *Inductive limit of multi-spectra*, J. of Comm. in Algebra, **27**, No. 4, (1999), 1895-1902.

for any $U \supseteq V \supseteq W$ open subsets of X , and any $i \in I_{UV}$ then $ie = ei = i$.

G₃: For any $U \supseteq V, W$ in τ and $i \in I_{UV}$ resp. $j \in I_{VW}$, there exists $W' \in \tau$ with $W' \subseteq V, W$ and some $k \in I_{VW'}$ resp. $\ell \in I_{WW'}$ with the property that $ik = j\ell$ in $I_{UW'}$.

For each $x \in X$, the family $\mathfrak{I}_x = \bigcup \{I_{UV}, x \in V \subseteq U\}$ is called local monoid.

Definition 2.2. A ($\mathfrak{I}$ -indexed) multi-presheaf F of sets (abelian groups) on the space X consists the following data:

1. For each open subset U of X , there is given a set (abelian groups) $F(U)$.
2. For each pair of open subsets $U \supseteq V$ of X , there is given a non-empty set $F_{UV} = \{F_{UV}^i, i \in I_{UV}\}$, the so called restriction maps (homeomorphisms).

These data should satisfy the following conditions:

P1. For any open subset U of X , we have $id_{F(U)} = F_{UU}^e$, the identity map on F_{UU} .

P2. For any open subsets $W \subseteq V \subseteq U$ of X , and any F_{UV}^i resp. F_{VW}^j for some $i \in I_{UV}$ resp. $j \in I_{VW}$, the composition $F_{VW}^j F_{UV}^i$ belongs to F_{UW} .

P3. Whenever $x \in V \subseteq U$, any $s \in F(U)$ and any $i, j \in I_{UV}$, there exists $W \in \mathcal{N}_x$ with $W \subseteq V$ and indices $k, \ell \in I_{VW}$ such that $ik = j\ell$ and

$$F_{VW}^k F_{UV}^i(s) = F_{VW}^\ell F_{UV}^j(s).$$

P4. For any open neighborhoods $W \subseteq V \subseteq U$ of $x \in X$ and any $i \in I_{UV}$ resp. $j \in I_{VW}$, there exists some open set $W' \in \mathcal{N}_x$ with $W' \subseteq W$ such that

$$F_{VW'}^j F_{UV}^i = F_{UW'}^{ij}.$$

Note that once an open neighborhood $W' \in \mathcal{N}_x$ as in (P4) has been found, the same conclusion remains valid for any open neighborhood $W'' \subseteq W'$ of x . Indeed, it is sufficient to note that $F_{VW'}^j F_{UV}^i = F_{UW'}^{ij}$ which implies that

$$F_{VW''}^j F_{UV}^i = F_{W'W''}^e F_{VW'}^j F_{UV}^i = F_{W'W''}^e F_{UW'}^{ij} = F_{UW''}^{ij}.$$

Let us also point out that, in the terminology of [7], any multi-presheaf may be viewed as a contravariant multifunctor from the category of open subsets of X to the category of sets.

Example 1. Let X be the closed interval $[0, 1] \subseteq \mathbb{R}$, endowed with a relative topology τ on X with respect to usual topology on $\mathbb{R}$. Assume that τ is directed

by inclusion, i.e., $U \leq V$ if $U \supseteq V$. We will construct a multi-presheaf F on X , indexed by a constant global monoid $\mathfrak{I} = N$, i.e., $I_{UV} = N$ for all $U \supseteq V$ with the composition

$$N \times N \rightarrow N : (i, j) \mapsto \max(i, j).$$

For any strictly positive integer $m \in N$, let $f_m : X \rightarrow R$ be defined by

$$f_m(x) = \left(1 - \frac{1}{m}\right)x, \forall x \in X.$$

If U is an open subset of X , denote f_m^U the restriction $f_m|_U$ of f_m on U , i.e., $f_m^U : U \rightarrow R : x \mapsto \left(1 - \frac{1}{m}\right)x$. Let $F(U) = \{f_m^U, m \in N\}$ and whenever $V \subseteq U$ open subsets of X , we put $F_{UV} = \{F_{UV}^i, i \in N\}$ where $F_{UV}^i : F(U) \rightarrow F(V)$ is given by

$$F_{UV}^i(f_m^U) = \begin{cases} f_{m-1}^V & \text{if } m \text{ is even and } \frac{m}{2} \leq i, \\ f_m^V & \text{otherwise.} \end{cases}$$

It is clear that F_{UU}^0 is the identity on $F(U)$ for any open subset U of X and straightforward verification shows that for any $W \subseteq V \subseteq U$ open subsets of X , and $i, j \in N$, we have $F_{VW}^j F_{UV}^i = F_{UW}^{\ell^*}$ where $\ell^* = \max(i, j)$. From this one easily deduces that F is a multi-presheaf, Indeed.

Example 2. Let X be the unite circle C^1 endowed with a relative topology τ on X , with respect to usual topology on R^2 . Let τ be directed by inclusion, i.e., $U \leq V$ if $U \supseteq V$. Let H be the helix in R^3 parameterized by

$$R \rightarrow H : t \mapsto (t, \sin(t), \cos(t)),$$

both endowed with their standard topology, let

$$p : H \rightarrow X : (t, \sin(t), \cos(t)) \mapsto (\sin(t), \cos(t)),$$

be the canonical projection. We construct a multi-presheaf F^H of abelian groups on X , indexed by a constant global monoid $\mathfrak{I} = Z$, i.e., $I_{UV} = Z$ for all $U \supseteq V$, with composition

$$Z \times Z \rightarrow Z : (i, j) \mapsto i + j.$$

For any open subset U on X , denote $F^H(U)$ for $\left\{ \text{Continuous functions } s : U \rightarrow H, p \circ s = id_U \right\}$, and whenever $V \subseteq U$, consider $F_{UV} = \{F_{UV}^i, i \in Z\}$ where the

$R_{\text{ef.}}$

[7] B. R. Tennison, *Sheaf Theory*, Cambridge Univ. Press, London, New York, Melbourne, 1975.

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III. STALKS OF MULTI-PRESHEAVES

Just as ordinary presheaves give rise to stalks, which are used as a main ingredient of construction and study of sheaves. In this section we show how the multi-presheaves we introduce above allow for the construction of stalks. The actual construction of this stalks is of course the idea of inductive limit of multi-spectra,[8], and its properties and thus depends heavily upon these results.

Construction 1. Let F be a multi-presheaf on X and $x \in X$. Consider the set $F(U)$ of all multi-sections of F on U , where U is running over all open subsets of X such that $U \in \mathcal{N}_x$. Then, as mentioned before, the system $\{F(U), F_{UV}, \mathcal{N}_x\}$ is a multi-spectrum over a local monoid $\mathfrak{I}_x = \bigcup \{I_{UV}, x \in V \subseteq U\}$. Let $P = \coprod_{U \in \mathcal{N}_x} F(U)$ the disjoint union of sets $F(U)$. We define a relation " $\sim$ " on P as follows. Let $s \in F(U)$ and $t \in F(V)$, be two multi-section of F on U, V

respectively, then $s \sim t$ if there exists $W \in \mathcal{N}_x$, $W \subseteq U \cap V$ and $i \in I_{UW} \cap I_{VW}$ such that $F_{UW}^i(s) = F_{VW}^i(t)$. This is actually an equivalence relation.

Definition 3.1. The multi-stalk (abbreviated: M^* -stalk), F_x at $x \in X$ is the quotient of $P = \coprod_{U \in \mathcal{N}_x} F(U)$ under the equivalence relation " $\sim$ ". For any $s \in F(U)$, let s_x denote the equivalence class contain $s \in F(U)$ in F_x , which is called multi-germ (abbreviated: M^* -germ) of multi-section s at x , we thus obtain the map $F_{U,x}^e : F(U) \rightarrow F_x : s \mapsto s_x$. For any $U \in \mathcal{N}_x$, we put $I_U = \bigcup \{I_{UV}, U \leq V, V \in \mathcal{N}_x\}$, then any $i \in I_U$ where $i \in I_{UV}$ yields an obvious map $F_{U,x}^i : F(U) \rightarrow F_x$ defined

$$\begin{array}{ccc} F(U) & \xrightarrow{F_{U,x}^i} & F_x \\ F_{UV}^i \downarrow & \nearrow F_{V,x}^e & \\ F(V) & & \end{array}$$

or equivalently, $F_{U,x}^i(s) = [F_{UV}^i(s)]$ where $U \supseteq V \ni x$ is chosen such that $i \in I_{UV}$.

It easy to say that the M^* -stalk F_x of F at x is none other than the inductive limit of multi-spectrum $\{F(U), F_{UV}, \mathcal{N}_x\}$. We write

$$F_x = \varinjlim_{x \in U} F(U).$$

This comes equipped with maps $F(U) \rightarrow F_x : s \mapsto s_x$, whenever an open $U \ni x$.

IV. MORPHISMS OF MULTI-PRESHEAVES

In this section, we give the definition of morphisms between multi-presheaves and we discuss the composition of such morphisms of multi-presheaves and their stalks.

Definition 4.1. Let F, G be a multi-presheaves of sets (abelian groups) over X . A morphism of multi-presheaves $f : F \rightarrow G$ is the family $\{f(U)\}$ of maps (homomorphisms) $f(U) : F(U) \rightarrow G(U)$, one for each $U \in \tau$ such that whenever $U \supseteq V$ and $i \in I_{UV}$, the following diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{F_{UV}^i} & F(V) \\ f(U) \downarrow & & \downarrow f(V) \\ G(U) & \xrightarrow{G_{UV}^i} & G(V) \end{array}$$

is commutative. The composition of such morphisms of multi-presheaves is defined in the same way: If $f : F \rightarrow G$ and $g : G \rightarrow H$ are two morphisms of multi-presheaves, then the composition $gf : F \rightarrow H$ is defined by $(gf)(U) = g(U) \circ f(U)$, for each open subset U of X . The identity morphism $id_F : F \rightarrow F$ of multi-sheaf is defined by $id_F(U) = id_{F(U)}$ for any open subset U of X .

Example 3. Let $X = N$ endowed with the topology τ_0 , consisting of open sets of the form $U_\infty = \phi$ and $U_n = \{n, n+1, \dots\}$, where $n \in N$. Assume that τ_0 is directed by inclusion, i.e., $U_n \leq U_m$ if $U_n \supseteq U_m$, for any $n \leq m$. Let $\mathfrak{I} = N$ be the constant global monoid, i.e., $I_{U_n U_m} = N$ for all $U_n \supseteq U_m$ with the mapping $N \times N \rightarrow N : (i, j) \mapsto ij$. If U_n is an open subset of X , consider $G(U_\infty) = \{1\}$ and $G(U_n) = \{2^{nz}, z \in Z\}$, whenever $U_n \supseteq U_m$ open subsets of X . Let $G_{nm} = \{G_{nm}^i, i \in N\}$ where the homomorphisms $G_{nm}^i : G(U_n) \rightarrow G(U_m)$ are defined by $G_{nm}^i(2^{nz}) = 2^{(iz)m}$. In case of $m = \infty$, let $G_{nm}^i(2^{nz}) = 1$ (constant homomorphism) for all $i \in N$. It is clear that G_{nn}^1 is the identity on $G(U_n)$ for any open subset U_n of X . Moreover, if $n \leq m \leq p$, then

$$\begin{aligned} G_{mp}^j G_{nm}^i(2^{nz}) &= G_{mp}^j(2^{(iz)m}) \\ &= 2^{(ij)zp} = G_{np}^{ij}(2^{nz}) \end{aligned}$$

i.e., $G_{mp}^j G_{nm}^i = G_{np}^{ij}$. It easy to see that the conditions **(P3)** and **(P3)** of Definition 2.2 are verified. Thus the system $G = \{G(U_n), G_{nm}, \tau_0\}$ is a multi-presheaf of abelian groups.

Definition 4.2. Let F, G be multi-presheaves of sets (abelian groups) indexed by a local monoid $\mathfrak{I}_x = \bigcup \{I_{UV}, x \in V \subseteq U\}$. A morphism $f : F \rightarrow G$ of multi-presheaves of sets (abelian groups) on a space X , induces for each point $x \in X$, a map (homomorphism) of M^* -stalks $f_x : F_x \rightarrow G_x$ such that for any $s \in F(U)$, $i \in I_{UV}$, we have $f_x(s_x^i) = f(U)(s)_x^i$, where for simplicity $s_x^i, f(U)(s)_x^i$ means $F_{U,x}^i, (f(U)(s))_x^i$ respectively.

It is not difficult to show that f well is defined. For, let $s_x^i = t_x^i$, where $s \in F(U)$ and $t \in F(V)$, then there exists $W \in \mathcal{N}_x, W \subseteq U \cap V$ and $i \in I_{UW} \cap I_{VW}$

such that $F_{UW}^i(s) = F_{VW}^i(t)$, then $f(W)F_{UW}^i(s) = f(W)F_{VW}^i(t)$. Since f is a morphism of multi-presheaves then the following diagrams

$$\begin{array}{ccc} F(U) & \xrightarrow{F_{UW}^i} & F(W) \\ f(U) \downarrow & & \downarrow f(W) \\ G(U) & \xrightarrow{G_{UW}^i} & G(W) \end{array} \quad \begin{array}{ccc} F(V) & \xrightarrow{F_{VW}^i} & F(W) \\ f(V) \downarrow & & \downarrow f(W) \\ G(V) & \xrightarrow{G_{VW}^i} & G(W) \end{array}$$

commute. Therefore,

$$G_{UW}^i(f(U)(s)) = f(W)F_{UW}^i(s) = f(W)F_{VW}^i(t) = G_{VW}^i(f(V)(t)),$$

then $f(U)(s)_x^i = f(V)(t)_x^i$. Which proves that f_x is well defined.

Theorem 4.1. Let F, G, H be multi-presheaves of sets (abelian groups) on a space X , then for any $x \in X$

1. $f : F \rightarrow G$ and $g : G \rightarrow H$ are morphisms of multi-presheaves, then $(gf)_x = g_x f_x$.
2. $(id_F)_x = id_{F_x}$.

Proof. Consider M^* -germ $(s)_x^i$ of multi-section $s \in F(U)$ for some $U \in \mathcal{N}_x$ and some $i \in I_{UV}$, then

1.
$$\begin{aligned} (g_x f_x)(s)_x^i &= g_x(f_x(s)_x^i) = g_x(f(U)(s)_x^i) \\ &= g(U)f(U)(s)_x^i = (gf)(U)(s)_x^i = (gf)_x(s)_x^i \end{aligned}$$
2.
$$(id_F)_x(s)_x^i = id_{F(U)}(U)(s)_x^i = s_x^i = id_{F_x}(s)_x^i.$$

V. MULTI-SHEAVES

In this section we introduce and study the notion of multi-sheaf, which is just that a multi-presheaf satisfying some additional properties guaranteeing a nice, well-behaving local-global structure. The axioms for multi-sheaves, just as from the ordinary sheaf case, [7].

R_{ef.}

[7] B. R. Tennison, *Sheaf Theory*, Cambridge Univ. Press, London, New York, Melbourne, 1975.

Definition 5.1. Suppose $\mathfrak{I}_x = \bigcup \{I_{UV}, I_{UV} \subseteq I\}$ is a fixed global monoid indexed by the open subsets of the topological space X . An ($\mathfrak{I}$ -indexed) multi-presheaf F of sets on X is called multi-sheaf if it satisfies the following conditions:

- S1.** For any open subset $U \subseteq X$, any $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ open covering of U and any two multi-sections $s, t \in F(U)$. If for any $\alpha \in \mathcal{A}$ there exists some $i \in I_{UU_\alpha}$ such that $F_{UU_\alpha}^i(s) = F_{UU_\alpha}^i(t)$, then $s = t$.
- S2.** For any open subset $U \subseteq X$, any $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ open covering of U and any family $(s_\alpha)_{\alpha \in \mathcal{A}}$ of multi-sections $s_\alpha \in F(U_\alpha)$. If for any $(\alpha, \beta) \in \mathcal{A} \times \mathcal{A}$, there exists some $i \in I_{U_\alpha U_\alpha \cap U_\beta} \cap I_{U_\beta U_\alpha \cap U_\beta}$ such that

$$F_{U_\alpha U_\alpha \cap U_\beta}^i(s_\alpha) = F_{U_\beta U_\alpha \cap U_\beta}^i(s_\beta),$$

then there exists some $s \in F(U)$ and some $j \in I_{UU_\alpha}$ such that $F_{UU_\alpha}^j(s) = s_\alpha$, for all $\alpha \in \mathcal{A}$.

- S3.** For any open subset $U \subseteq X$, any $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ open covering of U and any two multi-sections $s, t \in F(U)$. If for any $\alpha \in \mathcal{A}$ there exists some $i, j \in I_{UU_\alpha}$ with the property that $F_{UU_\alpha}^i(s) = F_{UU_\alpha}^j(t)$, then there exists $k, \ell \in I_{UU}$ such that $F_{UU}^k(s) = F_{UU}^\ell(t)$, then we call F a strong multi-sheaf.

Note that the condition (**S3**) asserts that the result of the glueing procedure inferred in (**S2**) is essentially unique. Also we can define in the obvious way the notion of abelian multi-sheaf (multi-sheaf of abelian groups). In this case the condition (**S1**) can be simplified by putting $t = 0$.

Definition 5.2. (1) The global monoid $\mathfrak{I} = \bigcup \{I_{UV}, I_{UV} \subseteq I\}$ is called divisible if for any $V \subseteq U$ open subsets of X and any pair of indices $i, j \in I_{UV}$, there exists some $k \in I_{UU}$ with $i = kj$ or $j = ki$.

- (2) The global monoid $\mathfrak{I}$ is called commutative if for any pair of open subsets $V \subseteq U$ and any $i \in I_{UV}$ resp. $k \in I_{VV}$, there exists some $k' \in I_{UU}$ such that $ik = k'i \in I_{UV}$.

- (3) The global monoid $\mathfrak{I}$ is called torsion free if for any pair of open subsets $V \subseteq U$ and any $i \in I_{UV}$ resp. any $k', k'' \in I_{UU}$ such that $k''i = k'i$ then $k' = k''$.

Note that the constant global monoid $\mathfrak{J}$, where $\mathfrak{J}$ is additive monoid N_0 or any commutative group, then $\mathfrak{J}$ satisfies the pervious properties.

Theorem 5.1. Assume that the global monoid $\mathfrak{J}$ satisfies the following properties:

1. $\mathfrak{J}$ is divisible
2. $\mathfrak{J}$ is commutative and torsion free.

Then any ($\mathfrak{J}$ -indexed) multi-sheaf F on X is a strong multi-sheaf.

Proof. Assume that $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is open covering of $U \subseteq X$. Choose $x \in U$ and some $\alpha \in \mathcal{A}$ with $x \in U_\alpha$, and suppose that for some $i, j \in I_{UU_\alpha}$, and some $s, t \in F(U)$, we have $F_{UU_\alpha}^i(s) = F_{UU_\alpha}^j(t)$ for all $\alpha \in \mathcal{A}$. Let us first suppose that $\mathfrak{J}$ is divisible. Pick $k \in I_{UU}$ such that, for example, $i = kj$. Up to shrinking the open neighborhood U_α of x , we may assume that

$$F_{UU_\alpha}^j(t) = F_{UU_\alpha}^i(s) = F_{UU_\alpha}^j F_{UU}^k(s).$$

Since U_α covers U , it follows from (S1) that $t \in F_{UU}^k(s)$, which proves our claim. Next, let us assume that condition (2) holds. Choose some open neighborhood $U'_\alpha \subseteq U_\alpha$ of x with the property that there exists some $k, \ell \in I_{U_\alpha U'_\alpha}$ with $ik = j\ell \in I_{UU'_\alpha}$. shrink U'_α such that $F_{U_\alpha U'_\alpha}^k F_{UU_\alpha}^i = F_{UU'_\alpha}^{ik}$ resp. $F_{U_\alpha U'_\alpha}^\ell F_{UU_\alpha}^j = F_{UU'_\alpha}^{j\ell}$. Since $\mathfrak{J}$ is commutative, there exist $k', \ell' \in I_{UU}$ such that $ik = k'i$ resp. $j\ell = \ell'j$, within $I_{U_\alpha U'_\alpha}$, note that $i, j \in I_{U_\alpha U'_\alpha}$. Since $\mathfrak{J}$ is assumed to be torsion free, k' and ℓ' are actually independent of the particular choice of x and U_α . Again shrinking the open neighborhood U'_α of x , we may assume that $F_{UU'_\alpha}^i F_{UU}^{k'} = F_{UU'_\alpha}^{k'i}$ resp. $F_{UU'_\alpha}^j F_{UU}^{\ell'} = F_{UU'_\alpha}^{j\ell'}$. We now obtain:

$$\begin{aligned} F_{UU'_\alpha}^i F_{UU}^{k'}(t) &= F_{UU'_\alpha}^{k'i}(t) = F_{UU'_\alpha}^{ik}(t) = F_{UU'_\alpha}^{j\ell'}(t) \\ &= F_{U_\alpha U'_\alpha}^\ell F_{UU_\alpha}^j(t) = F_{U_\alpha U'_\alpha}^\ell F_{UU_\alpha}^i(s) \\ &= F_{UU'_\alpha}^{i\ell}(s) = F_{UU'_\alpha}^{\ell'i}(s) = F_{UU'_\alpha}^i F_{UU}^{\ell'}(s). \end{aligned}$$

Since U'_α is a covering of U , applying (S1) now clearly yields that $F_{UU}^{k'}(t) = F_{UU}^{\ell'}(s)$, which proves our claim.

Example 4. Consider a multi-presheaf F^H of abelian groups on the space C^1 , given in Example 2. Assume that $(U_\alpha)_{\alpha \in \mathcal{A}}$ is an open covering of an open set $U \in C^1$, let for all $\alpha \in \mathcal{A}$, $s, t \in F^H(U)$, we have $F_{UU_\alpha}^i(s) = F_{UU_\alpha}^i(t)$ for some $i \in I_{UU_\alpha}$, then $s|_{U_\alpha} + (0, 0, 2\pi i) = t|_{U_\alpha} + (0, 0, 2\pi i)$ thus $s|_{U_\alpha} = t|_{U_\alpha}$ which means that $s(x) = t(x)$ for all $x \in U_\alpha$. Since $(U_\alpha)_{\alpha \in \mathcal{A}}$ is an open covering of U , then $s(x) = t(x)$ for all $x \in U$, this implies that $s = t$. In order to verify the condition (S2), for any family of multi-sections $(s_\alpha)_{\alpha \in \mathcal{A}}$ with $s_\alpha \in F(U_\alpha)$ and some $i \in I_{U_\alpha U_\alpha \cap U_\beta} \cap I_{U_\beta U_\alpha \cap U_\beta}$, we have for any $(\alpha, \beta) \in \mathcal{A} \times \mathcal{A}$

$$F_{U_\alpha U_\alpha \cap U_\beta}^i(s_\alpha) = F_{U_\beta U_\alpha \cap U_\beta}^i(s_\beta),$$

then we have $s_\alpha|_{U_\alpha \cap U_\beta} + (0, 0, 2\pi i) = s_\beta|_{U_\alpha \cap U_\beta} + (0, 0, 2\pi i)$ which yields to $s_\alpha(x) = s_\beta(x)$ for all $x \in U_\alpha \cap U_\beta$, since $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ we define $s \in F(U)$ by $s(x) = s_\alpha(x)$ for all $x \in U$, then we have $s|_{U_\alpha}(x) = s_\alpha(x)$ which yields to $s|_{U_\alpha} + (0, 0, 2\pi i) = s_\alpha$, then $F_{UU_\alpha}^j(s) = s_\alpha$, for some $j \in I_{UU_\alpha}$.

Theorem 5.2. If X is a topological space and F is a multi-sheaf on X . Then for any pair of multi-section $s, t \in F(U)$, we have $s = t$ if and only if for all $x \in U$, there is $U_x \in \mathcal{N}_x$ and $i \in I_{UU_x}$ such that $s_x^i = t_x^i$.

Proof. Let $s = t$ this implies that for any $W_x \in \mathcal{N}_x$ with $W_x \subseteq U$ and for some $i \in I_{UW_x}$, we have $F_{UW_x}^i(s) = F_{UW_x}^i(t)$, which means that $s_x^i = t_x^i$ for all $x \in U$. Conversely, if $s, t \in F(U)$ such that for all $x \in U$, there is $U_x \in \mathcal{N}_x$ and $i \in I_{UU_x}$ such that $s_x^i = t_x^i$. It follows that we may find some open neighborhood $V_x \subseteq U_x$ of x such that $i \in I_{UV_x}$ with $F_{UV_x}^i(s) \sim F_{UV_x}^i(t)$, and up to making V_x even smaller and changing i if necessary, we may assume that we actually have $F_{UV_x}^i(s) = F_{UV_x}^i(t)$. Since $(V_x)_{x \in U}$ is obviously an open cover for U , if we make x vary through U , it follows from the multi-sheaf axioms that $s = t$.

Theorem 5.3. A morphism $f : F \rightarrow G$ of multi-sheaves on a topological space X satisfies the condition $f_x : F_x \rightarrow G_x$ is injective for all $x \in X$ if and only if the map $f(U) : F(U) \rightarrow G(U)$ is injective for all open subset U of X .

Proof. Assume that $f(U) : F(U) \rightarrow G(U)$ is injective, and suppose that $t \in F_x$ is such that $f_x(t) = 0$, then there exists an open set U , and $s \in F(U)$ such that s

has germ t at x , for some $i \in I_{UV}$, so that $f(U)(s)$ has germ 0 at x . Then there exists $V \subseteq U$, $V \in \mathcal{N}_x$ and some $i \in I_{UV}$ such that $G_{UV}^i(f(U)(s)) = 0$, but f is a morphism of multi-sheaf, therefore

$$G_{UV}^i(f(U)(s)) = 0 = f(V)(F_{UV}^i(s)).$$

But $f(V)$ is injective by hypothesis, so $F_{UV}^i(s) = 0$ and thus $t = 0$.

Conversely, assume that $f_x : F_x \rightarrow G_x$ is injective and suppose $s \in F(U)$ is such that $f(U)(s) = 0 \in G(U)$, therefore $G_{U,x}^i(f(U)(s)) = 0$, for some $i \in I_{UV}$, then $f_x(F_{U,x}^i(s)) = 0$, since f_x is injective by hypothesis. Then there exists $U_x \in \mathcal{N}_x$ with $U_x \subseteq U$ and some $i \in I_{UU_x}$ such that $F_{UU_x}^i(s) = 0$. Since $(U_x)_{x \in U}$ is an open covering for U , by a multi-sheaf condition (S2), $s = 0$ on U . Thus $f(U)$ is injective.

Definition 5.3. Let X be a topological space. A sheaf space over X is a pair $\mathbf{E} = (E, p)$, where E is a topological space and the projection p is a continuous onto map $E \rightarrow X$ which has an additional structure, that is, p is a local homeomorphism, i.e., any $e \in E$ has a neighborhood $\mathcal{N}$ such that $p|_{\mathcal{N}}$ maps $\mathcal{N}$ homeomorphically onto a neighborhood of $p(e)$.

Definition 5.4. Let $\mathbf{E} = (E, p)$ and $\mathbf{E}^1 = (E^1, p^1)$ are two sheaf spaces, then a morphism $\mathbf{f} : \mathbf{E} \rightarrow \mathbf{E}^1$ is by definition a map $f : E \rightarrow E^1$, which is continuous and makes the following diagram commutative:

$$\begin{array}{ccc} E & \xrightarrow{p} & X \\ f \downarrow & \nearrow p^1 & \\ E^1 & & \end{array}$$

Construction 2. Consider a sheaf space $\mathbf{E} = (E, p)$. We can construct a multi-sheaf of sets $\Gamma E = \tilde{E}$ (the multi-sheaf of sections of E) in such a way that a morphism $\mathbf{f} : \mathbf{E} \rightarrow \mathbf{E}^1$ of sheaf spaces gives rise to a morphism $\tilde{f} : \tilde{E} \rightarrow \tilde{E}^1$ of multi-sheaves. Now, let $\mathcal{I} = \bigcup \{I_{UV}, U \supseteq V\}$ be a local monoid indexed by open subsets of X . We say that $\mathcal{I}$ acts on the sheaf space $\mathbf{E}$, if there is given for any pair of open subsets $V \subseteq U$ in X a map

$$E|_U \times I_{UV} \rightarrow E|_V : (m, i) \mapsto m^i,$$

where $E|_U = p^{-1}(U)$, with the property that for any open subsets $W \subseteq V \subseteq U$ of X , the induced diagram

$$\begin{array}{ccc} E|_U \times I_{UV} \times I_{VW} & \longrightarrow & E|_V \times I_{VW} \\ \downarrow & & \downarrow \\ E|_U \times I_{UW} & \longrightarrow & E|_W \end{array}$$

Commutates, where the left vertical map is induced by the composition $I_{UV} \times I_{VW} \rightarrow I_{UW}$. This means that for any $m \in E|_U$ and any $i \in I_{UV}$ resp. $j \in I_{VW}$, we have $m^{ij} = (m^i)^j$. If $\mathcal{I}$ acts on a sheaf space $\mathbf{E}$, then it easy to verify that we may associate to it a multi-sheaf $\tilde{E}$ on X , by letting $\tilde{E}(U) = \Gamma(U, E) = \{\text{continuous maps } s : U \rightarrow E, p \circ s = id_U\}$, for any open subset $U \subseteq X$ and putting $\tilde{E}_{UV} = \{\tilde{E}_{UV}^i, i \in I_{UV}\}$, for any $U \supseteq V$ and $i \in I_{UV}$, where $\tilde{E}_{UV}^i : \tilde{E}(U) \rightarrow \tilde{E}(V)$ is defined by $s \rightarrow i(s|_V)$, where $i(s)(y) = s(y)^i$. One can easily show that $\{\tilde{E}(U), \tilde{E}_{UV}, \tau\}$ is a multi-sheaf of sets.

Construction 3. For each multi-presheaf F of sets on X over a local monoid $\mathcal{I} = \bigcup\{I_{UV}, U \supseteq V\}$, we can construct a sheaf space LF in such a way that a morphism $f : F \rightarrow G$ of multi-presheaves gives rise to a morphism $Lf : LF \rightarrow LG$ of sheaf spaces. Consider $LF = \coprod_{x \in X} F_x$ (the disjoint union of the stalks F_x) with $p : LF \rightarrow X$ the natural projection. We topologize LF as follows: Let U be an open set in X , and $s \in F(U)$. Define a map

$$\tilde{s}^e : U \rightarrow LF : x \mapsto s_x; s_x \in F_{U,x}^e(s).$$

Moreover, for any $i \in I_{UV}$ there is a map

$$\tilde{s}^i : V \rightarrow LF : x \mapsto F_{V,x}^i(s) = F_{V,x}^e(F_{UV}^i(s)).$$

From this, it also follows that we have a commutative diagram

$$\begin{array}{ccc} s \in F(U) & \longrightarrow & \Gamma(U, LF) \ni F_{U,x}^e(s) \\ F_{UV}^i \downarrow & & \downarrow \Gamma(F_{UV}^i) \\ F_{UV}^i(s) \in F(V) & \longrightarrow & \Gamma(V, LF) \ni F_{U,x}^i(s) \end{array}$$

Therefor, we obtain a multi-sheaf ΓLF of sets called the associated multi-sheaf (for simplicity we call $\Gamma LF = \underline{a}F$).

Theorem 5.4. F is a multi-sheaf if and only if $F \rightarrow \underline{a}F$ is an isomorphism of multi-sheaves.

Proof. First, we shall prove that if $F \rightarrow \underline{a}F$ is an isomorphism of multi-presheaves and $\underline{a}F$ is a multi-sheaf, then F is a multi-sheaf. Let U be an open set of X and $\{U_x\}_{x \in X}$ be an open cover of U , i.e., $U = \bigcup_{x \in X} U_x$, let s and t be two sections in $F(U)$ and suppose that $F_{UU_x}^i(s) = F_{UU_x}^i(t)$. Since $F \rightarrow \underline{a}F$ is a morphism of multi-presheaves, then the following diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{n_F(U)} & \underline{a}F(U) \\ F_{UU_x}^i \downarrow & & \downarrow \underline{a}F_{UU_x}^i \\ F(U_x) & \xrightarrow{G_{UV}^i} & \underline{a}F(U_x) \end{array}$$

is commutative. Then $\underline{a}F_{UU_x}^i(n_F(U)(s)) = \underline{a}F_{UU_x}^i(n_F(U)(t))$ for some $i \in I_{UU_x}$, but $\underline{a}F$ is a multi-sheaf, therefore, from the condition (S1) of Definition 5.1, we have

$$n_F(U)(s) = n_F(U)(t),$$

since n_F is an isomorphism, that is $n_F(U)$ is bijective, it follows that $s = t$. For the glueing condition (S2), let U be an open set of X and let $U = \bigcup_{x \in X} U_x$ be an open covering of U . Consider $(s^x)_{x \in X}$ the family of sections of F with $\forall x \in X$, $s^x \in F(U_x)$ such that

$$F_{U_x U_x \cap U_y}^i(s^x) = F_{U_y U_x \cap U_y}^i(s^y)$$

for some $i \in I_{U_x U_x \cap U_y} \cap I_{U_y U_x \cap U_y}$. Therefore

$$n_F(U_x \cap U_y) F_{U_x U_x \cap U_y}^i(s^x) = n_F(U_x \cap U_y) F_{U_y U_x \cap U_y}^i(s^y),$$

since n_F is morphism of multi-presheaves, therefore

$$\underline{a}F_{UU_x}^i(n_F(U)(s^x)) = \underline{a}F_{UU_x}^i(n_F(U)(s^y))$$

but $\underline{a}F$ is a multi-sheaf, thus there exists $\tilde{s} \in \underline{a}F(U)$ such that $\forall x \in U$, and for some $i \in I_{UU_x}$

$$\underline{a}F_{UU_x}^i(\tilde{s}) = n_F(U_x)(s^x)$$

$$F_{U,x}^i(s) = F_{U,x}^e(s^x)$$

$$F_{U_x, x}^e(F_{UU_x}^i(s)) = F_{U_x, x}^e(s^x)$$

In other words, for all $x \in U$ there exists $s \in F(U)$ such that $F_{U_x, x}^i(s) = s^x$ which prove that F satisfies the glueing condition (S2) of Definition 5.1. Now we want to prove that if F a multi-sheaf then $F(U) \rightarrow \underline{a}F(U)$ is bijective.

(i) the map is injective for

$$\tilde{s} = \tilde{t} \Leftrightarrow \forall x \in U \ F_{U_x, x}^i(s) = F_{U_x, x}^i(t) \text{ for some } i \in I_{UU_x}$$

$$\Leftrightarrow \forall x \in U \ F_{U_x, x}^e(F_{UU_x}^i(s)) = F_{U_x, x}^e(F_{UU_x}^i(t))$$

$$\Leftrightarrow U \text{ has open cover } U_x \text{ such that}$$

$$F_{UU_x}^i(s) = F_{UU_x}^i(t), \text{ for some restriction maps } F_{UU_x}^i$$

$$\Leftrightarrow s = t \text{ (since } F \text{ is multi-sheaf).}$$

(ii) Let $t \in \underline{a}F(U)$ and $x \in U_x \subseteq U$, if $\tilde{s}^x \in F(U_x)$ neighborhood $\tilde{s}^x(U_x) \subseteq t(U)$ for some open $U_x \subseteq U$ this mean that $s^x \in F(U_x)$ satisfies the glueing condition of Definition 5.1, for, let $s^y \in F(U_y)$, take $V = U_x \cap U_y$. Let $i \in I_{U_x V} \cap I_{U_y V}$ and $\tilde{s}^x : V \rightarrow LF$. Therefore

$$F_{V, z}^i(s^z) = F_{U_x, x}^i(s^x)F_{U_y, y}^i(s^y).$$

Thus $F_{U_x, x}^i(s^x)F_{U_y, y}^i(s^y)$, then $F_{U_x, V}^i(s^x)F_{U_y, V}^i(s^y)$ for some $i \in I_{U_x V} \cap I_{U_y V}$, but F is multi-sheaf then there exists $s \in F(U)$ such that for all $x \in U$, $F_{UU_x}^i(s) = s^x$. Therefore, there is $s \in F(U)$ such that $t \in F_{U_x, x}^e(s^x)$ which proves that the map is injective.

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