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Certain New Subclasses of Analytic Univalent Functions in the Unit Disk

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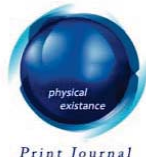
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Certain New Subclasses of Analytic Univalent Functions in the Unit Disk

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Abstract - Introduction of new subclasses $K(\omega, \alpha, \lambda)$ and $H(\omega, \alpha, \lambda)$ of analytic functions with negative co-efficients defined in the unit disk. We estimate some properties of functions in these classes and obtain Co-efficient bounds, Neighbourhood and Integral means inequality for the function $f(z)$ under these classes.

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1. INTRODUCTION AND DEFINITIONS

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Let S denote the subclass of A consisting of univalent functions $f(z)$ in E .

Let $A(\omega) \subset A$ denote the class of functions of the form

$$f(z) = (z - \omega) + \sum_{k=2}^{\infty} a_k (z - \omega)^k \quad (1.2)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$ and normalised with $f(\omega) = 0$ and $f'(\omega) - 1 = 0$ and ω is a fixed point in E .

Let $B(\omega, \alpha, \lambda)$ and $Q(\omega, \alpha, \lambda)$ denote the class of analytic function of order α in E defined as follows:

$$S(\omega) = \{f \in A(\omega) : f \text{ is univalent in } E\}$$

$$B(\omega, \alpha, \lambda) = \left\{ f \in S(\omega) : \operatorname{Re} \left(\frac{\lambda(z - \omega)^2 f''(z) + (z - \omega) f'(z)}{(1 - \lambda) f(z) + \lambda(z - \omega) f'(z)} \right) > \alpha, 0 \leq \lambda \leq 1, 0 \leq \alpha < 1, \right. \\ \left. z \in E \right\}$$

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$$Q(\omega, \alpha, \lambda) = \left\{ f \in S(\omega) : \operatorname{Re} \left(\frac{\lambda(z - \omega)^3 f'''(z) + (1 + 2\lambda)(z - \omega)^2 f''(z) + (z - \omega)f'(z)}{\lambda(z - \omega)^2 f''(z) + (z - \omega)f'(z)} \right) > \alpha, \right. \\ \left. 0 \leq \lambda \leq 1, 0 \leq \alpha < 1, z \in E \right\}$$

Let $T(\omega)$ denote subclass of $S(\omega)$ whose elements can be expressed in the form

$$f(z) = (z - \omega) - \sum_{k=2}^{\infty} a_k (z - \omega)^k. \quad (1.3)$$

Here we denote by $H(\omega, \alpha, \lambda)$ and $K(\omega, \alpha, \lambda)$ respectively the subfamilies of $B(\omega, \alpha, \lambda)$ and $Q(\omega, \alpha, \lambda)$ obtained by taking intersection of $B(\omega, \alpha, \lambda)$ and $Q(\omega, \alpha, \lambda)$ with $T(\omega)$ that is,

$$H(\omega, \alpha, \lambda) = B(\omega, \alpha, \lambda) \cap T(\omega),$$

and

$$K(\omega, \alpha, \lambda) = Q(\omega, \alpha, \lambda) \cap T(\omega).$$

Also let $P(\omega) \subset P(\text{class of Cartheodory functions})$ denote the class of functions of the form

$$P_\omega(z) = 1 + \sum_{k=1}^{\infty} B_k (z - \omega)^k \quad (1.4)$$

that are regular in E and satisfy $P_\omega(\omega) = 1$, $\operatorname{Re} P_\omega(z) > 0$ for $z \in E$ and ω is a fixed point in E and

$$|B_k| \leq \frac{2}{(1+d)(1-d)^k}, \quad k \geq 1, \quad d = |\omega|.$$

II. CO-EFFICIENT ESTIMATES

For our main results we first derive the following:

Lemma 2.1. A function $f(z) \in T(\omega)$ is in the class $H(\omega, \alpha, \lambda)$ if and only if

$$\sum_{k=2}^{\infty} (k - \alpha)(1 - d)^{k-1}(1 - \lambda + \lambda k)a_k \leq 1 - \alpha. \quad (2.1)$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z - \omega| = 1 - d < 1$ then we have

$$\left| \frac{\lambda(z - \omega)^2 f''(z) + (z - \omega)f'(z)}{(1 - \lambda)f(z) + \lambda(z - \omega)f'(z)} - 1 \right| = \left| \frac{-\sum_{k=2}^{\infty} (k - 1)a_k (z - \omega)^{k-1}(1 - \lambda + \lambda k)}{1 - \sum_{k=2}^{\infty} a_k (z - \omega)^{k-1}(1 - \lambda + \lambda k)} \right| \\ \leq \frac{\sum_{k=2}^{\infty} (k - 1)a_k (1 - d)^{k-1}(1 - \lambda + \lambda k)}{1 - \sum_{k=2}^{\infty} a_k (1 - d)^{k-1}(1 - \lambda + \lambda k)} \\ \leq 1 - \alpha.$$

This shows that the values of $\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)}$ lie in the circle centered at $\gamma = 1$ whose radius is $1 - \alpha$. Hence $f(z)$ is in the class $H(\omega, \alpha, \lambda)$. Then

$$\operatorname{Re} \left(\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)} \right) = \operatorname{Re} \left(\frac{1 - \sum_{k=2}^{\infty} k(1-\lambda + \lambda k)a_k(z-\omega)^{k-1}}{1 - \sum_{k=2}^{\infty} (1-\lambda + \lambda k)a_k(z-\omega)^{k-1}} \right) > \alpha. \quad (2.2)$$

for $z \in E$ and ω is a fixed point in E . Choose values of z on the real axis so that $\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)}$ is real. Upon clearing the denominator and letting $z \rightarrow 1^-$ through real values, we have

$$\alpha \left(1 - \sum_{k=2}^{\infty} a_k(1-d)^{k-1}(1-\lambda + \lambda k) \right) \leq 1 - \sum_{k=2}^{\infty} k(1-\lambda + \lambda k)a_k(1-d)^{k-1} \quad (2.3)$$

which is the required result.

Finally, we note that the assertion (2.1) of lemma 2.1 is sharp, with the external function

$$f(z) = (z-\omega) - \frac{1-\alpha}{(k-\alpha)(1-\lambda + \lambda k)(1-d)^{k-1}}(z-\omega)^k. \quad (2.4)$$

Corollary 2.2. Let $f(z) \in T(\omega)$ be in the class $H(\omega, \alpha, \lambda)$. Then we have

$$a_k \leq \frac{1-\alpha}{(k-\alpha)(1-\lambda + \lambda k)(1-d)^{k-1}}. \quad (2.5)$$

Equality in (2.5) holds true for the function $f(z)$ given by (2.4).

Lemma 2.3. A function $f(z) \in T(\omega)$ is in the class $K(\omega, \alpha, \lambda)$ if and only if

$$\sum_{k=2}^{\infty} k(k-\alpha)(1-d)^{k-1}(1-\lambda + \lambda k)a_k \leq 1-\alpha. \quad (2.6)$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z-\omega| = 1-d < 1$ then we have

$$\left| \frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)} - 1 \right|$$

$$= \left| \frac{-\sum_{k=2}^{\infty} k(k-1)a_k(z-\omega)^{k-1}(1-\lambda+\lambda k)}{1-\sum_{k=2}^{\infty} ka_k(z-\omega)^{k-1}(1-\lambda+\lambda k)} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty} k(k-1)a_k(1-d)^{k-1}(1-\lambda+\lambda k)}{1-\sum_{k=2}^{\infty} ka_k(1-d)^{k-1}(1-\lambda+\lambda k)} \leq 1-\alpha.$$

This shows that the values of $\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}$ lie in the circle centered at $\gamma = 1$ whose radius is $1 - \alpha$. Hence $f(z)$ is in the class $K(\omega, \alpha, \lambda)$. Then

$$Re \left(\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)} \right)$$

$$= Re \left(\frac{1 - \sum_{k=2}^{\infty} k^2(1-\lambda+\lambda k)a_k(z-\omega)^{k-1}}{1 - \sum_{k=2}^{\infty} k(1-\lambda+\lambda k)a_k(z-\omega)^{k-1}} \right) > \alpha \quad (2.7)$$

for $z \in E$ and ω is a fixed point in E . Choose values of z on the real axis so that $\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}$ is real. Upon clearing the denominator and letting $z \rightarrow 1^-$ through real values, we have

$$\alpha \left(1 - \sum_{k=2}^{\infty} ka_k(1-d)^{k-1}(1-\lambda+\lambda k) \right) \leq 1 - \sum_{k=2}^{\infty} k^2(1-\lambda+\lambda k)a_k(1-d)^{k-1} \quad (2.8)$$

which is the required result.

Finally, we note that the assertion (2.6) of Lemma 2.3 is sharp, with the external function

$$f(z) = (z-\omega) - \frac{1-\alpha}{k(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}(z-\omega)^k. \quad (2.9)$$

Corollary 2.4. Let $f(z) \in T(\omega)$ be in the class $K(\omega, \alpha, \lambda)$. Then we have

$$a_k \leq \frac{1-\alpha}{k(k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1}}. \quad (2.10)$$

Equality in (2.10) holds true for the function $f(z)$ given by (2.9).

Theorem 2.5. Let $f(z) \in H(\omega, \alpha, \lambda)$ and $f(z) = (z-\omega) - a_2(z-\omega)^2 - \dots$ for $0 \leq \alpha < 1$, and ω is a fixed point in E . Then

$$|a_2| \leq \frac{-2(1-\alpha)}{(1+\lambda)(1-d^2)}, \quad (2.11)$$

$$|a_3| \leq \frac{-1}{1+2\lambda} \left[\frac{1-\alpha}{(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{(1-d^2)^2} \right],$$

$$|a_4| \leq \frac{-1}{1+3\lambda} \left[\frac{2(1-\alpha)}{3(1+d)(1-d)^3} + \frac{2(1-\alpha)^2}{(1-d)(1-d^2)^2} + \frac{4(1-\alpha)^3}{3(1-d^2)^3} \right].$$

Proof. Let us define

$$\frac{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{(1-\lambda)f(z) + \lambda(z-\omega)f'(z)} = \alpha + (1-\alpha)P_\omega(z). \quad (2.12)$$

That is,

$$\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z) = [(1-\lambda)f(z) + \lambda(z-\omega)f'(z)][\alpha + (1-\alpha) \sum_{k=1}^{\infty} B_k(z-\omega)^k]. \quad (2.13)$$

On comparing the coefficient in (2.13) the result follows.

We define the δ -neighbourhood of the function $f(z) \in T(\omega)$ by

$$N_\delta = \{g \in T(\omega) : g(z) = (z-\omega) - \sum_{k=2}^{\infty} b_k(z-\omega)^k, \sum_{k=2}^{\infty} k(1-d)^{k-1}|b_k| \leq \delta\}, \quad (2.14)$$

and in particular, for the identity function

$$e(z) = \left(1 - \frac{\omega}{z}\right)z, \quad (2.15)$$

we immediately have

$$N_\delta(e) = \{g \in T(\omega) : g(z) = (z-\omega) - \sum_{k=2}^{\infty} b_k(z-\omega)^k, \sum_{k=2}^{\infty} k(1-d)^{k-1}|b_k| \leq \delta\}. \quad (2.16)$$

Theorem 2.6. Let $f(z) \in K(\omega, \alpha, \lambda)$ and $f(z) = (z-\omega) - a_2(z-\omega)^2 - \dots$ for $0 \leq \alpha < 1$, and ω is a fixed point in E . Then

$$|a_2| \leq \frac{-(1-\alpha)}{(1+\lambda)(1-d^2)}, \quad (2.17)$$

$$|a_3| \leq \frac{-1}{1+2\lambda} \left[\frac{1-\alpha}{3(1-d)(1-d^2)} + \frac{2(1-\alpha)^2}{3(1-d^2)^2} \right],$$

$$|a_4| \leq \frac{-1}{1+3\lambda} \left[\frac{1-\alpha}{6(1+d)(1-d)^3} + \frac{(1-\alpha)^2}{2(1-d)(1-d^2)^2} + \frac{(1-\alpha)^3}{3(1-d^2)^3} \right].$$

Proof. Let us define

$$\frac{\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z)}{\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)} = \alpha + (1-\alpha)P_\omega(z). \quad (2.18)$$

That is,

$$\lambda(z-\omega)^3 f'''(z) + (1+2\lambda)(z-\omega)^2 f''(z) + (z-\omega)f'(z) = [\lambda(z-\omega)^2 f''(z) + (z-\omega)f'(z)] \left[\alpha + (1-\alpha) \sum_{k=1}^{\infty} B_k(z-\omega)^k \right] \quad (2.19)$$

comparing the coefficients we get the required result.

From (2.14) and (2.15) we immediately have

$$N_\delta(e) = \{g \in T(\omega) : g(z) = (z-\omega) - \sum_{k=2}^{\infty} b_k(z-\omega)^k, \sum_{k=2}^{\infty} k^2(1-d)^{k-1}|b_k| \leq \delta\}. \quad (2.20)$$

Theorem 2.7. $H(\omega, \alpha, \lambda) \subset N_\delta(e)$ where $\delta = \frac{2(1-\alpha)}{(2-\alpha)(1-d)(1+\lambda)}$.

Proof. Let $f(z) \in H(\omega, \alpha, \lambda)$. Then in view of lemma 2.1, since $(k-\alpha)(1+\lambda)(1-d)^{k-1}$ is an increasing function of k ($k \geq 2$), we have

$$(1-d)(1+\lambda)(2-\alpha) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} (k-\alpha)(1-\lambda+\lambda k)(1-d)^{k-1} a_k \leq 1-\alpha, \quad (2.21)$$

which immediately yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{(2-\alpha)(1-d)(1+\lambda)}. \quad (2.22)$$

On the other hand, we also find from (2.3) that

$$\begin{aligned} (1-d) \sum_{k=2}^{\infty} k a_k (1-\lambda+\lambda k) - \alpha(1-d) \sum_{k=2}^{\infty} a_k (1-\lambda+\lambda k) &\leq \sum_{k=2}^{\infty} (k-\alpha) a_k (1-d)^{k-1} (1-\lambda+\lambda k) \\ &\leq 1-\alpha. \end{aligned} \quad (2.23)$$

From (2.22) and (2.23), we have

$$(1-d)(1+\lambda) \sum_{k=2}^{\infty} k a_k \leq (1-\alpha) + \alpha(1-d)(1+\lambda) \sum_{k=2}^{\infty} a_k \quad (2.24)$$

$$\begin{aligned}
&\leq (1 - \alpha) + \frac{\alpha(1 - \alpha)}{(2 - \alpha)} \\
&\leq \frac{2(1 - \alpha)}{2 - \alpha}. \\
\sum_{k=2}^{\infty} ka_k &\leq \frac{2(1 - \alpha)}{(2 - \alpha)(1 - d)(1 + \lambda)}. \tag{2.25}
\end{aligned}$$

which proved the theorem.

Theorem 2.8. $K(\omega, \alpha, \lambda) \subset N_{\delta}(e)$ where $\delta = \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)}$.

Proof. Let $f(z) \in K(\omega, \alpha, \lambda)$. Then in view of lemma 2.3, since

$k(k - \alpha)(1 - \lambda + \lambda k)(1 - d)^{k-1}$ is an increasing function of $k(k \geq 2)$, we have

$$2(1 - d)(1 + \lambda)(2 - \alpha) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} k(k - \alpha)(1 - \lambda + \lambda k)(1 - d)^{k-1} a_k \leq 1 - \alpha, \tag{2.26}$$

which immediately yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{1 - \alpha}{2(2 - \alpha)(1 - d)(1 + \lambda)}. \tag{2.27}$$

On the other hand, we also find from (2.8) that

$$\begin{aligned}
2(1 - d)(1 + \lambda) \sum_{k=2}^{\infty} ka_k - 2\alpha(1 - d)(1 + \lambda) \sum_{k=2}^{\infty} a_k &\leq \sum_{k=2}^{\infty} k(k - \alpha)(1 - d)^{k-1} a_k (1 - \lambda + \lambda k) \\
&\leq 1 - \alpha. \tag{2.28}
\end{aligned}$$

From (2.27) and (2.28), we have

$$2(1 - d)(1 + \lambda) \sum_{k=2}^{\infty} ka_k \leq (1 - \alpha) + 2\alpha(1 - d)(1 + \lambda) \sum_{k=2}^{\infty} a_k \tag{2.29}$$

$$\leq (1 - \alpha) + \frac{\alpha(1 - \alpha)}{(2 - \alpha)} = \frac{2(1 - \alpha)}{2 - \alpha}.$$

$$\sum_{k=2}^{\infty} ka_k \leq \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)} \tag{2.30}$$

which proved the theorem.

III. INTEGRAL MEAN INEQUALITY

Lemma3.1. If f and g are analytic in E with $f < g$ then

$$\int_0^{2\pi} |g(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \quad (3.1)$$

where $\delta > 0, z = re^{i\theta}, \omega = de^{i\theta}$ and $0 < r + d < 1$.

Applying Lemma 3.1 and (1.2) we prove the following.

Theorem3.2. Let $\delta > 0$. If $f(z) \in H(\omega, \alpha, \lambda)$, then $z = re^{i\theta}, \omega = de^{i\theta}$ and $0 \leq d < r < 1$, we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\delta d\theta \quad (3.2)$$

where

$$f_2(z) = (z - \omega) - \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega)^2. \quad (3.3)$$

Proof. Let $f(z)$ defined by (1.3) and $f_2(z)$ be given by (3.3) we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} a_k (z - \omega)^{k-1} \right|^\delta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega) \right|^\delta d\theta. \quad (3.4)$$

By Lemma 3.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k (z - \omega)^{k-1} < 1 - \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega), \quad (3.5)$$

setting

$$1 - \sum_{k=2}^{\infty} a_k (z - \omega)^{k-1} = 1 - \frac{1 - \alpha}{(2 - \alpha)(1 - d)(1 + \lambda)} h(z). \quad (3.6)$$

From (3.6) and (2.1) we obtain

$$\begin{aligned} |h(z)| &= \left| \sum_{k=2}^{\infty} \frac{(2 - \alpha)(1 - d)(1 + \lambda)}{(1 - \alpha)} a_k (z - \omega)^{k-1} \right| \\ &\leq |z - \omega| \sum_{k=2}^{\infty} \frac{(k - \alpha)(1 - d)^{k-1}(1 + \lambda)}{(1 - \alpha)} a_k \leq |z - \omega|. \end{aligned} \quad (3.7)$$

This completes the proof of theorem 3.2.

Theorem 3.3. Let $\delta > 0$. If $f(z) \in K(\omega, \alpha, \lambda)$, then $z = re^{i\theta}$, $\omega = de^{i\theta}$ and $0 \leq d < r < 1$, we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\delta d\theta \quad (3.8)$$

where

$$f_2(z) = (z - \omega) - \frac{1 - \alpha}{2(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega)^2. \quad (3.9)$$

Proof. Let $f(z)$ defined by (1.3) and $f_2(z)$ be given by (3.9) we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} a_k(z - \omega)^{k-1} \right|^\delta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1 - \alpha}{2(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega) \right|^\delta d\theta. \quad (3.10)$$

By lemma 3.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k(z - \omega)^{k-1} < 1 - \frac{1 - \alpha}{2(2 - \alpha)(1 - d)(1 + \lambda)} (z - \omega), \quad (3.11)$$

setting

$$1 - \sum_{k=2}^{\infty} a_k(z - \omega)^{k-1} = 1 - \frac{1 - \alpha}{2(2 - \alpha)(1 - d)(1 + \lambda)} h(z). \quad (3.12)$$

From (3.12) and (2.6) we obtain

$$\begin{aligned} |h(z)| &= \left| \sum_{k=2}^{\infty} \frac{2(2 - \alpha)(1 - d)(1 + \lambda)}{(1 - \alpha)} a_k(z - \omega)^{k-1} \right| \\ &\leq |z - \omega| \sum_{k=2}^{\infty} \frac{2(k - \alpha)(1 - d)^{k-1}(1 + \lambda)}{(1 - \alpha)} a_k \\ &\leq |z - \omega|. \end{aligned} \quad (3.13)$$

This completes the proof of theorem 3.3.

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