GLOBAL JOURNAL

OF SCIENCE FRONTIER RESEARCH - A: PHYSICS & SPACE SCIENCE

Volume 26 | Issue 2 | 2026

Article ID: GJSFR-258904

Language: English

The Relationship between the Momentum of an Electron and the Momentum of a Photon Possessed by the Electron, Derived from an Energy-Momentum Relationship Applicable to an Electron inside a Hydrogen Atom

Article Record

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RECEIVED	ACCEPTED	ONLINE PUBLISHED	PUBLISHED	PEER REVIEW
2026-06-26	2026-07-09	2026-07-21	2026-09-23	Double Blind

Abstract

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compton wavelength | einstein's energy-momentum relationship | energy-momentum relationship in a hydrogen atom | fine structure constant | momentum of a electron | momentum of a photon

AI USE STATEMENT
No generative AI was used for analysis or results.

FUNDING
No external funding was declared for this work.

CONFLICT OF INTEREST
The authors declare no conflict of interest.

DATA AVAILABILITY
Not applicable for this article.

ETHICS
No ethics committee approval was required for this article type.

CONSENT
Not applicable for this article.

TRIAL REG.
Not applicable.

Crossref DOI: 10.34257/GJSFRA258904

How to Cite: Suto (2026). The Relationship between the Momentum of an Electron and the Momentum of a Photon Possessed by the Electron, Derived from an Energy-Momentum Relationship Applicable to an Electron inside a Hydrogen Atom. Global Journal of Science Frontier Research, 26(2), 1-5. DOI: 10.34257/GJSFRA258904

AR Experience Web Page DOI

Print ISSN 0975-5896

Online ISSN 2249-4626

ARCHIVAL RECORD

GJSFR · Vol 26 · Issue 2 · 2026

Article ID GJSFR-258904 · DOI 10.34257/GJSFRA258904

Print ISSN 0975-5896 · Online ISSN 2249-4626

Global Journal of Science Frontier Research - Section A
GJSFR Volume 26 Issue 2

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Abstract

Einstein's energy-momentum relationship, which holds in an isolated system in free space, cannot be applied to an electron in a hydrogen atom, where potential energy exists. Thus, the author has previously derived an energy-momentum relationship applicable to an electron in a hydrogen atom. It is possible, by using this relationship, to derive the relationships between the momentum of an electron, and the momentum of a photon possessed by the electron. There is a close relationship between these two physical quantities and the fine structure constant. Also, if the de Broglie relationship is applied to the energy-momentum relationship derived by the author, it is possible to derive the relationship between the wavelength of a moving electron, the wavelength of a photon possessed by the electron, and the Compton wavelength of the electron.

Keywords: *compton wavelength, einstein's energy-momentum relationship, energy-momentum relationship in a hydrogen atom, fine structure constant, momentum of a electron, momentum of a photon*

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DOI
10.34257/GJSFRA258904

1. Introduction

N.Bohr was the first to derive the energy levels of a hydrogen atom [1]. Bohr considered the case where an electron moves in a circle at constant speed around the atomic nucleus (a proton).The following formula of motion of Newton holds when an electron moves at velocity v on a circular orbit with radius r.

$$\frac{m_e v^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} \tag{1}$$

This formula indicates that the centrifugal force acting on an electron (left side) is equal to the Coulomb attraction exerted on an electron by the atomic nucleus (right side). Here, m_e is the rest mass of the electron.

Multiplying both sides of Formula (1) by 1/2 yields the following.

$$\frac{1}{2}m_e v^2 = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \tag{2}$$

Incidentally, the potential energy of a hydrogen atom can be expressed with the following formula.

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \tag{3}$$

Therefore, Formula (1) can be written as follows.

$$\frac{1}{2}m_e v^2 = -\frac{1}{2}V(r) \tag{4}$$

Also, in classical quantum theory, the total mechanical energy of a hydrogen atom is defined as the sum of the kinetic energy K and potential energy $V(r)$ of the electron, and thus the energy can be described with the following formula.

$$E = \frac{m_e v^2}{2} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \tag{5}$$

Hence, the energy of a hydrogen atom is given by the following formula.

$$E = -\frac{1}{2}V(r) + V(r) = \frac{1}{2}V(r) = -\frac{1}{2}m_e v^2 = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \tag{6}$$

When an electron with energy $m_e c^2$ is taken into an energy level of a hydrogen atom, the electron emits a photon of energy $h\nu$, but the electron simultaneously acquires the same amount of kinetic energy as $h\nu$. The source which supplies these energies can only be the rest mass energy of the electron.

The author has previously pointed out that the reduction in rest mass energy of the electron corresponds to the potential energy of the hydrogen atom [2, 3, 4].

Here, if the reduction in rest mass energy of the electron is represented as $-\Delta m_e c^2$, the potential energy can be defined as follows.

$$V(r) = -\Delta m_e c^2 \tag{7}$$

Next, the relativistic energy of an electron in a hydrogen atom is defined as follows.

$$m_n c^2 = m_e c^2 - \Delta m_e c^2 + K_{re,n} = m_e c^2 + \frac{1}{2}V(r_n) = m_e c^2 - K_{re,n} \tag{8}$$

Here, $m_n c^2$ is the relativistic energy of the electron, described with an absolute scale.

$m_n c^2$ is the sum of the residual part of the rest mass energy of the electron ($m_e c^2 + V(r_n)$) and the kinetic energy $K_{re,n}$.

Also, $E_{re,n}$ are the relativistic energy levels of a hydrogen atom. The "re" subscript of $E_{re,n}$ stands for "relativistic."

These energies can be illustrated as follows (see *Fig. 1*).

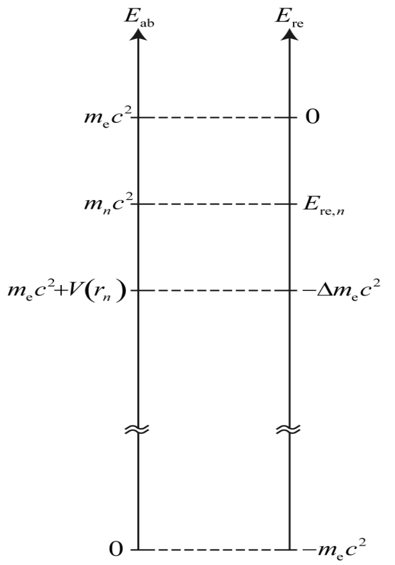

Figure 1. For $E_{re,n}$, the potential energy when the electron is at a position infinitely distant from the atomic nucleus (proton) is set to zero, just as for the energy levels $E_{BO,n}$ of a hydrogen atom derived nonrelativistically by Bohr. The relationship between $E_{BO,n}$ and $E_{re,n}$ is elucidated in Section 2, but $E_{BO,n}$ is an approximation for $E_{re,n}$. The "ab" subscript of E_{ab} stands for "absolute."

However, the energy of an electron placed at an infinitely distant position is not originally zero, it is $m_e c^2$ (however, the potential energy is zero in this case too). Energy in its original sense should be described with an absolute scale. In classical quantum theory, the potential energy of an electron at an infinite distance and the energy of the hydrogen atom are both regarded as zero. However, that idea is mistaken. There is a problem with classical quantum theory which describes the energy levels of a hydrogen atom in relative terms, ignoring the special theory of relativity (STR).

Also, $E_{re,n}$ has the following relationship with the energy $h\nu$ of the photon emitted to the outside from the electron.

$$E_{re,n} + h\nu = 0 \tag{9}$$

This completes preparations for deriving an energy-momentum relationship applicable to an electron in a hydrogen atom.

The author has already derived an energy-momentum relationship applicable to the electron in a hydrogen atom using five types of methods [5, 6, 7, 8, 9, 10]. One of those is reviewed in Section 2.

2. Energy-Momentum Relationship Applicable to an Electron in a Hydrogen Atom

According to the STR, the following relation holds between the energy and momentum of a body moving in free space [11].

$$(mc^2)^2 = (m_0 c^2)^2 + c^2 p^2 \tag{10}$$

Here, $m_0 c^2$ is the rest mass energy of the body. And mc^2 is the relativistic energy.

Incidentally, Einstein and Sommerfeld defined the relativistic kinetic energy as follows [12].

$$K_{re} = mc^2 - m_0 c^2 \tag{11}$$

The "re" subscript of K_{re} stands for "relativistic."

Taking Formula (11) into account, Formula (10) can be rewritten as follows.

$$\begin{aligned} c^2 p^2 &= (mc^2 - m_0 c^2)(mc^2 + m_0 c^2) \\ &= K_{re}(mc^2 + m_0 c^2) \end{aligned} \tag{12}$$

From this, the following formula for relativistic kinetic energy can be derived.

$$K_{re} = \frac{p_{re}^2}{m + m_0} \tag{13}$$

Here, the subscript "re" is attached to p, just as in K_{re}.

Incidentally, Einstein's energy-momentum relationship (10) holds when the energy absorbed by a body is all converted to kinetic energy of that body. However, an electron in an atom acquires kinetic energy through emission of energy. Therefore, Einstein's relationship (10) cannot be applied to an electron in an atom.

Here, the relativistic kinetic energy of an electron inside a hydrogen atom is defined as follows by referring to Formulas (11) and (13).

$$K_{re,n} = m_e c^2 - m_n c^2 \tag{14}$$

$$K_{re,n} = \frac{p_{e,n}^2}{m_e + m_n} \tag{15}$$

Here, $p_{e,n}$ is the relativistic momentum of the electron. (Due to the situation in this paper, $p_{re,n}$ is rewritten here as $p_{e,n}$.)

The term "relativistic" as used in this paper is not based on the STR. It means that the fact that the mass of the electron changes due to the electron's motion is taken into account. It must be noted that the mass of the electron decreases when the velocity of the electron inside the atom increases [13].

Linking the right sides of Formulas (14) and (15) with an equals sign and rearranging, the following relationship can be derived.

$$(m_n c^2)^2 + c^2 p_{e,n}^2 = (m_e c^2)^2 \tag{16}$$

where,

$$m_n c^2 = m_e c^2 + E_{re,n} = m_e c^2 - K_{re,n} = m_e c^2 + \frac{1}{2} V(r_n) \tag{17}$$

The following formula can be derived from Formula (16).

$$m_n c^2 = m_e c^2 \left(1 + \frac{v_n^2}{c^2}\right)^{-1/2} \tag{18}$$

To change Formula (18) into a formula of quantum theory, the discreteness of energy must be incorporated into Formula (18).

Previously, the author has shown that the following relationship holds for an electron in a hydrogen atom [14, 15].

$$\frac{v_n}{c} = \frac{\alpha}{n} \tag{19}$$

Here, α is the following fine-structure constant.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = 7.2973525693 \times 10^{-3} \tag{20}$$

where α is a dimensionless constant introduced by A. Sommerfeld in 1916.

Using the relationship in Formula (19), Formula (18) can be written as follows.

$$m_n c^2 = m_\mathrm{e} c^2 \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2} \tag{21}$$

Taking Formula (21) into account, $E_{\mathrm{re,n}}$ in Formula (17) can be written as follows.

$$E_{\mathrm{re},n} = m_n c^2 - m_\mathrm{e} c^2 = m_\mathrm{e} c^2 \left[\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2} - 1\right], \quad n = 0, 1, 2, \ldots \tag{22}$$

Formula (22) gives the energy levels of a hydrogen atom, taking relativity into account (see *Fig. 2*).

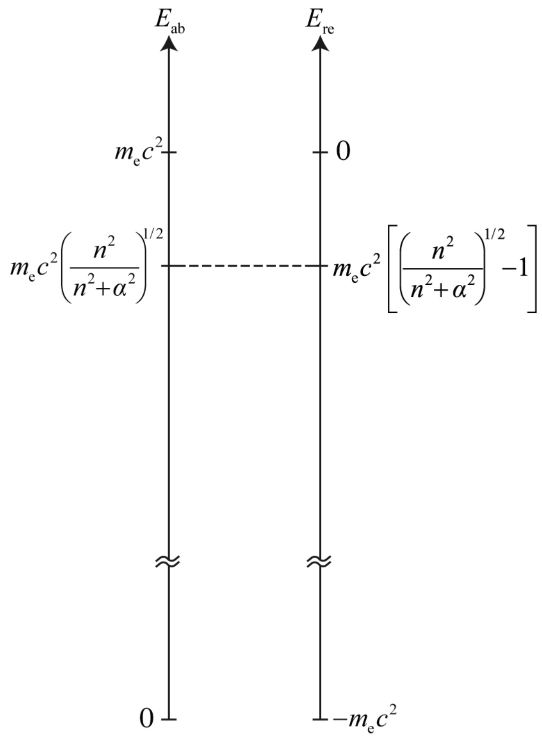

Figure 2. In this figure, the energies in Figure 1, $m_n c^2$ and $E_{\mathrm{re},n}$, have been rewritten using Formulas (21) and (22).

Formula (22) predicts the existence of a state with n=0 [16].

Next, when the part of Formula (22) in parentheses is expressed as a Taylor expansion,

$$\begin{aligned} E_{\mathrm{re},n} &\approx m_\mathrm{e} c^2 \left[\left(1 - \frac{\alpha^2}{2n^2} + \frac{3\alpha^4}{8n^4} - \frac{5\alpha^6}{16n^6}\right) - 1\right] \\ &\approx -\frac{\alpha^2 m_\mathrm{e} c^2}{2n^2}. \end{aligned} \tag{23}$$

Incidentally, the nonrelativistic energy levels of a hydrogen atom, derived by Bohr, are given by the following formula.

$$E_{\mathrm{BO},n} = -\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_\mathrm{e} e^4}{2\hbar^2} \cdot \frac{1}{n^2}, \quad n = 1, 2, \ldots \tag{24}$$

The “BO” subscript of $E_{\mathrm{BO},n}$ stands for “Bohr.”

However, when discussing the energy levels of a hydrogen atom, it is best if the rest mass energy of the electron is included in the formula. Thus Formula (24) is rewritten as follows.

$$E_{\mathrm{BO},n} = -\frac{m_\mathrm{e} c^2}{2} \left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right)^2 \frac{1}{n^2} = -\frac{\alpha^2 m_\mathrm{e} c^2}{2n^2}, \quad n = 1, 2, \ldots \tag{25}$$

From this, it is evident that Formula (24) is an approximation of Formula (22).

The energies of the hydrogen atom and electron treated thus far are summarized in the following table (see *Table 1*) [17].

Table 1. Comparison of the energies of a hydrogen atom predicted by Bohr's classical quantum theory and this paper. The difference between the two is whether or not the STR is taken into consideration.

	Classical Quantum Theory	This Paper
Potential Energy $V(r_\infty)$ when $r = \infty$	$V(r_\infty) = 0$	$V(r_\infty) = 0$
Energy of Hydrogen Atom when $r = \infty (n = \infty)$	$E_{\mathrm{BO},\infty} = 0$	$E_{\mathrm{re},\infty} = 0$
Energy of Electron when $r = \infty$	0 ?	$m_\mathrm{e} c^2$
Energy of Hydrogen Atom when $r = r_n$	$E_{\mathrm{BO},n} = \frac{1}{2} V(r_n)$	$E_{\mathrm{re},n} = \frac{1}{2} V(r_n)$
Energy of Electron Described with an Absolute Scale $m_n c^2$	Not Discussed	$m_\mathrm{e} c^2 + \frac{1}{2} V(r_n)$
STR	Not Considered	Considered
Electron Mass	m_e (Constant)	m_n (Varies)
Principal Quantum Number	$n = 1, 2, \ldots$	$n = 0, 1, 2, \ldots$

Now, in preparation for the discussion in Section 3, let us rewrite Formula (16) as a relationship for momentum. To do that, it is enough to divide both sides of Formula (16) by c^2. That is,

$$(m_n c)^2 + p_{\mathrm{e},n}^2 = (m_\mathrm{e} c)^2 \tag{26}$$

3. Discussion

It is known that the momentum and wavelength of an electron have the following relation.

$$p = \frac{h}{\lambda} \tag{27}$$

Here, if the momentum of the photon possessed by an electron in a state with principal quantum number n is $p_{\mathrm{p},n}$, and the momentum of a moving electron is $p_{\mathrm{e},n}$, then these two types of momentum can be expressed as follows.

$$p_{\mathrm{p},n} = m_n c = m_\mathrm{e} c \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2} = \frac{h}{\lambda_{\mathrm{p},n}} \tag{28}$$

$$p_{\mathrm{e},n} = m_n v_n = m_\mathrm{e} c \left(\frac{\alpha^2}{n^2+\alpha^2}\right)^{1/2} = \frac{h}{\lambda_{\mathrm{e},n}} \tag{29}$$

Here, $\lambda_{\mathrm{p},n}$ is the wavelength of a photon whose momentum is $p_{\mathrm{p},n}$, and $\lambda_{\mathrm{e},n}$ is the wavelength of an electron whose momentum is $p_{\mathrm{e},n}$.

The following formula can be obtained from the definition of the Planck constant [18].

$$m_\mathrm{e} c = \frac{h}{\lambda_\mathrm{C}}. \tag{30}$$

Here, λ_C is the Compton wavelength of the electron.

Incidentally, Formula (26) can also be written as follows.

$$p_{\mathrm{p},n}^2 + p_{\mathrm{e},n}^2 = (m_\mathrm{e}c)^2 \quad (31)$$

The following relationship can be derived by substituting the three types of momentum in Formulas (28) to (30) into Formula (31) and rearranging.

$$\frac{1}{\lambda_{\mathrm{p},n}^2} + \frac{1}{\lambda_{\mathrm{e},n}^2} = \frac{1}{\lambda_\mathrm{C}^2} \quad (32)$$

Formula (16) has already been derived in another paper, but Formula (32) has been derived for the first time in this paper.

4. Conclusions

Due to Formulas (28) and (29), the following relationship holds between the two types of wavelengths corresponding to the momentum of an electron in a hydrogen atom and the momentum of a photon possessed by the electron.

$$\frac{\lambda_{\mathrm{p},n}}{\lambda_{\mathrm{e},n}} = \frac{p_{\mathrm{e},n}}{p_{\mathrm{p},n}} = \frac{\alpha}{n}. \quad (33)$$

In the case where $n = 1$,

$$\frac{\lambda_{\mathrm{p},1}}{\lambda_{\mathrm{e},1}} = \frac{p_{\mathrm{e},1}}{p_{\mathrm{p},1}} = \alpha. \quad (34)$$

The fine structure constant α has previously been understood as the ratio of the velocity of an electron, as the particle in the ground state of a hydrogen atom, and the speed of light. However, it was not understood why the ratio of the two is so important for physics.

However, if the electron is considered as a wave, the fine structure constant can be understood as the ratio of the wavelength $\lambda_{\mathrm{p},1}$ of light possessed by an electron in the ground state and the wavelength $\lambda_{\mathrm{e},1}$ of a moving electron [19].

Also, this paper has derived the fact that the following relationship holds between the wavelength of an electron moving inside a hydrogen atom, the wavelength of a photon possessed by the electron, and the Compton wavelength of the electron.

$$\frac{1}{\lambda_{\mathrm{p},n}^2} + \frac{1}{\lambda_{\mathrm{e},n}^2} = \frac{1}{\lambda_\mathrm{C}^2}. \quad (35)$$

However, the only quantum number addressed in this paper is the principal quantum number n. This does not mean that all energy levels can be explained with this quantum condition (33) alone. However, Formula (35) is believed to be a new discovery.

In this paper, I was unable to derive a relationship incorporating the spin wave of the electron. It is hoped that, in the future, a relationship will be derived incorporating the electron spin.

For this purpose, I believe it will be necessary to solve the relativistic wave equation previously derived by the author [6, 20].

ACKNOWLEDGEMENTS

I would like to express my thanks to the staff at ACN Translation Services for their translation assistance. Also, I wish to express my gratitude to Mr. H. Shimada for drawing figures.

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