



GLOBAL JOURNAL OF SCIENCE FRONTIER RESEARCH
PHYSICS AND SPACE SCIENCE
Volume 13 Issue 1 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Reference Frame and Lorentz Transformation

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GJSFR-A Classification : FOR Code: 029904, 029999



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I. GEOMETRIC OBJECT AND INVARIANCE PRINCIPLE

A measurement of a spatial interval and a time length is one of the major tasks of general relativity. This is a physical process that allows the study of geometry in a certain area of spacetime. From a geometric point of view, the observer uses an orthonormal basis in a tangent plane as his measurement tool because an orthonormal basis leads to the simplest local geometry. When the observer moves from point to point he brings his measurement tool with him.

Notion of a geometric object is closely related with physical values measured in space time. The invariance principle allows expressing physical laws independently from the selection of a basis. On the other hand if we want to examine a relationship obtained in the test, we need to select measurement tool. In our case this is basis. Choosing the basis we can define coordinates of the geometric object corresponding to studied physical value. Hence we can define the measured value.

Every reference frame is equipped by anholonomic coordinates. For instance, synchronization of a reference frame is an anholonomic time coordinate. Simple calculations show how synchronization influences time measurement in the vicinity of the Earth. Measurement of Doppler shift from the star orbiting the black hole helps to determine mass of the black hole.

Sections 8 and 9 show importance of calculations in orthogonal basis. Coordinates that we use in event space are just labels and calculations that we make in coordinates may appear not reliable. For instance in papers [8, 9] authors determine coordinate speed of light. This leads to not reliable answer and as consequence of this to the difference of speed of light in different directions.

Paper [10] drew my attention. To explain anomalous acceleration of Pioneer 10 and Pioneer 11 ([6]) Antonio Ranada incorporated the old Einstein's view on nature of gravitational field and considered Einstein's idea about variability of speed of light. When Einstein started to study the gravitational field he tried to keep the Minkowski geometry, therefore he assumed that scale of space and time does not change. As result he had to accept the idea that speed of light should vary in gravitational field. When Grossman introduced Riemann geometry to Einstein, Einstein realized that the initial idea was wrong and Riemann geometry solves his problem better. Einstein never returned to idea about variable speed of light.

Indeed, three values: scale of length and time and speed of light are correlated in present theory and we cannot change one without changing another. The presence of gravitational field changes this relation. We have two choices. We keep a priori given geometry (here, Minkowski geometry) and we accept that the speed of light changes from point to point. The Riemann geometry gives us another option. Geometry becomes the result of observation and the measurement tool may change from point to point. In this case we can keep the speed of light constant. Geometry becomes a background which depends on physical processes. Physical laws become background independent.

II. TORSION TENSOR IN GENERAL RELATIVITY

Close relationship between the metric tensor and the connection is the basis of the Riemann geometry. At the same time, connection and metric as any geometric object are objects of measurement. When Hilbert derived Einstein equation, he introduced

the lagrangian where the metric tensor and the connection are independent. Later on, Hilbert discovered that the connection is symmetric and found dependence between connection and metric tensor. One of the reasons is in the simplicity of the lagrangian.

Since an errors of measurement are inescapable, analysis of quantum field theory shows that either symmetry of connection or dependence between connection and metric may be broken. This assumption leads to metric-affine manifold which is space with torsion and nonzero covariant derivative of metric (section 12).

The effects of torsion are cumulative. They may be small but measurable. We can observe their effects not only in strong fields like black hole or Big Bang but in regular conditions as well. Studying geometry and dynamics of point particle gives us a way to test this point of view. There is mind to test this theory in condition when spin of quantum field is accumulated.

To test if the spacetime has the torsion we can test the opportunity to build a parallelogram in spacetime. We can get two particles or two photons that start their movement from the same point and using a mirror to force them to move along opposite sides of the parallelogram. We can start this test when we do not have quantum field and then repeat the test in the presence of quantum field. If particles meet in the same place or we have the same interference then we have torsion equal 0 in this thread. In particular, the torsion may influence the behaviour of virtual particles.

III. TIDAL ACCELERATION

Observations in Solar system and outside are very important. They give us an opportunity to see where general relativity is right and to find out its limitation. It is very important to be very careful with such observations. NASA provided very interesting observation of Pioneer 10 and Pioneer 11 and managed complicated calculation of their accelerations. However, one interesting question arises: what kind of acceleration did we measure?

Pioneer 10 and Pioneer 11 performed free movement in solar system. Therefore they move along their trajectory without acceleration. However, it is well known that two bodies moving along close geodesics have relative acceleration that we call tidal acceleration. Tidal acceleration in general relativity has form

$$\frac{D^2 \delta x^k}{ds^2} = R_{lnk}^k \delta x^k v^n v^l \quad (3.1)$$

where v^l is the speed of body 1 and δx^k is the deviation of geodesic of body 2 from geodesic of body 1. We see from this expression that tidal acceleration depends on movement of body 1 and how the trajectory of body 2 deviates from the trajectory of body 1. But this means that even for two bodies that are at the same distance from a central body we can measure different acceleration relative an observer.

Section 14 is dedicated to the problem what kind of changes the tidal force experiences on metric affine manifolds.

Finally the question arises. Can we use equation (14.5) to measure torsion? We get tidal acceleration from direct measurement. There is a method to measure curvature (see for instance [5]). However, even if we know the acceleration and curvature we still have differential equation to find torsion. However, this way may give direct answer to the question of whether torsion exists or not.

Deviation from tidal acceleration (3.1) predicted by general relativity may have different reason. However we can find answer by combining different type of measurement.

IV. REFERENCE FRAME ON MANIFOLD

When we study manifold $\mathcal{V}$ the geometry of tangent space is one of important factors. In this section, we will make the following assumption.

- All tangent spaces have the same geometry.
- Tangent space is vector space $\mathcal{V}$ of finite dimension n .
- Symmetry group of tangent space is Lie group G .

Any homomorphism of the vector space maps one basis into another. Thus we can extend a representation of the symmetry group to the set of bases. However not every two bases can be mapped by a transformation from the symmetry group because not every nonsingular linear transformation belongs to the representation of group G . The basis that belong to selected orbit of group G is called G -basis.

Definition 4.1 Set $\bar{e} = \langle e_{(i)}, i \in I \rangle$ of vector fields $e_{(i)}$ is called G -reference frame, if for any $x \in \mathcal{V}$ set $\bar{e}(x) = \langle e_{(i)}(x), i \in I \rangle$ is a G -basis in tangent space T_x .

Vector field a has expansion

$$a = a^{(i)} e_{(i)} \quad (4.1)$$

relative reference frame $\bar{e}$.

If we do not limit definition of a reference frame by symmetry group, then at each point of the manifold we can select reference frame $\bar{\partial} = \langle \partial_i \rangle$ based on vector fields tangent to lines $x^i = const$. We call this field of bases the **coordinate reference frame**. Vector field a has expansion

$$a = a^i \partial_i \quad (4.2)$$

relative coordinate reference frame. Then standard coordinates of reference frame $\bar{e}$ have form $e_{(i)}^k$

$$e_{(i)} = e_{(i)}^k \partial_k \quad (4.3)$$

Because vectors $e_{(i)}$ are linearly independent at each point matrix $\|e_{(i)}^k\|$ has inverse matrix $\|e_k^{(i)}\|$

$$\partial_k = e_k^{(i)} e_{(i)} \quad (4.4)$$

We use also a more extensive definition for reference frame on manifold, presented in form $\bar{e} = (e_{(k)}, e^{(k)})$ where we use the set of vector fields $e_{(k)}$ and dual forms $e^{(k)}$ such that

$$e^{(k)}(e_{(l)}) = \delta_{(l)}^{(k)} \quad (4.5)$$

at each point. Forms $e^{(k)}$ are defined uniquely from (4.5).

In a similar way, we can introduce a coordinate reference frame (∂_i, ds^i) . These reference frames are linked by the relationship

$$e_{(k)} = e_{(k)}^i \partial_i \quad (4.6)$$

$$e^{(k)} = e_i^{(k)} dx^i \quad (4.7)$$

From equations (4.6), (4.7), (4.5) it follows

$$e_i^{(k)} e_{(l)}^i = \delta_{(l)}^{(k)} \quad (4.8)$$

In particular we assume that we have $GL(n)$ -reference frame (∂, dx) raised by n differentiable vector fields ∂_i and 1-forms dx^i , that define field of bases ∂ and cobases dx dual them.

If we have function φ on V then we define **paffian derivative**

$$d\varphi = \partial_i \varphi dx^i$$

V. REFERENCE FRAME IN EVENT SPACE

Starting from this section, we consider orthogonal reference frame $\bar{e} = (e_{(k)}, e^{(k)})$ in Riemann space with metric g_{ij} . According to definition, at each point of Riemann space vector fields of orthogonal reference frame satisfy to equation

$$g_{ij} e_{(k)}^i e_{(l)}^j = g_{(k)(l)}$$

where $g_{(k)(l)} = 0$, if $(k) \neq (l)$, and $g_{(k)(k)} = 1$ or $g_{(k)(k)} = -1$ depending on signature of metric.

We can define the **reference frame in event space V** as $O(3,1)$ -reference frame. To enumerate vectors, we use index $k = 0, \dots, 3$. Index $k = 0$ corresponds to time like vector field.

Remark 5.1. We can prove the existence of a reference frame using the orthogonalization procedure at every point of space time. From the same procedure we get that coordinates of basis smoothly depend on the point.

A smooth field of time like vectors of each basis defines congruence of lines that are tangent to this field. We say that each line is a world line of an observer or a **local reference frame**. Therefore a reference frame is set of local reference frames.

We define the **Lorentz transformation** as transformation of a reference frame

$$x'^i = f^i(x^0, x^1, x^2, x^3) \\ e'^i_{(k)} = a_j^i b_{(k)}^{(l)} e_{(l)}^j \quad (5.1)$$

where

$$a_j^i = \frac{\partial x'^i}{\partial x^j}$$

$$\delta_{(i)(l)} b_{(j)}^{(i)} b_{(k)}^{(l)} = \delta_{(j)(k)}$$

We call the transformation a_j^i the holonomic part and transformation $b_{(k)}^{(l)}$ the anholonomic part.

VI. ANHOLONOMIC COORDINATES

Let $E(V, G, \pi)$ be the principal bundle, where V is the differential manifold of dimension n and class not less than 2. We also assume that G is symmetry group of tangent plain.

We define connection form on principal bundle

$$\omega^L = \lambda_N^L da^N + \Gamma_i^L dx^i \quad \omega = \lambda_N da^N + \Gamma dx \quad (6.1)$$

We call functions Γ_i connection components.

If fiber is group $GL(n)$, then connection has form

$$\omega_b^a = \Gamma_{bc}^a dx^c \\ \Gamma_i^A = \Gamma_{bi}^a \quad (6.2)$$

A vector field a has two types of coordinates: **holonomic coordinates** a^i relative coordinate reference frame and **anholonomic coordinates** $a^{(i)}$ relative reference frame. These two forms of coordinates also hold the relation

$$a^i(x) = e_{(i)}^i(x) a^{(i)}(x) \quad (6.3)$$

at any point x .

We can study parallel transfer of vector fields using any form of coordinates. Because (5.1) is a linear transformation we expect that parallel transfer in anholonomic coordinates has the same representation as in holonomic coordinates. Hence we write

$$da^k = -\Gamma_{ij}^k a^i dx^j$$

$$da^{(k)} = -\Gamma_{(i)(j)}^{(k)} a^{(i)} dx^{(j)}$$

It is required to establish link between **holonomic coordinate of connection** Γ_{ij}^k and **anholonomic coordinates of connection** $\Gamma_{(i)(j)}^{(k)}$

$$a^i(x+dx) = a^i(x) + da^i = a^i(x) - \Gamma_{kp}^i a^k(x) dx^p \quad (6.4)$$

$$a^{(i)}(x+dx) = a^{(i)}(x) + da^{(i)} = a^{(i)}(x) - \Gamma_{(k)(p)}^{(i)} a^{(k)}(x) dx^{(p)} \quad (6.5)$$

Considering (6.4), (6.5), and (6.3) we get

$$\begin{aligned} & a^i(x) - \Gamma_{kp}^i a^k(x) dx^p \\ &= e_{(i)}^i(x+dx) \left(a^{(i)}(x) - \Gamma_{(k)(p)}^{(i)} e_i^{(k)}(x) a^i(x) e_p^{(p)}(x) dx^p \right) \end{aligned} \quad (6.6)$$

It follows from (6.6) that

$$\begin{aligned} \Gamma_{(k)(p)}^{(i)} e_i^{(k)}(x) e_p^{(p)}(x) a^i(x) dx^p &= a^{(i)}(x) - e_i^{(i)}(x+dx) (a^i(x) - \Gamma_{kp}^i a^k(x) dx^p) \\ &= a^i(x) e_i^{(i)}(x) - e_i^{(i)}(x+dx) (a^i(x) - \Gamma_{kp}^i a^k(x) dx^p) \\ &= a^i(x) \left(e_i^{(i)}(x) - e_i^{(i)}(x+dx) \right) + e_j^{(i)}(x) \Gamma_{ip}^j a^i(x) dx^p \\ &= e_j^{(i)}(x) \Gamma_{ip}^j a^i(x) dx^p - a^i(x) \frac{\partial e_i^{(i)}(x)}{\partial x^p} dx^p \\ &= \left(e_j^{(i)}(x) \Gamma_{ip}^j - \frac{\partial e_i^{(i)}(x)}{\partial x^p} \right) a^i(x) dx^p \end{aligned}$$

Because $a^i(x)$ and dx^p are arbitrary we get

$$\Gamma_{(k)(p)}^{(i)} e_i^{(k)}(x) e_p^{(p)}(x) = e_j^{(i)}(x) \Gamma_{ip}^j - \frac{\partial e_i^{(i)}(x)}{\partial x^p}$$

$$\Gamma_{(k)(p)}^{(i)} = e_{(k)}^i e_{(p)}^p e_j^{(i)} \Gamma_{ip}^j - e_{(k)}^i e_{(p)}^p \frac{\partial e_i^{(i)}}{\partial x^p} \quad (6.7)$$

We introduce symbolic operator

$$\frac{\partial}{\partial x^{(p)}} = e_{(p)}^p \frac{\partial}{\partial x^p} \quad (6.8)$$

From (4.8) it follows

$$e_{(l)}^i \frac{\partial e_i^{(k)}}{\partial x^p} + e_i^{(k)} \frac{\partial e_{(l)}^i}{\partial x^p} = 0 \quad (6.9)$$

We substitute (6.8) and (6.9) into (6.7)

$$\Gamma_{(k)(p)}^{(i)} = e_{(k)}^i e_{(p)}^p e_j^{(i)} \Gamma_{ip}^j - e_i^{(i)} \frac{\partial e_i^{(k)}}{\partial x^{(p)}} \quad (6.10)$$

Equation (6.10) shows some similarity between holonomic and anholonomic coordinates. We introduce symbol $\partial_{(k)}$ for the derivative along vector field $e_{(k)}$

$$\partial_{(k)} = e_{(k)}^i \partial_i$$

Then (6.10) takes the form

$$\Gamma_{(l)(p)}^{(k)} = e_{(l)}^i e_{(p)}^r e_j^{(k)} \Gamma_{ir}^j - e_{(l)}^i \partial_{(p)} e_i^{(k)}$$

Therefore when we move from holonomic coordinates to anholonomic, the connection transforms the way similarly to when we move from one coordinate system to another. This leads us to the model of anholonomic coordinates.

The vector field $e_{(k)}$ generates lines defined by the differential equations

$$e_{(l)}^j \frac{\partial t}{\partial x^j} = \delta_{(l)}^{(k)}$$

or the symbolic system

$$\frac{\partial t}{\partial x^{(l)}} = \delta_{(l)}^{(k)} \quad (6.11)$$

Keeping in mind the symbolic system (6.11) we denote the functional t as $x^{(k)}$ and call it the **anholonomic coordinate**. We call the regular coordinate holonomic.

From here we can find derivatives and get

$$\frac{\partial x^{(i)}}{\partial x^k} = e_k^{(i)} \quad (6.12)$$

The necessary and sufficient conditions of complete integrability of system (6.12) are

$$c_{(k)(l)}^{(i)} = 0$$

where we introduced anholonomy object

$$c_{(k)(l)}^{(i)} = e_{(k)}^k e_{(l)}^l \left(\frac{\partial e_k^{(i)}}{\partial x^l} - \frac{\partial e_l^{(i)}}{\partial x^k} \right) \quad (6.13)$$

Therefore each reference frame has n vector fields

$$\partial_{(k)} = \frac{\partial}{\partial x^{(k)}} = e_{(k)}^i \partial_i$$

which have commutator

$$[\partial_{(i)}, \partial_{(j)}] = \left(e_{(i)}^k \partial_k e_{(j)}^l - e_{(j)}^k \partial_k e_{(i)}^l \right) e_l^{(m)} \partial_{(m)} = e_{(i)}^k e_{(j)}^l \left(-\partial_k e_l^{(m)} + \partial_l e_k^{(m)} \right) \partial_{(m)} = c_{(k)(l)}^{(m)} \partial_{(m)}$$

For the same reason we introduce forms

$$dx^{(k)} = e^{(k)} = e_l^{(k)} dx^l$$

and an exterior differential of this form is

$$\begin{aligned} d^2 x^{(k)} &= d \left(e_i^{(k)} dx^i \right) \\ &= \left(\partial_j e_i^{(k)} - \partial_i e_j^{(k)} \right) dx^i \wedge dx^j \\ &= -c_{(k)(l)}^{(m)} dx^{(k)} \wedge dx^{(l)} \end{aligned} \quad (6.14)$$

Therefore when $c_{(k)(l)}^{(i)} \neq 0$, the differential $dx^{(k)}$ is not an exact differential and the system (6.12) in general cannot be integrated. However we can create a meaningful object that models the solution. We can study how function $x^{(i)}$ changes along different lines. We call such coordinates **anholonomic coordinates on manifold**.

Remark 6.1. Function $x^{(i)}$ is a natural parameter along a flow line of vector field $\bar{e}_{(i)}$. The time coordinate along a local reference frame is the observer's proper time. Because the reference frame consists of local reference frames, we expect that their proper times are synchronized.

We introduce Ricci tensor

$$R_{ij}^D = R_{bij}^a = \partial_i \Gamma_{bj}^a - \partial_j \Gamma_{bi}^a + \Gamma_{ci}^a \Gamma_{bj}^c - \Gamma_{cj}^a \Gamma_{bi}^c + \Gamma_{bk}^a c_{ij}^k \quad (6.17)$$

$$R_{bj} = R_{ba j}^a = \partial_a \Gamma_{bj}^a - \partial_j \Gamma_{ba}^a + \Gamma_{ca}^a \Gamma_{bj}^c - \Gamma_{cj}^a \Gamma_{ba}^c + \Gamma_{bk}^a c_{aj}^k$$

VII. ANHOLONOMIC COORDINATES IN CENTRAL BODY GRAVITATIONAL FIELD

We will study an observer orbiting around a central body. The results are only estimation and are good when eccentricity is near 0 because we study circular orbits. However, the main goal of this estimation

We introduce the **synchronization of reference frame** as the anholonomic time coordinate.

Because synchronization is the anholonomic coordinate it introduces new physical phenomena that we should keep in mind when working with strong gravitational fields or making precise measurements. I describe one of these phenomena in sections 7 - 11. To synchronize clocks of local reference frames we use classical procedure of exchange light signals.

Somebody may have impression that we cannot synchronize clock, however this conflicts with our observation. We accept that synchronization is possible until we introduce time along non closed lines. Synchronization breaks when we try synchronize clocks along closed line. Namely, a change of function along a loop is

$$\begin{aligned} \Delta x^{(i)} &= \oint dx^{(i)} \\ &= \int \int c_{(k)(l)}^{(i)} dx^{(k)} \wedge dx^{(l)} \\ &= \int \int c_{(k)(l)}^{(i)} e_k^{(k)} e_l^{(l)} dx^k \wedge dx^l \end{aligned} \quad (6.15)$$

This means ambiguity in definition of anholonomic coordinates.

From now on we will not make a difference between holonomic and anholonomic coordinates. Also, we will denote $b_{(k)}^{(l)}$ as $a^{-1}_{(k)}^{(l)}$ in the Lorentz transformation (5.1).

We define the curvature form for connection (6.1)

$$\begin{aligned} \Omega &= d\omega + [\omega, \omega] \\ \Omega^D &= d\omega^D + C_{AB}^D \omega^A \wedge \omega^B = R_{ij}^D dx^i \wedge dx^j \end{aligned}$$

where we defined a curvature object

$$R_{ij}^D = \partial_i \Gamma_j^D - \partial_j \Gamma_i^D + C_{AB}^D \Gamma_i^A \Gamma_j^B + \Gamma_k^D c_{ij}^k$$

The curvature form for the connection (6.2) is

$$\Omega_c^a = d\omega_c^a + \omega_b^a \wedge \omega_c^b \quad (6.16)$$

where we defined a curvature object

is to show that we have a measurable effect of anholonomy.

We use the Schwarzschild metric of a central body

$$\begin{aligned} ds^2 &= \frac{r-r_g}{r} c^2 dt^2 - \frac{r}{r-r_g} dr^2 - r^2 d\phi^2 - r^2 \sin^2 \phi d\theta^2 \\ r_g &= \frac{2Gm}{c^2} \end{aligned} \quad (7.1)$$

G is the gravitational constant, m is the mass of the central body, c is the speed of light.

Connection in this metric is

$$\left\{ \begin{array}{ll} \Gamma_{10}^0 = \frac{r_g}{2r(r-r_g)} & \\ \Gamma_{00}^1 = \frac{r_g(r-r_g)}{2r^3} & \Gamma_{11}^1 = -\frac{r_g}{2r(r-r_g)} \\ \Gamma_{22}^1 = -(r-r_g) & \Gamma_{33}^1 = -(r-r_g)\sin^2\phi \\ \Gamma_{12}^2 = -\frac{1}{r} & \Gamma_{33}^2 = -\sin\phi\cos\phi \\ \Gamma_{13}^3 = -\frac{1}{r} & \Gamma_{23}^3 = \cot\phi \end{array} \right. \quad (7.2)$$

I want to show one more way to calculate the Doppler shift. The Doppler shift in gravitational field is well known issue, however the method that I show is useful to better understand physics of gravitational field.

Then the equation (7.3) gets form

$$\begin{aligned} dk^0 &= -\Gamma_{10}^0(k^1 dt + k^0 dr) \\ d\left(\frac{\omega}{c}\sqrt{\frac{r}{r-r_g}}\right) &= -\frac{r_g\omega}{2r(r-r_g)}\left(\sqrt{\frac{r-r_g}{r}}\frac{r}{r-r_g} + \sqrt{\frac{r}{r-r_g}}\right)\frac{dr}{c} \\ d\omega\sqrt{\frac{r}{r-r_g}} - \omega\frac{1}{2}\sqrt{\frac{r-r_g}{r}}\frac{r_g dr}{(r-r_g)^2} &= -\frac{r_g\omega dr}{r(r-r_g)}\sqrt{\frac{r}{r-r_g}} \\ \frac{d\omega}{\omega} &= -\frac{r_g}{2r(r-r_g)}dr \\ \ln\omega &= \frac{1}{2}\ln\frac{r}{r-r_g} + \ln C \end{aligned}$$

If we define $\omega = \omega_0$ when $r = \infty$, we get finally

$$\omega = \omega_0\sqrt{\frac{r}{r-r_g}}$$

VIII. TIME DELAY IN CENTRAL BODY GRAVITATIONAL FIELD

We will study orbiting around a central body. The results are only an estimation and are good when eccentricity is near 0 because we study circular orbits. However, the main goal of this estimation is to show that we have a measurable effect of anholonomy.

Let us compare the measurements of two observers. The first observer fixed his position in the gravitational field

$$t = \frac{s}{c}\sqrt{\frac{r}{r-r_g}}$$

$$r = \text{const}, \phi = \text{const}, \theta = \text{const}$$

The second observer orbits the center of the field with constant speed

We can describe the movement of photon in gravitational field using its wave vector k^i . The length of this vector is 0; $\frac{k^i}{dx^i} = \text{const}$; a trajectory is geodesic and therefore coordinates of this vector satisfy to differential equation

$$dk^i = -\Gamma_{kl}^i k^k dx^l \quad (7.3)$$

We looking for the frequency ω of light and k^0 is proportional ω . Let us consider the radial movement of a photon. In this case wave vector has form $k = (k^0, k^1, 0, 0)$. In the central field with metric (7.1) we can choose

$$\begin{aligned} k^0 &= \frac{\omega}{c}\sqrt{\frac{r}{r-r_g}} \\ k^1 &= \omega\sqrt{\frac{r-r_g}{r}} \\ dt &= \frac{k^0}{k^1}dr = \frac{1}{c}\frac{r}{r-r_g}dr \end{aligned}$$

$$t = s\sqrt{\frac{r}{(r-r_g)c^2 - \alpha^2 r^3}}$$

$$\phi = \alpha s\sqrt{\frac{r}{(r-r_g)c^2 - \alpha^2 r^3}}$$

$$r = \text{const}, \theta = \text{const}$$

We choose a natural parameter for both observers.

The second observer starts his travel when $s = 0$ and finishes it when returning to the same spatial point. Because ϕ is a cyclic coordinate the second observer finishes his travel when $\phi = 2\pi$. We have at this point

$$\begin{aligned} s_2 &= \frac{2\pi}{\alpha}\sqrt{\frac{(r-r_g)c^2 - \alpha^2 r^3}{r}} \\ t &= T = \frac{2\pi}{\alpha} \end{aligned}$$

The value of the natural parameter for the first observer at this point is

$$s_1 = \frac{2\pi}{\alpha}c\sqrt{\frac{r-r_g}{r}}$$

The difference between their proper times is

$$\Delta s = s_1 - s_2 = \frac{2\pi}{\alpha} \left(c\sqrt{\frac{r-r_g}{r}} - \sqrt{\frac{(r-r_g)c^2 - \alpha^2 r^3}{r}} \right)$$

We have a difference in centimeters. To get this difference in seconds we should divide both sides by c .

$$\Delta t = \frac{2\pi}{\alpha} \left(\sqrt{\frac{r-r_g}{r}} - \sqrt{\frac{r-r_g}{r} - \frac{\alpha^2 r^2}{c^2}} \right)$$

Now we get specific data.

The mass of the Sun is $1.989_{10}33$ g, the Earth orbits the Sun at a distance of $1.495985_{10}13$ cm from its center during 365.257 days. In this case we get $\Delta t = 0.155750625445089$ s. Mercury orbits the Sun at a distance of $5.791_{10}12$ cm from its center during 58.6462 days. In this case we get $\Delta t = 0.145358734930827$ s.

The mass of the Earth is $5.977_{10}27$ g. The spaceship that orbits the Earth at a distance of $6.916_{10}8$ cm from its center during 95.6 mins has $\Delta t = 1.8318_{10} - 6$ s. The Moon orbits the Earth at a distance of $3.84_{10}10$ cm from its center during 27.32 days. In this case we get $\Delta t = 1.372_{10} - 5$ s.

For better presentation I put these data to tables 8.1, 8.2, and 8.3.

Because clocks of first observer show larger time at meeting, first observer estimates age of second one older then real. Hence, if we get parameters of S2 orbit from [1], we get that if first observer estimates age of S2 as 10 Myr then S2 will be .297 Myr younger.

Table 8.1 : Sun is central body, mass is $1.989_{10}33$ g

Sputnik	Earth	Mercury
distance, cm	$1.495985_{10}13$	$5.791_{10}12$
orbit period, days	365.257	58.6462
Time delay s	0.15575	0.14536

Table 8.2 : Earth is central body, mass is $5.977_{10}27$ g

Sputnik	spaceship	Moon
distance, cm	$6.916_{10}8$	$3.84_{10}10$
orbit period	95.6 mins	27.32 days
Time delay, s	$1.8318_{10} - 6$	$1.372_{10} - 5$

Table 8.3 : Sgr A is central body, S2 is sputnik

mass, $M_{\odot}$	$4.1_{10}6$	$3.7_{10}6$
distance sm	$1.4692_{10}16$	$1.1565_{10}16$
orbit period, years	15.2	15.2
Time delay, min	164.7295	153.8326

IX. LORENTZ TRANSFORMATION IN ORBITAL DIRECTION

The reason for the time delay that we estimated above is in Lorentz transformation between stationary and orbiting observers. This means that we have rotation in plain $(e_{(0)}, e_{(2)})$. The basis vectors for stationary observer are

$$e_{(0)} = \left(\frac{1}{c} \sqrt{\frac{r}{r-a}}, 0, 0, 0 \right)$$

$$e_{(2)} = \left(0, 0, \frac{1}{r}, 0 \right)$$

We assume that for orbiting observer changes of ϕ and t are proportional and

$$d\phi = \omega dt$$

Unit vector of speed in this case should be proportional to vector

$$(1, 0, \omega, 0) \quad (9.1)$$

The length of this vector is

$$L^2 = \frac{r-a}{r} c^2 - r^2 \omega^2 \quad (9.2)$$

We see in this expression very familiar pattern and expect that linear speed of orbiting observer is $V = \omega r$.

However we have to remember that we make measurement in gravitational field and coordinates are just tags to label points in spacetime. This means that we need a legal method to measure speed.

If an object moves from point (t, ϕ) to point $(t+dt, \phi+d\phi)$ we need to measure spatial and time intervals between these points. We assume that in both points there are observers A and B . Observer A sends the same time light signal to B and ball that has angular speed ω . Whatever observer B receives, he sends light signal back to A .

When A receives first signal he can estimate distance to B . When A receives second signal he can estimate how long ball moved to B .

The time of travel of light in both directions is the same. Trajectory of light is determined by equation $ds^2 = 0$. In our case we have

$$\frac{r-r_g}{r} c^2 dt^2 - r^2 d\phi^2 = 0$$

When light returns back to observer A the change of t is

$$dt = 2 \sqrt{\frac{r}{r-r_g}} c^{-1} r d\phi$$

The proper time of first observer is

$$ds^2 = \frac{r-r_g}{r} c^2 4 \frac{r}{r-r_g} c^{-2} r^2 d\phi^2$$

Therefore spatial distance is

$$L = r d\phi$$

When object moving with angular speed ω gets to B change of t is $\frac{d\phi}{\omega}$. The proper time at this point is

$$ds^2 = \frac{r-r_g}{r} c^2 d\phi^2 \omega^{-2}$$

$$L = \sqrt{\frac{r-r_g}{r} \left(c^2 - \frac{r}{r-r_g} r^2 \omega^2 \right)} = \sqrt{\frac{r-r_g}{r}} c \sqrt{\left(1 - \frac{V^2}{c^2} \right)}$$

Therefore time ort of moving observer is

$$e'_{(0)} = \left(\frac{1}{L}, 0, \frac{\omega}{L}, 0 \right)$$

$$e'_{(0)} = \left(\sqrt{\frac{r}{r-r_g}} c^{-1} \frac{1}{\sqrt{\left(1 - \frac{V^2}{c^2} \right)}}, 0, \omega \sqrt{\frac{r}{r-r_g}} c^{-1} \frac{1}{\sqrt{\left(1 - \frac{V^2}{c^2} \right)}}, 0 \right)$$

Spatial ort $e'_{(2)} = (A, 0, B, 0)$ is orthogonal $e'_{(0)}$ and has length -1 . Therefore

$$\frac{r-r_g}{r} c^2 \frac{1}{L} A - r^2 \frac{\omega}{L} B = 0 \quad (9.3)$$

$$\frac{r-r_g}{r} c^2 A^2 - r^2 B^2 = -1 \quad (9.4)$$

We can express A from (9.3)

$$A = c^{-2} \frac{r}{r-r_g} r^2 \omega B$$

and substitute into (9.4)

$$c^{-2} \frac{r}{r-r_g} r^4 \omega^2 B^2 - r^2 B^2 = -1$$

$$\frac{V^2}{c^2} r^2 B^2 - r^2 B^2 = -1$$

Finally spatial ort in direction of movement is

$$e'_{(2)} = \left(c^{-2} \frac{r}{r-r_g} r \omega \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, 0, \frac{1}{r} \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, 0 \right)$$

$$e'_{(2)} = \left(c^{-2} \sqrt{\frac{r}{r-r_g}} \frac{V}{\sqrt{1 - \frac{V^2}{c^2}}}, 0, \frac{1}{r} \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, 0 \right)$$

Therefore we get transformation

$$e'_{(0)} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} e_{(0)} + \frac{V}{c} \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} e_{(2)}$$

$$T = \sqrt{\frac{r-r_g}{r}} \omega^{-1} d\phi$$

Therefore the observer A measures speed

$$V = \frac{L}{T} = \sqrt{\frac{r}{r-r_g}} r \omega$$

We can use speed V as parameter of Lorentz transformation. Then length (9.2) of vector (9.1) is

$$e'_{(2)} = \frac{V}{c} \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} e_{(0)} + \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} e_{(2)} \quad (9.5)$$

If a stationary observer sends light in a radial direction, the orbiting observer observes Doppler shift

$$\omega' = \frac{\omega}{\sqrt{1 - \frac{V^2}{c^2}}}$$

We need to add Doppler shift for gravitational field if the moving observer receives a radial wave that came from infinity. In this case the Doppler shift will take the form

$$\omega' = \sqrt{\frac{r}{r-r_g}} \frac{\omega}{\sqrt{1 - \frac{V^2}{c^2}}}$$

We see the estimation for dynamics of star S2 that orbits Sgr A in tables 9.1 and 9.2. The tables are based on two different estimations for mass of Sgr A.

If we get mass Sgr $4.1_{10} 6 M_{\odot}$ [1] then in pericentre (distance $1.868_{10} 15$ cm) S2 has speed 738767495.4 cm/s and Doppler shift is $\omega'/\omega = 1.000628$. In this case we measure length $2:16474 \mu m$ for emitted wave with length $2.1661 \mu m$ (Br γ). In apocentre (distance $2.769_{10} 16$ cm) S2 has speed 49839993.28 cm/s and Doppler shift is $\omega'/\omega = 1.0000232$. We measure length $2.166049 \mu m$ for the same wave. Difference between two measurements of wave length is $13.098 \text{ \AA}$.

If we get mass Sgr $3.7_{10} 6 M_{\odot}$ [2] then in pericentre (distance $1.805_{10} 15$ cm) S2 has speed 713915922.3 cm/s and Doppler shift is $\omega'/\omega = 1.000587$. In this case we measure length $2.16483 \mu m$ for emitted

wave with length $2.1661\mu m (Br\gamma)$. In apocentre (distance $2.676_{10}16\text{ cm}$) S2 has speed 48163414.05 cm/s and Doppler shift is $\omega'/\omega = 1.00002171$. We measure length $2.1666052\mu m$ for the same wave. Difference between two measurements of wave length is $12.232\text{ \AA}$.

Difference between two measurements of wavelength in pericentre is $0.9\text{ \AA}$. Analyzing this data we can conclude that the use of Doppler shift can help improve estimation of the mass of Sgr A.

Table 9.1 : Doppler shift on the Earth of a wave emitted from S2; mass of Sgr A is $4.1_{10}6M_{\odot}$ [1]

	pericentre	apocentre
distance cm	$1.868_{10}15$	$2.769_{10}16$
speed cm/s	738767495.4	49839993.28
ω'/ω	1.000628	1.0000232
emitted wave ($Br\gamma$) μm	2.1661	2.1661
observed wave μm	2.16474	2.166049

Difference between two measurements of wavelength is $13.098\text{ \AA}$.

Table 9.2 : Doppler shift on the Earth of wave emitted from S2; mass of Sgr A is $3.7_{10}6M_{\odot}$ [2]

	pericentre	apocentre
distance cm	$1.805_{10}15$	$2.676_{10}16$
speed cm/s	713915922.3	48163414.05
ω'/ω	1.000587	1.00002171
emitted wave ($Br\gamma$) μm	2.1661	2.1661
observed wave μm	2.16483	2.1666052

Difference between two measurements of wavelength is $12.232\text{ \AA}$.

X. LORENTZ TRANSFORMATION IN ORBITAL DIRECTION

We see that the Lorentz transformation in orbital direction has familiar form. It is very interesting to see what form this transformation has for radial direction. We start from procedure of measurement speed and use coordinate speed v

$$dr = v dt \quad (10.1)$$

The time of travel of light in both directions is the same. Trajectory of light is determined by equation $ds^2 = 0$.

$$\frac{r-r_g}{r} c^2 dt^2 - \frac{r}{r-r_g} dr^2 = 0$$

When light returns back to observer A the change of t is

$$dt = 2 \frac{r}{r-r_g} c^{-1} dr$$

The proper time of observer A is

$$ds^2 = \frac{r-r_g}{r} c^2 4 \frac{r^2}{(r-r_g)^2} c^{-2} dr^2 =$$

$$= 4 \frac{r}{r-r_g} dr^2$$

Therefore spatial distance is

$$L = \sqrt{\frac{r}{r-r_g}} dr$$

When object moving with speed (10.1) gets to B change of t is $\frac{dr}{v}$. The proper time of observer A at this point is

$$ds^2 = \frac{r-r_g}{r} c^2 dr^2 v^{-2}$$

$$T = \sqrt{\frac{r-r_g}{r}} v^{-1} dr$$

Therefore the observer A measures speed

$$V = \frac{L}{T} = \frac{\sqrt{\frac{r}{r-r_g}} dr}{\sqrt{\frac{r-r_g}{r}} v^{-1} dr} = \frac{r}{r-r_g} v$$

Now we are ready to find out Lorentz transformation. The basis vectors for stationary observer are

$$e_{(0)} = \left(\frac{1}{c} \sqrt{\frac{r}{r-r_g}}, 0, 0, 0 \right)$$

$$e_{(1)} = \left(0, \sqrt{\frac{r-r_g}{r}}, 0, 0 \right)$$

Unit vector of speed should be proportional to vector

$$(1, v, 0, 0) \quad (10.2)$$

Therefore time ort of moving observer is

$$\begin{aligned} e'_{(0)} &= \left(\frac{1}{L}, \frac{v}{L}, 0, 0 \right) = \\ &= \left(\sqrt{\frac{r}{r-r_g}} c^{-1} \frac{1}{\sqrt{(1-\frac{V^2}{c^2})}}, v \sqrt{\frac{r}{r-r_g}} c^{-1} \frac{1}{\sqrt{(1-\frac{V^2}{c^2})}}, 0, 0 \right) \\ &= \left(\sqrt{\frac{r}{r-r_g}} c^{-1} \frac{1}{\sqrt{(1-\frac{V^2}{c^2})}}, \frac{V}{c} \sqrt{\frac{r-r_g}{r}} \frac{1}{\sqrt{(1-\frac{V^2}{c^2})}}, 0, 0 \right) \end{aligned}$$

Spatial ort $e'_{(1)} = (A, B, 0, 0)$ is orthogonal

$e'_{(0)}$ and has length -1 . Therefore

$$\frac{r-r_g}{r} c^2 \frac{1}{L} A - \frac{r}{r-r_g} \frac{v}{L} B = 0 \quad (10.4)$$

$$\frac{r-r_g}{r} c^2 A^2 - \frac{r}{r-r_g} B^2 = -1 \quad (10.5)$$

We can express A from (10.4)

$$A = c^{-2} \frac{r^2}{(r-r_g)^2} v B = c^{-2} \frac{r}{r-r_g} V B$$

and substitute into (10.5)

$$\frac{r-r_g}{r} c^2 c^{-4} \frac{r^2}{(r-r_g)^2} V^2 B^2 - \frac{r}{r-r_g} B^2 = -1$$

$$\frac{r}{r-r_g} B^2 \left(1 - \frac{V^2}{c^2} \right) = 1$$

$$B^2 = \frac{r-r_g}{r} \frac{1}{1-\frac{V^2}{c^2}}$$

$$B = \sqrt{\frac{r-r_g}{r}} \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$A = c^{-2} \frac{r}{r-r_g} V \sqrt{\frac{r-r_g}{r}} \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= c^{-2} V \sqrt{\frac{r}{r-r_g}} \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$$

Finally spatial ort in direction of movement is

The length of this vector is

$$\begin{aligned} L^2 &= \frac{r-r_g}{r} c^2 - \frac{r}{r-r_g} v^2 = \\ &= \frac{r-r_g}{r} c^2 \left(1 - \frac{V^2}{c^2} \right) \end{aligned} \quad (10.3)$$

$$e'_{(1)} = \left(c^{-2} V \sqrt{\frac{r}{r-r_g}} \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}, \sqrt{\frac{r-r_g}{r}} \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}, 0, 0 \right)$$

Therefore we get transformation in in familiar form

$$\begin{aligned} e'_{(0)} &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} e_{(0)} + \frac{V}{c} \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} e_{(1)} \\ e'_{(1)} &= \frac{V}{c} \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} e_{(0)} + \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} e_{(1)} \end{aligned} \quad (10.6)$$

XI. DOPPLER SHIFT IN FRIEDMAN SPACE

We consider another example in Friedman space. Metric of the space is

$$ds^2 = a^2(dt^2 - d\chi^2 - \sin^2 \chi (d\theta^2 - \sin^2 \theta d\phi^2))$$

for closed model and

$$ds^2 = a^2(dt^2 - d\chi^2 - \sinh^2 \chi (d\theta^2 - \sin^2 \theta d\phi^2))$$

for open one. Connection in this space is (α, β get values 1, 2, 3)

$$\Gamma_{00}^0 = \frac{\dot{a}}{a}$$

$$\Gamma_{\alpha\alpha}^0 = -\frac{\dot{a}}{a^2} g_{\alpha\alpha}$$

$$\Gamma_{0\beta}^\alpha = \frac{\dot{a}}{a} \delta_\beta^\alpha$$

Because space is homogenous we do not care about direction of light. In this case

$$dk^0 = -\Gamma_{ij}^0 k^i k^j$$

Because k is isotropic vector tangent to its trajectory we have

$$dx^\alpha = \frac{k^\alpha}{k^0} dt$$

Because $k^0 = \frac{\omega}{a}$, then

$$\begin{aligned} d\frac{\omega}{a} &= -\frac{da}{a^2}\omega + \frac{da}{\omega a}g_{\alpha\alpha}k^\alpha k^\alpha = -2\frac{da}{a^2}\omega \\ ad\omega + \omega da &= 0 \\ a\omega &= \text{const} \end{aligned}$$

Therefore when a grows ω becomes smaller and length of waves grows as well.

a grows during light travel through spacetime and this leads to red shift. We observe red shift because geometry changes, but not because galaxies runs away one from other.

Now we want to see how red shift changes with time if initial and final points do not move. For simplicity I will change only χ . Initial value is χ_1 and nal value is χ_2 . Because $dt = d\chi$ on light trajectory we have

$$\chi = \chi_1 + t - t_1 \quad t_2 = \chi_2 - \chi_1 + t_1$$

Therefore $a(t_1)\omega_1 = a(t_2)\omega_2$. Doppler shift is

$$T^a = T_{cb}^a dx^c \wedge dx^b = -c_{cb}^a dx^c \wedge dx^b + (\Gamma_{bc}^a - \Gamma_{cb}^a) dx^c \wedge dx^b \quad (12.3)$$

where we defined **torsion tensor**

$$T_{cb}^a = \Gamma_{bc}^a - \Gamma_{cb}^a - c_{cb}^a \quad (12.4)$$

Commutator of second derivatives has form

$$u_{;kl}^\alpha - u_{;lk}^\alpha = R_{\beta lk}^\alpha u^\beta - T_{lk}^p u_{;p}^\alpha \quad (12.5)$$

From (12.5) it follows that

$$\xi_{;cb}^a - \xi_{;bc}^a = R_{dbc}^a \xi^d - T_{bc}^p \xi_{;p}^a \quad (12.6)$$

In Rieman space we have metric tensor g_{ij} and connection Γ_{ij}^k . One of the features of the Rieman space is symetricity of connection and covariant derivative of metric is 0. This creates close relation between metric and connection. However the connection is not necessarily symmetric and the covariant derivative of the metric tensor may be different from 0. In latter case we introduce the **nonmetricity**

$$Q_k^{ij} = g_{;k}^{ij} = g_{,k}^{ij} + \Gamma_{pk}^i g^{pj} + \Gamma_{pk}^j g^{ip} \quad (12.7)$$

Due to the fact that derivative of the metric tensor is not 0, we cannot raise or lower index of a tensor under derivative as we do it in regular Riemann space. Now this operation changes to next

$$a_{;k}^i = g^{ij} a_{j;k} + g_{;k}^{ij} a_j$$

$$K(t_1) = \frac{\omega_2}{\omega_1} = \frac{a(t_1)}{a(t_2)}$$

If initial time changes $t'_1 = t_1 + dt$ then $K(t_1+dt) = a(t_1+dt)/a(t_2+dt)$

Time derivative of K is

$$\dot{K} = \frac{\dot{a}_1 a_2 - a_1 \dot{a}_2}{a_2^2}$$

For closed space $a = \cosh t$. Then $\dot{a} = \sinh t$.

$$\dot{K} = \frac{\sinh t_1 \cosh t_2 - \sinh t_2 \cosh t_1}{\cosh^2 t_2} = \frac{\sinh(t_1 - t_2)}{\cosh^2 t_2}$$

K decreases when t_1 increases.

XII. METRIC-AFFINE MANIFOLD

For connection (6.2) we defined the **torsion form**

$$T^a = d^2 x^a + \omega_b^a \wedge dx^b \quad (12.1)$$

From (6.2) it follows

$$\omega_b^a \wedge dx^b = (\Gamma_{bc}^a - \Gamma_{cb}^a) dx^c \wedge dx^b \quad (12.2)$$

Putting (12.2) and (6.14) into (12.1) we get

This equation for the metric tensor gets the following form

$$g_{;k}^{ab} = -g^{ai} g^{bj} g_{ij;k}$$

Definition 12.1: We call a manifold with a torsion and a nonmetricity the **metric - affine manifold** [3].

Nonmetricity dramatically changes law how orthogonal basis moves in space time. However learning of parallel transport in space with nonmetricity allows us to introduce the Cartan transport and the connection compatible with the metric tensor (section [11]-10). The Cartan transport holds the basis orthonormal and this makes it valuable tool because the observer uses an orthonormal basis as his measurement tool. We will assume that nonmetricity of space time is equal 0.

XIII. GEOMETRIC MEANING OF TORSION

Suppose that a and b are non collinear vectors in a point A (see gure 13.1).

We draw the geodesic L_a through the point A using the vector a as a tangent vector to L_a in the point A . Let τ be the canonical parameter on L_a and

$$\frac{dx^k}{d\tau} = a^k$$

We transfer the vector b along the geodesic L_a from the point A into a point B that defined by any value of the parameter $\tau = \rho > 0$. We mark the result as b' .

We draw the geodesic L_b through the point A using the vector b as a tangent vector to L_b in the point A . Let φ be the canonical parameter on L_b and

$$\frac{dx^k}{d\varphi} = b^k$$

We transfer the vector a along the geodesic L_b from the point A into a point D that defined by any value of the parameter $\varphi = \rho > 0$. We mark the result as a' .

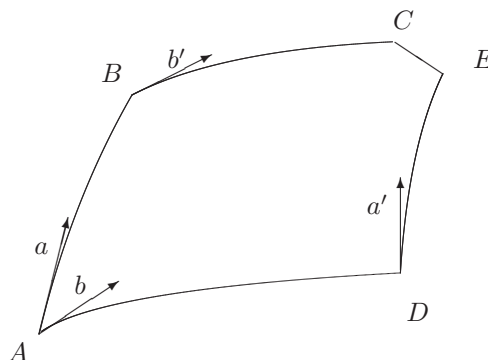


Figure 13.1 : Meaning of Torsion

We draw the geodesic $L_{b'}$ through the point B using the vector b' as a tangent vector to $L_{b'}$ in the point B . Let φ' be the canonical parameter on $L_{b'}$ and

$$\frac{dx^k}{d\varphi'} = b'^k$$

We define a point C on the geodesic $L_{b'}$ by parameter value $\varphi' = \rho$

We draw the geodesic $L_{a'}$ through the point D using the vector a' as a tangent vector to $L_{a'}$ in the point D . Let τ' be the canonical parameter on $L_{a'}$ and

$$\frac{dx^k}{d\tau'} = a'^k$$

We define a point E on the geodesic $L_{a'}$ by parameter value $\tau' = \rho$

Formally lines AB and DE as well as lines AD and BC are parallel lines. Lengths of AB and DE are the same as well as lengths of AD and BC are the same. We call this figure a **parallelogram** based on vectors a and b with the origin in the point A .

Theorem 13.1: Suppose $CBADE$ is a parallelogram with a origin in the point A ; then the resulting figure will not be closed [4]. The value of the difference of coordinates of points C and E is equal to surface integral of the torsion over this parallelogram¹

$$\Delta_{CE}x^k = \iint T_{mn}^k dx^m \wedge dx^n$$

with precision of small value of first level. Putting (13.3) into (13.2) and (13.2) into (13.1) we will receive

$$\Delta_{BC}x^k = b^k \rho - \Gamma_{mn}^k(A) b^m a^n \rho^2 - \frac{1}{2} \Gamma_{mn}^k(B) b^m b^n \rho^2 + O(\rho^2)$$

Proof : We can find an increase of coordinate x^k along any geodesic as

$$\begin{aligned} \Delta x^k &= \frac{dx^k}{d\tau} \tau + \frac{1}{2} \frac{d^2 x^k}{d\tau^2} \tau^2 + O(\tau^2) = \\ &= \frac{dx^k}{d\tau} \tau - \frac{1}{2} \Gamma_{mn}^k \frac{dx^m}{d\tau} \frac{dx^n}{d\tau} \tau^2 + O(\tau^2) \end{aligned}$$

where τ is canonical parameter and we take values of derivatives and components Γ_{mn}^k in the initial point. In particular we have

$$\Delta_{AB}x^k = a^k \rho - \frac{1}{2} \Gamma_{mn}^k(A) a^m a^n \rho^2 + O(\rho^2)$$

along the geodesic L_a and

$$\Delta_{BC}x^k = b'^k \rho - \frac{1}{2} \Gamma_{mn}^k(B) b'^m b'^n \rho^2 + O(\rho^2) \quad (13.1)$$

along the geodesic $L_{b'}$. Here

$$b'^k = b^k - \Gamma_{mn}^k(A) b^m dx^n + O(dx) \quad (13.2)$$

is the result of parallel transfer of b^k from A to B and

$$dx^k = \Delta_{AB}x^k = a^k \rho \quad (13.3)$$

Common increase of coordinate x^K along the way ABC has form

$$\begin{aligned}\Delta_{ABC}x^k &= \Delta_{AB}x^k + \Delta_{BC}x^k = \\ &= (a^k + b^k)\rho - \Gamma_{mn}^k(A)b^mb^n\rho^2 - \\ &\quad - \frac{1}{2}\Gamma_{mn}^k(B)b^mb^n\rho^2 - \frac{1}{2}\Gamma_{mn}^k(A)a^ma^n\rho^2 + O(\rho^2)\end{aligned}\quad (13.4)$$

Similar way common increase of coordinate x^K along the way ADE has form

$$\begin{aligned}\Delta_{ADE}x^k &= \Delta_{AD}x^k + \Delta_{DE}x^k = \\ &= (a^k + b^k)\rho - \Gamma_{mn}^k(A)a^ma^n\rho^2 - \\ &\quad - \frac{1}{2}\Gamma_{mn}^k(D)a^ma^n\rho^2 - \frac{1}{2}\Gamma_{mn}^k(A)b^mb^n\rho^2 + O(\rho^2)\end{aligned}\quad (13.5)$$

From (13.4) and (13.5), it follows that

$$\begin{aligned}\Delta_{ADE}x^k - \Delta_{ABC}x^k &= \\ &= \Gamma_{mn}^k(A)b^mb^n\rho^2 + \frac{1}{2}\Gamma_{mn}^k(B)b^mb^n\rho^2 - \frac{1}{2}\Gamma_{mn}^k(A)a^ma^n\rho^2 - \\ &\quad - \Gamma_{mn}^k(A)a^ma^n\rho^2 - \frac{1}{2}\Gamma_{mn}^k(D)a^ma^n\rho^2 - \frac{1}{2}\Gamma_{mn}^k(A)b^mb^n\rho^2 + O(\rho^2)\end{aligned}$$

For small enough value of ρ underlined terms annihilate each other and we get integral sum for expression

$$\Delta_{ADE}x^k - \Delta_{ABC}x^k = \iint_{\Sigma} (\Gamma_{nm}^k - \Gamma_{mn}^k) dx^m \wedge dx^n$$

However it is not enough to find the difference

$$\Delta_{ADE}x^k - \Delta_{ABC}x^k$$

to find the difference of coordinates of points C and E . Coordinates may be anholonomic and we have to consider that coordinates along closed loop change (6.15)

$$\Delta x^k = \oint_{ECBADE} dx^k = - \iint_{\Sigma} c_{mn}^k dx^m \wedge dx^n$$

where c is anholonomy object.

Finally the difference of coordinates of points C and E is

$$\Delta_{CE}x^k = \Delta_{ADE}x^k - \Delta_{ABC}x^k + \Delta x^k = \iint_{\Sigma} (\Gamma_{nm}^k - \Gamma_{mn}^k - c_{mn}^k) dx^m \wedge dx^n$$

Using (12.4) we prove the statement.

XIV. TIDAL EQUATION

Assume that considered bodies perform not geodesic but arbitrary movement.

We assume that both observers start their travel from the same point² and their speed satisfy to differential equations

$$\frac{Dv_I^i}{ds_I} = a_I^i \quad (14.1)$$

¹Proof of this statement I found in [7]

where $I = 1, 2$ is the number of the observer and ds_I is infinitesimal arc on geodesic I . Observer I follows the geodesic of connection [11]-(10.1) when $a_I = 0$. We assume also that $ds_1 = ds_2 = ds$.

Deviation of trajectories (14.1) δx^k is vector connecting observers. The lines are infinitesimally close in the neighborhood of the start point

$$x_2^i(s_2) = x_1^i(s_1) + \delta x^i(s_1)$$

$$v_2^i(s_2) = v_1^i(s_1) + \delta v^i(s_1)$$

Derivative of vector δx^i has form

$$\frac{d\delta x^i}{ds} = \frac{d(x_2^i - x_1^i)}{ds} = v_2^i - v_1^i = \delta v^i$$

Speed of deviation δx^i is covariant derivative

$$\begin{aligned} \frac{d\delta x^i}{ds} &= \frac{D\delta x^i}{ds} + \Gamma_{kl}^i \delta x^k v_1^l \\ &= \delta v^i + \Gamma_{kl}^i \delta x^k v_1^l \end{aligned} \quad (14.2)$$

From (14.2) it follows that

$$\delta v^i = \frac{D\delta x^i}{ds} - \Gamma_{kl}^i \delta x^k v_1^l \quad (14.3)$$

Finally we are ready to estimate second covariant derivative of vector δx^i

$$\begin{aligned} \frac{D^2\delta x^i}{ds^2} &= \frac{d}{ds} \frac{D\delta x^i}{ds} + \Gamma_{kl}^i \frac{D\delta x^k}{ds} v_1^l \\ &= \frac{d(\delta v^i + \Gamma_{kl}^i \delta x^k v_1^l)}{ds} + \Gamma_{kl}^i \frac{D\delta x^k}{ds} v_1^l \\ &= \frac{d\delta v^i}{ds} + \frac{d\Gamma_{kl}^i}{ds} \delta x^k v_1^l + \Gamma_{kl}^i \frac{d\delta x^k}{ds} v_1^l + \Gamma_{kl}^i \delta x^k \frac{dv_1^l}{ds} + \Gamma_{kl}^i \frac{D\delta x^k}{ds} v_1^l \\ \frac{D^2\delta x^i}{ds^2} &= \frac{d\delta v^i}{ds} + \Gamma_{kl,n}^i v_1^n \delta x^k v_1^l + \Gamma_{kl}^i \delta v^k v_1^l + \Gamma_{kl}^i \delta x^k \frac{dv_1^l}{ds} + \Gamma_{kl}^i \frac{D\delta x^k}{ds} v_1^l \end{aligned} \quad (14.4)$$

Theorem 14.1. Tidal acceleration has form

$$\begin{aligned} \frac{D^2\delta x^i}{ds^2} &= T_{ln}^i \frac{D\delta x^n}{ds} v_1^l + (R_{klm}^i + T_{km;l}^i) \delta x^m v_1^k v_1^l \\ &\quad + a_2^i - a_1^i + \Gamma_{ml}^i \delta x^m a_1^l \end{aligned} \quad (14.5)$$

Proof. The trajectory of observer 1 satisfies equation

$$\frac{Dv_1^i}{ds} = \frac{dv_1^i}{ds} + \Gamma_{kl}^i(x_1) v_1^k v_1^l = a_1^i \quad (14.6)$$

$$\frac{dv_1^i}{ds} = a_1^i - \Gamma_{kl}^i v_1^k v_1^l \quad (14.7)$$

The same time the trajectory of observer 2 satisfies equation

$$\begin{aligned} \frac{Dv_2^i}{ds} &= \frac{dv_2^i}{ds} + \Gamma_{kl}^i(x_2) v_2^k v_2^l \\ &= \frac{d(v_1^i + \delta v^i)}{ds} + \Gamma_{kl}^i(x_1 + \delta x) (v_1^k + \delta v^k) (v_1^l + \delta v^l) \\ &= \frac{dv_1^i}{ds} + \frac{d\delta v^i}{ds} + (\Gamma_{kl}^i + \Gamma_{kl,m}^i \delta x^m) (v_1^k v_1^l + \delta v^k v_1^l + v_1^k \delta v^l + \delta v^k \delta v^l) \\ &= a_2^i \end{aligned}$$

We can rewrite this equation up to order 1

$$\frac{dv_1^i}{ds} + \frac{d\delta v^i}{ds} + \Gamma_{kl}^i v_1^k v_1^l + \Gamma_{kl}^i (\delta v^k v_1^l + v_1^k \delta v^l) + \Gamma_{kl,m}^i \delta x^m v_1^k v_1^l = a_2^i$$

²I follow [5], page 33

Using (14.6) we get

$$\begin{aligned}
 a_1^i + \frac{d\delta v^i}{ds} + \Gamma_{kl}^i \delta v^k v_1^l + \Gamma_{kl}^i v_1^k \delta v^l + \Gamma_{kl,m}^i \delta x^m v_1^k v_1^l &= a_2^i \\
 \frac{d\delta v^i}{ds} &= -\Gamma_{kl}^i \delta v^k v_1^l - \Gamma_{lk}^i \delta v^k v_1^l - \Gamma_{kl,m}^i \delta x^m v_1^k v_1^l + a_2^i - a_1^i
 \end{aligned} \tag{14.8}$$

We substitute (14.3), (14.7), and (14.8) into (14.4)

$$\begin{aligned}
 \frac{D^2 \delta x^i}{ds^2} &= -\Gamma_{kl}^i \delta v^k v_1^l - \Gamma_{ln}^i \left(\frac{D\delta x^n}{ds} - \Gamma_{mk}^n \delta x^m v_1^k \right) v_1^l - \Gamma_{kl,m}^i \delta x^m v_1^k v_1^l \\
 &\quad + a_2^i - a_1^i \\
 &\quad + \Gamma_{mk,l}^i v_1^k \delta x^m v_1^l + \Gamma_{kl}^i \delta v^k v_1^l \\
 &\quad + \Gamma_{mn}^i \delta x^m (a_1^n - \Gamma_{kl}^n v_1^k v_1^l) + \Gamma_{nl}^i \frac{D\delta x^n}{ds} v_1^l \\
 \frac{D^2 \delta x^i}{ds^2} &= (\Gamma_{mk,l}^i - \Gamma_{kl,m}^i + \Gamma_{ln}^i \Gamma_{mk}^n - \Gamma_{mn}^i \Gamma_{kl}^n) \delta x^m v_1^k v_1^l \\
 &\quad + \Gamma_{nl}^i \frac{D\delta x^n}{ds} v_1^l - \Gamma_{ln}^i \frac{D\delta x^n}{ds} v_1^l \\
 &\quad + a_2^i - a_1^i + \Gamma_{mn}^i \delta x^m a_1^n \\
 \frac{D^2 \delta x^i}{ds^2} &= (\Gamma_{mk,l}^i - \Gamma_{km,l}^i + \Gamma_{km,l}^i - \Gamma_{kl,m}^i \\
 &\quad + \Gamma_{ln}^i \Gamma_{mk}^n - \Gamma_{nl}^i \Gamma_{mk}^n + \Gamma_{nl}^i \Gamma_{mk}^n - \Gamma_{nl}^i \Gamma_{km}^n + \Gamma_{nl}^i \Gamma_{km}^n \\
 &\quad - \Gamma_{mn}^i \Gamma_{kl}^n + \Gamma_{nm}^i \Gamma_{kl}^n - \Gamma_{nm}^i \Gamma_{kl}^n) \delta x^m v_1^k v_1^l \\
 &\quad + T_{ln}^i \frac{D\delta x^n}{ds} v_1^l + a_2^i - a_1^i + \Gamma_{mn}^i \delta x^m a_1^n \\
 \frac{D^2 \delta x^i}{ds^2} &= T_{ln}^i \frac{D\delta x^n}{ds} v_1^l \\
 &\quad + (\Gamma_{km,l_2}^i + \Gamma_{km,l_1}^i - \Gamma_{kl,m_1}^i \\
 &\quad + T_{nl}^i \Gamma_{mk_2}^n + \Gamma_{nl}^i T_{km_2}^n + \Gamma_{nl}^i \Gamma_{km_1}^n \\
 &\quad - T_{nm}^i \Gamma_{kl_2}^n - \Gamma_{nm}^i \Gamma_{kl_1}^n) \delta x^m v_1^k v_1^l \\
 &\quad + a_2^i - a_1^i + \Gamma_{ml}^i \delta x^m a_1^l
 \end{aligned} \tag{14.9}$$

Terms underscored with symbol 1 are curvature and terms underscored with symbol 2 are covariant derivative of torsion. (14.5) follows from (14.9).

Remark 14.2 : The body 2 may be remote from body 1. In this case we can use procedure (like in [6]) based on parallel transfer. For this purpose we transport vector of speed of observer 2 to the start point of observer 1 and then estimate tidal acceleration. This procedure works in case of not strong gravitational field.

XV. TIDAL ACCELERATION AND LIE DERIVATIVE

(14.5) reminds expression of Lie derivative [11]-(7.4). To see this similarity we need to write equation (14.5) different way.

By definition

$$\frac{Da^k}{ds} = \frac{da^k}{ds} + \Gamma_{lp}^k a^l \frac{dx^p}{ds}$$

$$= a^k_{;p} v^p + \Gamma^k_{lp} a^l v^p \quad (15.1)$$

$$\frac{Da^k}{ds} = a^k_{;p} v^p$$

Because $\frac{Da^k}{ds}$ is vector, we can easy find second derivative

$$\begin{aligned} \frac{D^2 a^k}{ds^2} &= \frac{D \frac{Da^k}{ds}}{ds} = \frac{D(a^k_{;p} v^p)}{ds} \\ &= a^k_{;pr} v^p v^r + a^k_{;p} v^p_{;r} v^r \end{aligned} \quad (15.2)$$

On the last step we used (15.1) when $a^k = v^k$. When v^p is tangent vector of trajectory of observer 1 from (14.1) it follows that

Proof. We substitute (15.1) and (15.4) into (14.5).

$$\begin{aligned} \delta x^i_{;kl} v^k v^l + \delta x^k_{;p} a^p_1 &= T^i_{ln} \delta x^n_{;k} v^k v^l_1 + (R^i_{klm} + T^i_{km;l}) \delta x^m v^k_1 v^l_1 \\ &\quad + a^i_2 - a^i_1 + \Gamma^i_{ml} \delta x^m a^l_1 \\ 0 &= (T^i_{ln} \delta x^n_{;k} - \delta x^i_{;kl} + R^i_{klm} \delta x^m + T^i_{km;l} \delta x^m) v^k_1 v^l_1 \\ &\quad + a^i_2 - a^i_1 - \delta x^k_{;p} a^p_1 \end{aligned} \quad (15.6)$$

(15.5) follows from (15.6) and [11]-(7.4).

At a first glance one can tell that the speed of deviation of geodesics is the Killing vector of second type. This is an option, however equation

$$\mathcal{L} \frac{D \delta x^n}{ds} \Gamma^i_{kl} = 0$$

does not follow from equation

$$\mathcal{L} \frac{D \delta x^n}{ds} \Gamma^i_{kl} v^k v^l = 0 \quad (15.7)$$

However equation (15.7) shows a close relationship between deep symmetry of spacetime and gravitational field.

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$$\frac{Dv^i}{ds} = v^i_{;r} v^r = a^i_1 \quad (15.3)$$

From (15.2) and (15.3) it follows that

$$\frac{D^2 a^k}{ds^2} = a^k_{;pr} v^p v^r + a^k_{;p} a^p_1 \quad (15.4)$$

Theorem 15.1 : *Speed of deviation of two trajectories (14.1) satisfies equation*

$$\mathcal{L} \frac{D \delta x^n}{ds} \Gamma^i_{kl} v^k v^l = a^i_2 - a^i_1 + \Gamma^i_{ml} \delta x^m a^l_1 \quad (15.5)$$

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SPECIAL SYMBOLS AND NOTATIONS

$a^{(i)}$ anholonomic coordinates of vector 6
 a^i holonomic coordinates of vector 6
 (∂_i, ds^i) coordinate reference frame 5
 $\partial_{(k)}$ derivative $e_{(k)}$ 7
 $\frac{D\delta x^i}{ds}$ speed of deviation 22
 $e^{(k)}$ form of reference frame 4
 $e_{(i)}^k$ standard coordinates of reference frame 4
 $e_{(i)}$ vector field of reference frame 4
 $\bar{e} = \langle e_{(i)}, i \in I \rangle$ reference frame 4
 $\bar{e} = (e_{(k)}, e^{(k)})$ reference frame, extensive definition 4
 $x^{(k)}$ anholonomic coordinate 7
 δx^k deviation of trajectories 22
 $\Gamma_{(i)(j)}^{(k)}$ anholonomic coordinates of connection 6
 Γ_{ij}^k holonomic coordinates of connection 6
 $e_{(k)(l)}^{(i)}$ anholonomy object 7



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