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Harmonic and Wave Maps Coupled with Einstein's Gravitation

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Harmonic and Wave Maps Coupled with Einstein's Gravitation

Rainer Schimming^α & Ragab M. Gad^σ

Abstract - In this paper we discuss the coupled dynamics, following from a suitable Lagrangian, of a harmonic or wave map ϕ and Einstein's gravitation described by a metric g . The main results concern energy conditions for wave maps, harmonic maps from warped product manifolds, and wave maps from wave-like Lorentzian manifolds.

Keywords : harmonic map; wave map; energy conditions.

I. INTRODUCTION

Scalar fields on a space or space-time manifold (X, g) , which satisfy a linear or nonlinear field equation, attract enduring attention; cf, e.g., [1, 2, 3]. The *Dirichlet Lagrangian* or *energy density*

$$e = \frac{1}{2}|d\phi|^2 = \frac{1}{2}g^{ab}(\partial_a\phi)(\partial_b\phi), \quad (1.1)$$

of a one-component scalar field $\phi = \phi(x)$ leads to a linear field equation of Laplace or D'Alembert type. Here we denote

$$g = g_{ab}dx^a dx^b, \quad (g^{ab}) := (g_{ab})^{-1}, \quad \partial_a := \frac{\partial}{\partial x^a}.$$

The Dirichlet Lagrangian e admits a natural generalization to a multi-component scalar field $\phi = \phi(x)$ if the range of ϕ is suitably geometrized, namely if $im\phi$ lies in a Riemannian manifold (Y, h) . That means, ϕ becomes a map $X \rightarrow Y$, $x \mapsto y$ between a source manifold (X, g) and a target manifold (Y, h) . The choice of the Lagrangian

$$e = e[\phi] = e[\phi, g] = \frac{1}{2}|d\phi|^2, \quad (1.2)$$

where now

$$|d\phi|^2 := tr(\phi^*h) \equiv g^{ab}(\partial_a\phi^i)(\partial_b\phi^j)h_{ij}(\phi^k),$$

leads to the generally semi-linear field equation with Laplace-like or D'Alembert-like principal part

$$tr(\nabla d\phi)^i \equiv g^{ab}\nabla_a\partial_b\phi^i = 0. \quad (1.3)$$

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Here we denote $h = h_{ij}dy^i dy^j$ and the special covariant derivative ∇ is built from g , h , ϕ as will be explained in section 2.

A map $\phi : X \rightarrow Y$ between properly Riemannian manifolds (X, g) , (Y, h) which satisfies (1.3) is called a *harmonic map* [4, 5]. A map $\phi : X \rightarrow Y$ from a Lorentzian manifold (X, g) to a properly Riemannian manifold (Y, h) which satisfies (1.3) is called a *wave map* [6, 7, 8, 9, 3].

In this paper, we take (Y, h) as a fixed background and consider g and ϕ as dynamical objects. The dynamics shall follow from the Lagrangian

$$L = \kappa R - e, \quad (1.4)$$

where R denotes the scalar curvature of g and $\kappa \neq 0$ is a coupling constant. Variation with respect to g yields, for $\dim X \geq 3$, the *Einstein equation* in the form

$$\kappa Ric = \phi^* h, \quad (1.5)$$

in components

$$\kappa R_{ab} = (\partial_a \phi^i)(\partial_b \phi^j) h_{ij}. \quad (1.6)$$

Variation with respect to ϕ yields (1.3).

A Lorentzian metric g can be interpreted as *gravitation*. According to the Kaluza-Klein principle, the space-time manifold (X, g) may have a higher dimension.

A positive definite metric g on X can be given a physical interpretation through a Lorentzian metric constructed from it as follows. Consider the product manifold $\tilde{X} := \mathbb{R} \times X$ with points (t, x) and equip it with the Lorentzian metric $\tilde{g} := dt^2 - g$. Extend $\phi : X \rightarrow Y$ to $\tilde{\phi} : \tilde{X} \rightarrow Y$ by setting $\tilde{\phi}(t, x) := \phi(x)$, that means $\tilde{\phi} := \phi \circ pr_2$, where pr_2 is the projection $\tilde{X} \rightarrow X$ to the second factor. The field equations (1.5), (1.3) for $(\tilde{X}, \tilde{g})$, $\tilde{\phi}$ reduce to (1.5), (1.3) for (X, g) , ϕ .

Let us sketch our main results.

- The energy-momentum tensor T of a wave map obeys several energy conditions. In particular, if v is a causal (i.e. non-spacelike) vector field then the momentum one-form $I := T(\cdot, v)$ is causal again.
- The Einstein equation (1.5) implies that the conditions $Ric(v, v) = 0$, $Ric(\cdot, v) = 0$, $\phi_* v = 0$ for a vector field v are equivalent to each other. Moreover, then v is a Ricci collineation, i.e. $\mathcal{L}_v Ric = 0$. Some conclusions are drawn from this latter fact.
- A submersive map $\phi : X \rightarrow Y$ between a pure manifold X and a Riemannian manifold (Y, h) can locally be made to a harmonic or wave map by a suitable choice of a metric g on X .
- We study the case of warped product $X = 'X \times ''X$, $g = 'g \oplus w^2 ''g$. Several propositions are proved by means of the argument that the integral of a Laplace expression over a closed manifold $'X$ vanishes.
- We study radiation conditions for a Lorentzian manifold (X, g) . The Einstein equation (1.5) leads from one condition to a stronger condition.

II. PRELIMINARIES

We consider dimensions

$$m := \dim X \geq 3, \quad n := \dim Y \geq 1$$

and adopt the following index convention: indices $a, b, c, \dots$ label the components of geometric objects on X ; indices $i, j, k, \dots$ label the components of geometric objects on Y .

Tensor fields on Y become multi-component scalar fields on X by insertion of $y = \phi(x)$, where $x \in X$, $y \in Y$. For covariant tensor fields on Y there is also the conventional pull-back map ϕ^* . For instance, the metric $h = h_{ij} dy^i dy^j$ yields $h \circ \phi$ with components $h_{ij}(\phi^k) = h_{ij}(\phi^k(x^a))$ on X and also the pull-back ϕ^*h on X with components $(\partial_a \phi^i)(\partial_b \phi^j)(h_{ij}(\phi^k))$. The object ϕ^*h is called *first fundamental form* of $\phi : X \rightarrow Y$.

We will occasionally write

$$\phi^*h = h(d\phi, d\phi)$$

and also use the bilinear symmetric form $h(\cdot, \cdot)$ with tensorial entries.

Some natural covariant derivative ∇ with components ∇_a is built from g , h , ϕ according to the following rules.

1- ∇ applied to tensor fields on X equals the Levi-Civita derivative ${}^g\nabla$ to g . For instance,

$$\nabla_a v^c := \partial_a v^c + {}^g\Gamma_{ab}^c v^b,$$

where $v = v^a \partial_a$ is a vector field on X and ${}^g\Gamma_{ab}^c$ are the Christoffel symbols to (g_{ab}) .

2- ∇ applied to tensor fields on Y equals some pull-back of the Levi-Civita derivative ${}^h\nabla$ to h . For instance,

$$\nabla_a w^k := (\partial_a \phi^i)(\partial_i w^k + {}^h\Gamma_{ij}^k w^j),$$

where $w = w^i \partial_i$ is a vector field on Y and ${}^h\Gamma_{ij}^k$ are the Christoffel symbols to (h_{ij}) . Here $\nabla_a w^k$ is understood to depend on $y^l = \phi^l(x^c)$.

3- There are natural product rules for mixed quantities, the components of which carry both indices $a, b, \dots$ and $i, j, \dots$. For instance, the *differential* $d\phi$ of $\phi : X \rightarrow Y$ with components $\partial_a \phi^i$ is a mixed tensor field. The covariant derivative $\nabla d\phi$ of $d\phi$ is called *second fundamental form* of ϕ . It has the components

$$\nabla_a \partial_b \phi^k = \partial_a \partial_b \phi^k - {}^g\Gamma_{ab}^c \partial_c \phi^k + {}^h\Gamma_{ij}^k (\partial_a \phi^i)(\partial_b \phi^j)$$

and the symmetry property

$$\nabla_a \partial_b \phi^k = \nabla_b \partial_a \phi^k.$$

More on calculus for maps between (pseudo-) Riemannian manifolds can be found in the literature, e.g. [4, 5].

III. THE FIELD EQUATIONS

The field theory for g and ϕ considered here is based on the Lagrangian

$$L = \kappa R - e, \quad (3.1)$$

which is the sum of the *gravitational Lagrangian* κR and the *matter Lagrangian* e . Here $R = R[g]$ denotes the scalar curvature of g , $\kappa \neq 0$ is a coupling constant and $e = e[\phi] = e[g, \phi]$ is given by (1.2).

The idea to couple a harmonic map, formerly also called *sigma model*, with Einstein's gravitation appeared in [10, 11, 12] and other early papers.

The following is well-known [10, 11, 12, 13, 14, 15, 16, 17, 18, 7, 8, 9]. We abbreviate $\det g := \det(g_{ab})$.

Proposition 3.1 *Variation of $L|\det g|^{\frac{1}{2}}$ with respect to g yields the Einstein equation in the form*

$$\kappa(\text{Ric} - \frac{1}{2}Rg) = \phi^*h - eg. \quad (3.2)$$

If $m = \dim X \geq 3$ then this is equivalent to

$$\kappa \text{Ric} = \phi^*h. \quad (3.3)$$

Variation of $L|\det g|^{\frac{1}{2}}$ with respect to ϕ yields

$$\text{tr}(\nabla d\phi) = 0, \quad (3.4)$$

where $\nabla d\phi$ is the second fundamental form of ϕ and the trace tr refers to the metric g .

The right-hand side of (3.2)

$$T := \phi^*h - eg \quad (3.5)$$

is called *energy-momentum tensor of ϕ* . There holds $e = \frac{1}{2}\text{tr}(\phi^*h)$ and (3.5) is equivalent to

$$T - (m-2)^{-1}(\text{tr}T)g = \phi^*h.$$

Proposition 3.2 *From the field equation (3.4) for ϕ there follows that T is divergence-free:*

$$\nabla^b T_{ab} = 0.$$

The *proof* follows from the identity

$$\nabla^b T_{ab} = h_{ij}(\partial_a \phi^i)(\text{tr} \nabla d\phi)^j.$$

IV. ENERGY CONDITIONS FOR A WAVE MAP

Let now the metric g have Lorentzian signature $(+ - \dots -)$.

Definition 4.1 Let a symmetric two-form $T = T_{ab}dx^a dx^b$ on X be interpreted as an energy-momentum tensor field and let $v = v^a \partial_a$ be a unit timelike vector field on X , i.e., $v_a v^a \equiv g_{ab} v^a v^b = 1$. Then $T(v, v) \equiv$

$T_{ab}v^av^b$ is called *energy density*, $I := T(\cdot, v)$ with components $I_a := T_{ab}v^b$ is called *momentum one-form*, and $J := I - T(v, v)v$ with components $J_a := I_a - T(v, v)v_a$ is called *proper momentum one-form*.

Physically, v is interpreted as the unit tangent vector to the world line of an observer. This observer measures the quantities $T(v, v)$, I , J .

Every unit timelike vector field $v = v^a\partial_a$ on X gives rise to a positive definite metric $g^+ = g_{ab}^+dx^adx^b$ on X with components $g_{ab}^+ = 2v_av_b - g_{ab}$. The inverse $(g_+^{ab}) := (g_{ab}^+)^{-1}$ has the representation $g_+^{ab} = 2v^av^b - g^{ab}$.

Theorem 4.1 *Consider the energy-momentum tensor*

$$T = \phi^*h - eg \quad (4.1)$$

of a map $\phi : X \longrightarrow Y$ between (X, g) and (Y, h) . The energy density equals

$$T(v, v) = e_+ := e[\phi, g_+] \equiv \frac{1}{2}g_+^{ab}(\partial_a\phi^i)(\partial_b\phi^j)h_{ij}. \quad (4.2)$$

It is a positive definite quadratic form in $d\phi$. The momentum one-form I obeys the estimate

$$e^2 \leq |I|^2 \leq e_+^2, \quad (4.3)$$

where

$$|I|^2 := I_aI^a \equiv g^{ab}I_aI_b. \quad (4.4)$$

Proof:

Let us abbreviate $f := \phi^*h$ with components $f_{ab} := (\phi^*h)_{ab} = (\partial_a\phi^i)(\partial_b\phi^j)h_{ij}$. We calculate

$$\begin{aligned} T(v, v) &= T_{av}v^av^b = (f_{ab} - eg_{ab})v^av^b \\ &= f_{ab}v^av^b - e = \frac{1}{2}(2v^av^b - g^{ab})f_{ab} = \frac{1}{2}g_+^{ab}f_{ab} = e_+. \end{aligned}$$

The proper momentum one-form J is orthogonal to v , that means $J_av^a = 0$. Considering that, we find that

$$\begin{aligned} 0 &\leq g_+^{ab}J_aJ_b = -g^{ab}J_aJ_b \\ &= -g^{ab}J_a(I_b - e_+v_b) = -g^{ab}J_aI_b \\ &= -g^{ab}(I_a - e_+v_a)I_b = -I_aI^a + e_+^2. \end{aligned}$$

Thus, the right-hand side inequality of (4.3) is proved. In order to prove the left-hand side of (4.3), we start with the remark that the matrix (f_{ab}) is positive semi-definite. Let us consider a fixed point $x_0 \in X$ and use coordinates x^a such that

$$v^a = \delta_0^a, \quad g_+^{ab} = \delta^{ab} \quad (4.5)$$

in that very point x_0 . Actually, such coordinates exist. The following 2×2 subdeterminants of (f_{ab}) are non-negative:

$$\begin{aligned} f_{00}f_{11} - f_{10}f_{10} &\geq 0, \\ f_{00}f_{22} - f_{20}f_{20} &\geq 0, \end{aligned}$$

...

$$f_{00}f_{mm} - f_{m0}f_{m0} \geq 0.$$

Let us sum up:

$$f_{00}f_{aa} - f_{a0}f_{a0} \geq 0. \quad (4.6)$$

Here a summation convention applies to the index a and the coordinate conditions (4.5) are assumed. The inequality (4.6) can be brought into a coordinate-invariant form

$$g_+^{ab}(f_{ab}v^cv^df_{cd} - v^cf_{ac}v^df_{bd}) \geq 0.$$

Here we insert

$$g_+^{ab}f_{ab} = 2e_+, \quad v^cv^df_{cd} = e + e_+,$$

$$v^cf_{ac} = v^c(T_{ac} + eg_{ac}) = I_a + ev_a,$$

$$g_+^{ab}(I_a + ev_a)(I_b + ev_b) = 2e_+(e + e_+) + e^2 - |I|^2.$$

Taking all this together, the left-hand side inequality of (4.3) follows. The conditions

$$T(v, v) \geq 0, \quad |I|^2 \geq 0$$

together form the *dominant energy condition*, which expresses that the energy density is non-negative and that the momentum I is causal. The latter condition physically means that the momentum I propagates with a velocity which is not greater than the velocity of light.

The so-called *strongy energy condition* also holds in the present situation, namely in the form

$$(m - 2)T(v, v) \geq trT.$$

Theorem 4.2 Consider the energy-momentum tensor $T = T_{ab}dx^a dx^b$ to ϕ as above and lightlike vector fields $l = l^a \partial_a$, $n = n^a \partial_a$ such that $l^a n^a \equiv g_{ab}l^a n^b = 1$. Then

$$T(l, l) \equiv T_{ab}l^a l^b = h(\phi_* l, \phi_* l) \geq 0, \quad (4.7)$$

and the one-form $I := T(\cdot, l)$ with components $I_a := T_{ab}l^b$ obeys

$$0 \leq |I|^2 \leq 2T(l, l)T(l, n). \quad (4.8)$$

Proof:

Assertion (4.7) follows from

$$T(l, l) = (f_{ab} - eg_{ab})l^a l^b = f_{ab}l^a l^b = h_{ij}(l^a \partial_a \phi^i)(l^b \partial_b \phi^j).$$

The projection tensor p with components

$$p_{ab} := l_a n_b + n_a l_b - g_{ab}$$

is a useful tool. It is orthogonal to l and n , i.e.

$$p_{ab}l^b = p_{ab}n^b = 0,$$

and it is positive semi-definite. Hence

$$0 \leq p^{ab} I_a I_b = (2l^a n^b - g^{ab}) I_a I_b = 2T(l, l)T(l, n) - |I|^2,$$

which proves the right-hand side inequality of (4.8)

Below we will also use

$$p^{ab} T_{ab} = (2l^a n^b - g^{ab}) T_{ab} = 2T(l, n) - trT = 2T(l, n) + (m - 2)e.$$

Let us, in order to complete the proof, consider a fixed point $x_0 \in X$ and use coordinates x^a such that

$$l^a = l_0^a, \quad n^a = \delta_1^a, \quad (p_{ab}) = diag(0, 0, 1, \dots, 1)$$

in that very point x_0 , where $diag$ indicates a diagonal matrix. Formally the same 2×2 subdeterminants of (f_{ab}) as in the preceding proof are non-negative. Their sum is now in another way transformed into a coordinate-invariant form

$$p^{ab} [f_{ab} (l^c l^d f_{cd}) - (l^c f_{ac}) (l^d f_{bd})] \geq 0.$$

Here we insert

$$\begin{aligned} p^{ab} f_{ab} &= p^{ab} (T_{ab} + e g_{ab}) = p^{ab} T_{ab} - (m - 2)e = 2T(l, n), \\ l^c l^d f_{cd} &= T(l, l), \quad p^{ab} (l^c f_{ac}) (l^d f_{bd}) = p^{ab} I_a I_b = 2T(l, l)T(l, n) - |I|^2. \end{aligned}$$

The result $|I|^2 = g^{ab} I_a I_b \geq 0$ follows.

Physically, theorem (4.2) can be interpreted in terms of a fictional observer which moves faster and faster. In the limit, he reaches the velocity of light and v turns to l . The energy density $T(l, l)$ then remains non-negative and the momentum I remains causal.

Corollary 4.3 *There holds $T(l, n) \geq 0$. Especially, $T(l, n) = 0$ iff $I_a = T(l, l)n_a$.*

Proof:

The formulas (4.7), (4.8) imply $T(l, n) \geq 0$. If $T(l, n) = 0$ then $|I|^2 = 0$ and

$$0 = p^{ab} (l^c f_{bc}) = p^{ab} I_b = (l^a n^b + n^a l^b - g^{ab}) I_b = T(l, l)n^a - I^a$$

V. IMPLICATIONS OF THE EINSTEIN EQUATION

Let us study

$$\kappa Ric = \phi^* h, \tag{5.1}$$

for given background (Y, h) as a relation between g and ϕ . Contraction with g^{-1} yields

$$\kappa R = 2e. \tag{5.2}$$

Double Contraction with a vector field $v = v^a \partial_a$ on X yields

$$\kappa Ric(v, v) = h(\phi_* v, \phi_* v), \tag{5.3}$$

with the interpretation that $y = \phi(x)$ is to be inserted into the right-hand side of (5.3); ϕ_*v denotes the push-forward of v with respect to ϕ . As a conclusion, $\kappa Ric(v, v)$ is positive definite in ϕ_*v and is positive semi-definite in v .

Proposition 5.1 *From (5.1) there follows that Ric and $d\phi$, interpreted as linear map, have the same rank:*

$$r := \text{rank}(Ric) = \text{rank}(d\phi). \quad (5.4)$$

In particular:

$r = 0$ iff ϕ is constant.

$r = m \equiv \dim X$ iff ϕ is an immersion.

$r = n \equiv \dim Y$ iff ϕ is a submersion.

$r = m = n$ iff ϕ is a local diffeomorphism.

The proof of (5.4) is an exercise in linear algebra.

Note that $r = 0$ means that (X, g) is Ricci flat, i.e., $Ric = 0$.

Proposition 5.2 *If (5.1) holds then the conditions*

$$Ric(v, v) = 0, \quad (5.5)$$

$$Ric(\cdot, v) = 0, \quad (5.6)$$

$$\phi_*v = 0 \quad (5.7)$$

for a vector field $v = v^a \partial_a$ on X are equivalent to each other. Moreover, they imply

$$\mathcal{L}_v Ric = 0, \quad (5.8)$$

where $\mathcal{L}_v$ denotes the Lie derivative with respect to v .

Proof:

Equation (5.3) in components reads

$$\kappa R_{ab} v^a v^b = h_{ij} (\phi_* v)^i (\phi_* v)^j,$$

where $(\phi_* v)^i = v^a \partial_a \phi^i$. Moreover, (5.1) implies

$$\kappa R_{ab} v^b = h_{ij} (\partial_a \phi^i) (\phi_* v)^j.$$

These formulas and the definiteness of h give the first assertion. Next, we use comoving coordinates x^a , which are adapted to v , that means $v^a = \delta_0^a$, and we get

$$(\phi_* v)^i = v^a \partial_a \phi^i = \partial_0 \phi^i,$$

$$\kappa \mathcal{L}_v R_{ab} = \kappa \partial_0 R_{ab} = \partial_0 (\partial_a \phi^i \partial_b \phi^j h_{ij}(\phi^k)).$$

If $\partial_0 \phi^i = 0$ then also $\partial_0 R_{ab} = 0$. This fact can be translated into the second assertion.

Proposition 5.3 *If the Ricci tensor vanishes on the vectors of some integrable distribution in the tangent bundle TX and (5.1) holds then ϕ is constant on each leaf of the foliation to the distribution.*

Proof:

A distribution of rank s in TX is integrable iff it admits adapted coordinates x^a , which means that the distribution is locally spanned by the coordinate vector fields $\partial_1, \partial_2, \dots, \partial_s$. The assumption becomes

$$Ric(\partial_a, \partial_b) = 0 \quad \text{for } a, b = 1, 2, \dots, s.$$

Proposition (5.2) then implies

$$\partial_1 \phi^i = 0, \partial_2 \phi^i = 0, \dots, \partial_s \phi^i = 0.$$

Hence ϕ^i does not depend on $x^1, x^2, \dots, x^s$ and is constant if the point x varies in a leaf of the foliation, i.e., if $x^{s+1} = \text{const.}, \dots, x^m = \text{const.}$

Proposition 5.4 *The Einstein equation (5.1) implies*

$$\kappa(\nabla_a R_{bc} + \nabla_b R_{ca} - \nabla_c R_{ab}) = 2h_{ij}(\nabla_a \partial_b \phi^i)(\partial_c \phi^j). \quad (5.9)$$

Proof:

Covariant differentiation of (5.1) gives

$$\kappa \nabla_c R_{ab} = h_{ij}[(\nabla_c \partial_a \phi^i)(\partial_b \phi^j) + (\partial_a \phi^i)(\nabla_c \partial_b \phi^j)].$$

Some rearrangement yields (5.9).

Proposition 5.5 *The Einstein equation (5.1) implies*

$$h(\text{tr}(\nabla d\phi), d\phi) = 0. \quad (5.10)$$

This equation for a submersion ϕ implies the harmonic or wave map equation (3.4).

Proof:

The Einstein tensor $Ric - \frac{1}{2}Rg$ is divergence-free. This fact and (5.9) imply

$$2h(\text{tr}(\nabla d\phi), d\phi)_c \equiv 2h_{ij} \text{tr}(\nabla d\phi)^i (\partial_c \phi^j) = \kappa g^{ab} (\nabla_a R_{bc} + \nabla_b R_{ca} - \nabla_c R_{ab}) = 0.$$

If ϕ is a submersion, then the matrix with elements $h(\cdot, d\phi)_{ic} = h_{ij} \partial_c \phi^j$ has maximal rank and therefore (5.10) implies (3.4).

Proposition 5.6 *If the Ricci tensor is covariantly constant, i.e., $\nabla Ric = 0$, and the Einstein equation (5.1) holds for a submersion ϕ , then ϕ is totally geodesic, that means*

$$\nabla d\phi = 0. \quad (5.11)$$

Proof:

If $\nabla Ric = 0$ then (5.9) reduces to $h_{ij}(\nabla_a \partial_b \phi^i)(\partial_c \phi^j) = 0$. If, additionally, ϕ is a submersion, then (5.11) follows by means of the rank argument already used in the preceding proof.

The next proposition needs some preparation. A diffeomorphism $f : X \rightarrow X$ is called a *Ricci symmetry* iff

$$f^* Ric = Ric. \quad (5.12)$$

A vector field $v = v^a \partial_a$ on X is called an *infinitesimal Ricci symmetry* or a *Ricci collineation* iff

$$\mathcal{L}_v Ric = 0. \quad (5.13)$$

It is well known that the flow f_t for $t \in I$ of a Ricci collineation v is a one-parameter family of Ricci symmetries, i.e. $f_t^* Ric = Ric$. Here $x = f_t(x_0)$ by definition represents the solution of the initial-value problem

$$\frac{dx}{dt} = v(x), \quad x|_{t=0} = x_0$$

and I denotes an open interval which contains 0.

Proposition 5.7 *Let $v = v^a \partial_a$ be a Ricci collineation of (X, g) . Then the Einstein equation (5.1) implies that $\phi : X \rightarrow Y$ is invariant under the flow f_t for $t \in I$ of v , that means*

$$\phi \circ f_t = \phi. \quad (5.14)$$

Proof:

Let us again use comoving coordinates such that $v^a = \delta_0^a$. In these coordinates, f_t is expressed by a translation $x^0 \mapsto x^0 + t$, $x^1 \mapsto x^1$, ..., $x^{m-1} \mapsto x^{m-1}$. We know already from the proof of proposition (5.2) $\partial_0 \phi^i = 0$, i.e. each ϕ^i is independent of x^0 . Hence $\phi^i = \phi^i(x^a)$ does not change under $x^0 \mapsto x^0 + t$, which is just expressed by (5.14).

The following definition is useful.

Definition 5.1 A property of subsets of a manifold X holds *globally* if it is valid for X . It holds *locally* if every point $x_0 \in X$ has a neighborhood U such that the property is valid for U .

Theorem 5.8 *Let $\phi : X \rightarrow Y$ be a submersion between smooth manifolds X, Y and let Y be equipped with a positive definite metric h .*

1- Locally there is a positive definite metric g on X such that ϕ becomes a harmonic map (X, g) and (Y, h) .

2- Locally there is a Lorentzian metric g on X such that ϕ becomes a wave map.

Proof:

We consider (5.1) and use the fact that the *problem of prescribed Ricci curvature* is locally solvable in the two cases [19, 20]. More precisely: cf., eg.

1- Einstein's equation (5.1) locally has a positive definite solution g . It can be constructed, e.g., through some boundary value problem [19]. By assumption, ϕ is a submersion; proposition 5.5 implies $tr(\nabla d\phi) = 0$.

2- Einstein's equation (5.1) locally has a Lorentzian solution g . It can be constructed, e.g., through some Cauchy initial value problem cf., e.g. [20]. An argument like in item 1 completes the proof.

VI. PRODUCT AND WARPED PRODUCT SOURCE MANIFOLDS

Definition 6.1: The *product* (X, g) of two (pseudo-) Riemannian manifolds $({}'X, {}'g)$, $({}''X, {}''g)$ is given by $X = {}'X \times {}''X$ as a product of manifolds and by

$$g(u, v) = {}'g({}'u, {}'v) + {}''g({}''u, {}''v)$$

where u, v are vector fields on X , ${}'u, {}'v$ are the push-forwards of u, v with respect to the projection $X \longrightarrow {}'X$, and ${}''u, {}''v$ are the push-forwards of u, v with respect to the projection $X \longrightarrow {}''X$.

We write then

$$g = {}'g \oplus {}''g,$$

$$\dim {}'X = {}'m, \quad \dim {}''X = {}''m, \quad m = {}'m + {}''m.$$

We apply the index convention

$${}'a, {}'b, {}'c, \dots = 1, 2, \dots, {}'m; \quad {}''a, {}''b, {}''c, \dots = {}'m + 1, {}'m + 2, \dots, m.$$

Definition 6.2: The *warped product* (X, g) is given by $X = {}'X \times {}''X$ like above and by

$$g(u, v) = {}'g({}'u, {}'v) + w^2 {}''g({}''u, {}''v),$$

where the *warping function* w is a map ${}'X \longrightarrow \mathfrak{R}$ with positive values $w > 0$. We write then

$$g = {}'g \oplus w^2 {}''g,$$

The following is known.

Proposition 6.1 *If (X, g) is the product of $({}'X, {}'g)$, $({}''X, {}''g)$ then the Einstein equation $\kappa Ric = \phi^* h \equiv h(d\phi, d\phi)$ decomposes into the two Einstein equations*

$$\kappa {}'Ric = h({}'d\phi, {}'d\phi), \quad \kappa {}''Ric = h({}''d\phi, {}''d\phi) \quad (6.1)$$

and the orthogonality condition with respect to h :

$$h({}'d\phi, {}''d\phi) = 0, \quad (6.2)$$

in components

$$\kappa {}'R_{a'b} = h_{ij}(\partial_a \phi^i)(\partial_b \phi^j), \quad \kappa {}''R_{a''b''} = h_{ij}(\partial_{a''} \phi^i)(\partial_{b''} \phi^j), \quad (6.3)$$

$$h_{ij}(\partial_a \phi^i)(\partial_{b''} \phi^j) = 0 \quad (6.4)$$

Proposition 6.2 *If (X, g) is the warped product of $({}'X, {}'g)$, $({}''X, {}''g)$ with warping function w then the Einstein equation $\kappa Ric = \phi^* h \equiv h(d\phi, d\phi)$ decomposes into*

$$\kappa({}'Ric - {}''mw^{-1}{}'\nabla' dw) = h({}'d\phi, {}'d\phi), \quad (6.5)$$

$$\kappa({}''Ric - \frac{1}{{}''m}w^{2-{}''m}{}''\Delta w {}''m g) = h({}''d\phi, {}''d\phi), \quad (6.6)$$

$$h({}'d\phi, {}''d\phi) = 0. \quad (6.7)$$

The trace parts of (6.5), (6.6) read

$$\kappa('R - {}''mw^{-1}'\Delta w) = 2'e, \quad (6.8)$$

$$\kappa('R - w^{2-''m}'\Delta w''^m) = 2''e. \quad (6.9)$$

The proof is an exercise in higher differential geometry

We say that $\phi: 'X \times ''X \rightarrow Y$ does not depend on $'x$ iff the restriction $\phi(\cdot, ''x): 'X \rightarrow Y$ is a constant map for every $''x \in ''X$. We say that ϕ does not depend on $''x \in ''X$ in the analogous situation.

Theorem 6.3 *Let in the situation of proposition 6.2 the restriction $\phi('x, \cdot): ''X \rightarrow X$ be a submersion for every $'x \in 'X$. Then ϕ does not depend on $'x$.*

Proof:

If every $\phi('x, \cdot)$ is a submersion then the quantities $h_{ij}\partial_{''b}\phi^j$ in (6.4) form a matrix of maximal rank, and (6.4) implies $\partial_a\phi^i = 0$, which gives the assertion.

Theorem 6.4 *Let in the situation of proposition 6.2 the first factor $('X, 'g)$ be properly Riemannian and closed. Then the following holds.*

- (i) *The symmetric two-form $\kappa''Ric$ is positive semi-definite.*
- (ii) *If $\kappa''Ric$ is not everywhere positive then $w = \text{const.}$*
- (iii) *$\kappa \int_X w'Rd'vol \geq 0$.*
- (iv) *If $\int_X w'Rd'vol = 0$ then $w = \text{const.}$ and ϕ does not depend on $'x$.*

Proof:

- (i) Multiply (6.6) by w''^{m-2} and evaluate the two-forms at a vector field $''v \neq 0$ on $''X$:

$$\kappa[w''^{m-2}''Ric(''v, ''v) - \frac{1}{''m}(''\Delta w''^m)|''v|^2] = w''^{m-2}h(''d\phi(''v), ''d\phi(''v)). \quad (6.10)$$

Integrate this equation over $'X$; the Laplacian term does not contribute, hence

$$\kappa''Ric(''v, ''v) \int_X w''^{m-2}d'vol = \int_X w''^{m-2}h(''d\phi(''v), ''d\phi(''v))d'vol.$$

Both the integrals are non-negative, hence $\kappa''Ric(''v, ''v) \geq 0$ for every $''v$.

- (ii) If $''Ric(''v, ''v)(''x_0) = 0$ for some point $''x_0 \in ''X$ then

$$\int_X w''^{m-2}h(''d\phi(''v(''x_0)), ''d\phi(''v(''x_0)))d'vol = 0,$$

which implies $''d\phi(''v(''x_0)) = 0$. Evaluation of (6.10) at the point $''x_0$ reduces (6.10) to

$$(''\Delta w''^m)|''v(''x_0)|^2 = 0.$$

We can assume $|''v(''x_0)|^2 \neq 0$; hence w''^m is a harmonic function. A harmonic function on a closed manifold is constant.

- (iii) Multiply (6.8) by w and integrate then over $'X$. The Laplacian term does not contribute; hence

$$\kappa \int_X w'Rd'vol = \int_X v'ed'vol \geq 0.$$

(iv) The last integral vanishes only if $'e = 0$, which implies $'d\phi = 0$, hence $\phi(\cdot, ''x) = \text{const}$ for fixed $''x \in ''X$. But then $''e(\cdot, ''x) = \text{const}$ and the standard separation argument can be applied to

$$\kappa w^{2-'m} \Delta w''^m = \kappa ''R - 2''e. \quad (6.11)$$

Thus both sides of (6.11) equal a constant c . Integration of

$$\kappa \Delta w''^m = c w''^{m-2}$$

yields $c = 0$. Hence w''^m is a harmonic function on the closed manifold $'X$. We find $w''^m = \text{const}$, $w = \text{const}$.

Proposition 6.5 *Let in the situation of the preceding theorem the second factor $(''X, ''g)$ be properly Riemannian with vanishing scalar curvature: $''R = 0$. Then $w = \text{const}$ and ϕ does not depend on $''x \in ''X$.*

Proof:

Now equation (6.9) reduces to

$$-\kappa \Delta w''^m = 2w''^{m-2} ''e$$

and we have $''e \geq 0$. Integration over the closed manifold $'X$ yields

$$0 = \int_{'X} w''^{m-2} ''e d'vol,$$

which implies

$$''e \equiv ''tr h(''d\phi, ''d\phi) = 0.$$

Hence $''d\phi = 0$, i.e. ϕ does not depend on $''x$, and w''^m becomes a harmonic function. We arrive at $w''^m = \text{const}$, $w = \text{const}$.

Example 6.1. Let in the situation of proposition 6.2 the first factor $('X, 'g)$ be the unit circle S^1 . It is a flat properly Riemannian closed manifold. Theorem 6.4 implies that $w = \text{const}$ and ϕ does not depend on $'x$.

Example 6.2. A static metric $g = w^2 dt^2 - 'g$ can be interpreted as a warped product metric on $X = 'X \times ''X$ where there the second factor $''X$ is one-dimensional. The proof of proposition 6.5 works, with a slight modification, for this case. Hence Einstein's equation $\kappa Ric = \phi^* h$ implies that $w = \text{const}$ and ϕ does not depend on $t \in ''X$.

VII. SOURCE MANIFOLDS WITH RADIATION CONDITIONS

Lichnerowicz [21] proposed *conditions of pure radiation* for a Lorentzian manifold (X, g) :

$$R_{abcd} l^d = 0, \quad R_{ab[cd} l_{e]} = 0,$$

where R_{abcd} are the components of the Riemann curvature tensor and $l = l^a \partial_a$ is a lightlike vector field. Bel [22] proposed weaker conditions:

$$R_{abcd} l^b l^d = 0, \quad l^b R_{ab[cd} l_{e]} = 0, \quad l_{[f} R_{ab][cd} l_{e]} = 0.$$

There are also radiation conditions for the Ricci tensor Ric or for the *Einstein tensor* $G := Ric - \frac{1}{2}Rg$ with components

$$G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab},$$

namely

$$R_{ab}l^b = 0, \quad R_{a[b}l_{c]} = 0,$$

$$G_{ab}l^b = 0, \quad G_{a[b}l_{c]} = 0.$$

One of the present authors studied radiation conditions in [23, 24].

Theorem 7.1 *If the Einstein equation $\kappa Ric = \phi^*h$ holds then the conditions*

$$G_{ab}l^b = 0, \tag{7.1}$$

$$R_{a[b}l_{c]} = 0 \tag{7.2}$$

for a lightlike vector field $l = l^a\partial_a$ are equivalent to each other.

Proof:

There is another lightlike vector field $n = n^a\partial_a$ such that $l^an_a = 1$. Then the sum $v := l + n$ is timelike with $v^av_a = 2$ and $g_{ab}^+ = v_av_b - g_{ab}$ are the components of a positive definite metric g_+ .

Let us consider

$$\psi_a^i := 2v^bl_{[a}\partial_{b]}\phi^i \equiv l_a(\phi_*v)^i - \partial_a\phi^i$$

and calculate

$$g_+^{ab}h_{ij}\psi_a^i\psi_b^j = -g^{ab}h_{ij}\psi_a^i\psi_b^j = h_{ij}(\phi_*v)^i(\phi_*l)^j - e = \kappa G_{ab}v^al^b.$$

Here we made use of

$$g_+^{ab} = v^av_b - g^{ab}, \quad v^a\psi_a^i = 0$$

and of $\kappa Ric = \phi_*h$. If $G_{ab}l^b = 0$ then the above positive definite expression vanishes and we get

$$\psi_a^i = 0, \quad \partial_a\phi^i = l_a(\phi_*v)^i,$$

$$\kappa R_{ab} = h_{ij}(\partial_a\phi^i)(\partial_b\phi^j) = l_al_bh(\phi_*v, \phi_*v).$$

Conversely, from $R_{ab} = rl_al_b$ with some scalar r there follow $R = 0$ and (7.1).

We here dwell again on the fact that to every smooth vector field $v = v^a\partial_a$ on X there are comoving coordinates $x^a = x^0, x^1, \dots, x^{m-1}$, which means $v^a = \delta_0^a$. The adaptedness of coordinates is preserved by coordinate transformations of the form

$$\bar{x}^0 = x^0 + f(x^i), \quad \bar{x}^i = \bar{x}^i(x^j). \tag{7.3}$$

We apply in this section the index convention

$$a, b, c, \dots = 0, 1, 2, \dots, m-1;$$

$$i, j, k, \dots = 1, 2, \dots, m-1;$$

$$I, J, K, \dots = 2, 3, \dots, m-1.$$

Proposition 7.2 *A Lorentzian manifold (X, g) admits a lightlike ... hypersurface-orthogonal Killing vector field $l = l^a \partial_a$, that means*

$$l_{[c} \nabla_a l_{b]} = 0, \quad \nabla_{(a} l_{b)} = 0,$$

iff in coordinates adapted to l the metric assumes the form

$$g = 2g_{01}dx^0dx^1 + g_{ij}dx^idx^j, \quad (7.4)$$

where g_{01} , g_{ij} do not depend on x^0 . The component $g_{01} = g_{01}(x^k)$ is invariant under gauge transformations (7.3) and the part $g_{IJ}dx^I dx^J$ of (7.4) shows tensorial behavior under the part $\bar{x}^i = \bar{x}^i(x^j)$ of (7.3). Moreover, the matrix $(g_{IJ}) = (g_{IJ}(x^k))$ is negative definite.

... covariantly constant vector field $l = l^a \partial_a$, that means $\nabla_a l_b = 0$, iff in coordinates adapted to l the metric assumes the form

$$g = 2dx^0dx^1 + g_{ij}dx^idx^j, \quad (7.5)$$

where the components g_{ij} do not depend on x^0 .

... covariantly constant vector field $l = l^a \partial_a$ such that the Bel condition

$$l_{[e} R_{ab][cd} l_{f]} = 0 \quad (7.6)$$

holds iff there are coordinates adapted to l such that

$$g = 2g_{01}dx^0dx^1 + g_{11}(dx^1)^2 + 2g_{1I}dx^1dx^I - dx^I dx^I, \quad (7.7)$$

where g_{11} , g_{1I} do not depend on x^0 and summation over I is applied.

... covariantly constant vector field $l = l^a \partial_a$ such that the Lichnerowicz condition

$$l_{[e} R_{ab]cd} = 0 \quad (7.8)$$

holds iff there are coordinates adapted to l such that

$$g = 2g_{01}dx^0dx^1 + g_{11}(dx^1)^2 - dx^I dx^I, \quad (7.9)$$

where g_{11} does not depend on x^0 and summation over I is applied.

All these facts together with *proof* and additional information are given in the papers [24, 23].

A Lorentzian manifold (X, g) which admits a covariantly constant light-like vector $l = l^a \partial_a$ is called a *plane-fronted gravitational wave with parallel rays*, abbreviated *pp-wave*. Note that from $\nabla_a l_b = 0$ and the Ricci identity there follows the Lichnerowicz condition

$$R_{abcd}l^d = 0.$$

Theorem 7.3 *A metric (7.4) satisfies an Einstein equation $\kappa Ric = \phi^* h$ iff $g_{01} = g_{01}(x^1, x^K)$ is a harmonic function of $x^K = x^2, x^3, \dots, x^{m-1}$ with respect to the positive definite metric (which depends on x^1 as a parameter) $-g_{IJ}dx^I dx^J$.*

Proof:

Some calculation gives the components

$$R_{00} = 0, \quad R_{01} = \frac{1}{2}\Delta g_{01}$$

of the Ricci tensor $Ric = R_{ab}dx^a dx^b$, where Δ denotes the Laplace operator with respect to $-g_{IJ}dx^I dx^J$. By proposition 5.2, from $R_{00} \equiv R_{ab}l^a l^b = 0$ and $\kappa Ric = \phi^*h$ there follows $R_{01} \equiv R_{1b}l^b = 0$. The assertion follows.

Theorem 7.4 *Let $m = \dim X = 4$. A metric of the form (7.7) satisfies an Einstein equation $\kappa Ric = \phi^*h$ iff it satisfies (7.8), that means iff it can be brought into the form (7.9).*

Proof:

Calculation of Ricci components gives $R_{IJ} = 0$; in particular $R_{II} = 0$ (without summation). By Proposition 5.2, from this and $\kappa Ric = \phi^*h$ there follows $R_{1I} = 0$. For $m = 4$, there are only two independent curvature components of type R_{iJKL} , namely

$$R_{1223} = -R_{13}, \quad R_{1323} = R_{12}.$$

Thus we get $R_{iJK2} = 0$ which is expressed by (7.9) in a coordinate invariant way.

VIII. DISCUSSION

The literature on harmonic or wave maps is very extensive. There are good surveys on harmonic maps [4, 5]. Work on such maps in the role of matter fields coupled with gravitation began about 1980 [10, 11]. One of the authors of this paper worked, with coauthors, already on this subject; we refer to the paper [17] and the unpublished preprint [25].

The Einstein equation $\kappa Ric = \phi^*h$ or $\kappa G = T$, where $G = Ric - \frac{1}{2}Rg$ denotes the Einstein tensor and $T = \phi^*h - eg$ the energy-momentum tensor of ϕ , exhibits some remarkable properties:

- The rank of Ric , taken as a linear map, equals the rank of the differential $d\phi$ (Proposition 5.1).
- The symmetries of the Ricci tensor Ric and of the map ϕ are closely related to each other (Propositions 5.2, 5.3, 5.7).
- If ϕ is submersive then the Einstein equation implies the harmonic or wave map equation (Proposition 5.5; cf. also Proposition 5.6).
- In the Lorentzian case there are identities and estimates for the energy-momentum tensor T which indicate a physically good behavior of T (Proposition 3.2, Theorems 4.1, 4.2).
- In the Lorentzian case there is a tendency to enhance radiation conditions. That means, the Einstein equation leads from one condition to a stronger condition (Section 7).

There is also a situation where the Einstein equation serves as an auxiliary construction: for a given submersion ϕ there locally exists a metric g on X which makes ϕ to a harmonic or wave map (Theorem 5.8).

One paper cannot touch all aspects of a subject. We did not discuss here:

- Bochner-Weitzenböck technique [5, 13, 14, 15],
- consequences of second variation formulas,
- factorizations of the map ϕ [18],
- exact solutions [10, 7, 8, 9, 12, 13, 14, 15, 16, 11],
- coupling of ϕ to a gravitational theory other than Einstein's theory.

These topics are by far not exhausted and could be subjects of further research.

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REFERENCES RÉFÉRENCES REFERENCIAS

- [1] A. Macias, J. L. Cervantes-Cota and C. Lämmerzahl, *Exact Solutions and Scalar Fields in Gravity*, Kluwer, Dordrecht (2001).
- [2] V. Faraoni, *Cosmology in Scalar-Tensor Gravity*, Kluwer, Dordrecht (2004).
- [3] A. D. Rendall, *Partial Differential Equations in General Relativity*, Oxford Univ. Press (2008).
- [4] J. Eells and J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109-160.
- [5] J. Eells and L. Lemaire, *Two Reports on Harmonic Maps*, World Scientific, Singapore (1993).
- [6] M. Struwe, *Wave maps*, Birkhäuser, Basel (1997).
- [7] P. Bizon, T. Chmaj and Z. Tabor, Dispersion and collapse of wave maps. *Nonlinearity* **13** (2000), 1411-1423.
- [8] P. Bizon and A. Wasserman, Self-similar spherically symmetric wave maps coupled to gravity, *Phys. Rev. D* **62** (2000), 084031.
- [9] P. Bizon and A. Wassermann, On the existence of self-similar spherically symmetric wave maps coupled to gravity, *Class. Quant. Grav.* **19** (2002), 3309-3321.
- [10] V. de Alfaro, S. Fubini and G. Furlan, Gauge theories and strong gravity, *Il Nuovo Cim. A* **50**, (1979), 523-554.
- [11] C. Omero and R. Percacci, Generalized non-linear σ -models in curved space and spontaneous compactification, *Nucl. Phys. B* **165** (1980), 351-364.
- [12] M. Gell-Mann and B. Zwiebach, Spacetime compactification induced by scalars, *Phys. Lett. B* **141** (1984), 333-336.
- [13] G. Ghika, Harmonic maps and submersions in local Euclidean gravity coupled to the σ -model, *Rev. Roum. Phys.* **31** (1986), 635-648.

- [14] G. Ghika and A. Corciovei, Static solutions for the sigma-model coupled to gravity, *Rev. Roum. Phys.* **32** (1987), 827-835.
- [15] G. Ghika and M. Visinescu, Four-dimensional σ -model coupled to the metric tensor field, *Il Nuovo Cim. B* **59** (1980), 59-73.
- [16] S. Ianus and M. Visinescu, Spontaneous compactification induced by non-linear dynamics, *Class. Quant. Grav.* **3** (1986), 889-896.
- [17] R. Schimming and T. Hirschmann, Harmonic maps from spacetimes and their coupling to gravitation, *Astron. Nachr.* **309** (1988), 311-321.
- [18] A. P. Whitman, R. J. Knill and W. R. Stoeger, Some harmonic maps on pseudo-Riemannian manifolds, *Internat. J. Theoret. Phys.* **25** (1986), 1139-1153.
- [19] D. DeTurck, Existence of metrics with prescribed Ricci curvature: local theory, *Invent. Math.* **65** (1981), 179-207.
- [20] H. Friedrich and A. D. Rendall, *The Cauchy Problem for the Einstein Equations*, Springer, Berlin (2000).
- [21] A. Lichnerowicz, Radiations en relativite generale, In *Colloque de Royaumont* (1959). Edition CNRS, Paris (1962).
- [22] L. Bel, La radiation gravitationnelle. In: *Colloque de Royaumont* (1959). Edition CNRS, Paris (1962).
- [23] R. Schimming, Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie, *Math. Nachr.* **59** (1974), 129-162.
- [24] B. Fiedler and R. Schimming, Exact solutions of the Bach field equations of general relativity, *Rep. Math. Phys.*, **17** (1980), 15-36.
- [25] T. Deck and R. Schimming, Harmonic maps coupled to the Einstein equation, Preprint Universität Mannheim (1991).

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