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Exploring b_I^{**}- Hyperconnectedness and b_I^*- Separation in Ideal Topological Spaces

By Donna Ruth Talo-Banga & Michael P. Baldado Jr.
Bohol Island State University

Abstract- We came up with the concept b^*-open set which has stricter condition with respect to the notion b-open sets, introduced by Andrijevic [3] as a generalization of Levine's [11] generalized closed sets. Anchoring on this concept, we defined b_I^{**}-hyperconnected sets and b^*-separated sets.

Topology is seen in many areas of science, for example, it is used to model the space-time notion of the universe. It is sometimes investigated in non-conventional ways, for example Donaldson [7] utilized mathematical concepts used by physicists to solve topological problems. These problems includes new topological sets like b^*-open set.

Keywords and phrases: b^*-open sets, b_I^*-open sets, ideals, b_I^{**}-hyperconnected sets, b_I^* -separated sets.

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Exploring b_I^{**}-Hyperconnectedness and b_I^*-Separation in Ideal Topological Spaces

Donna Ruth Talo-Banga [α] & Michael P. Baldado Jr. [σ]

Abstract- We came up with the concept $\boldsymbol{b}^*$-open set which has stricter condition with respect to the notion *b*-open sets, introduced by Andrijevic [3] as a generalization of Levine's [11] generalized closed sets. Anchoring on this concept, we defined $\boldsymbol{b}_I^{**}$-hyperconnected sets and $\boldsymbol{b}^*$ -separated sets.

Topology is seen in many areas of science, for example, it is used to model the space-time notion of the universe. It is sometimes investigated in non-conventional ways, for example Donaldson [7] utilized mathematical concepts used by physicists to solve topological problems.

These problems includes new topological sets like $\boldsymbol{b}^*$-open set.

A subset *B* of a topological space *W* is called a $\boldsymbol{b}^*$-*open relative to an ideal I* (or b_I^* -open), if there is an open set *P* with $P \subseteq \mathrm{Int}(B)$, and a closed set *S* with $\mathrm{Cl}(B) \subseteq S$ such that $(\mathrm{Int}(S) \cup \mathrm{Cl}(\mathrm{Int}(B))) \backslash B \in I$, and $B \backslash (\mathrm{Int}(\mathrm{Cl}(B)) \cup \mathrm{Cl}(P)) \in I$.

In this study, we gave some of the important properties of $\boldsymbol{b}_I^{**}$-hyperconnected sets and $\boldsymbol{b}^*$-separated sets.

Keywords and phrases: *b^*-open sets, b_I^* -open sets, ideals, $\boldsymbol{b}_I^{**}$ -hyperconnected sets, b_I^* -separated sets.*

I. Introduction

Topology is seen in many areas of science [14]. It is applied in biochemistry [5] and information systems [19]. Topology as a mathematical system is fundamentally comprised of open sets together with the operations union and intersection. Over time, open sets were generalized to different varieties. To name some, we have, Stone [20] introduced regular open set. Levine [10] introduced semi-open sets. Njasted [16] introduced α-open sets. Mashhour et al. [13] introduced pre-open sets. Abd El-Monsef et al. [1] introduced β-open set.

In the year 1970, Levine [11] introduced generalized closed sets, and achoring on this notion, Andrijevic [3] presented yet another generalization of open sets called b-open sets. This study uses the notion of b-open sets to come up with a new concept called b^*-open sets.

The concept ideal topological spaces was first seen in [9]. Vaidyanathaswamy [23] investigated this concept in point set topology. Tripathy and Shravan [17,18], Tripathy and Acharjee [21], Tripathy and Ray [22], Catalan et al. [6] also made investigations on ideal topological spaces.

Several concepts in topology were generalized using this structure. One of which is the concept b^*-open sets. Using the notion of b^*-open sets, we introduced the concepts b^*-compact sets, compatible b_I^*-compact sets, countably b_I^*-compact sets, b_I^*-connected sets, in ideal generalized topological spaces.

Author α: *Bohol Island State University - Bilar Campus, Philippines. e-mail: talodonnaruth@gmail.com*
Author σ: *Negros Oriental State University-Main Campus, Dumaguete City, Philippines. e-mail: michael.baldadojr@norsu.edu.ph*

Let W be a non-empty set. An ideal I on a set W is a non-empty collection of subsets of W which satisfies:

1. $B \in I$ and $D \subseteq B$ implies $D \in I$.
2. $B \in I$ and $D \in I$ implies $B \cup D \in I$.

Let W be a topological space and B be a subset of W. We say that B is *b^*-open set* if $B = \text{cl}(\text{int}(B)) \cup \text{int}(\text{cl}(B))$. For example, consider $W = \{a, b, c\}$ and the topology $\varsigma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, W\}$ on W. Then the b^*-open subsets are $\varnothing$, $\{a, b\}$, $\{c\}$ and W.

Let W be a topological space and B be a subset of W. The set B is called *b^*-open relative to an ideal I* (or *b_I^*-open*), if there is an open set P with $P \subseteq \text{Int}(B)$, and a closed set S with $\text{Cl}(B) \subseteq S$ such that

1. $(\text{Int}(S) \cup \text{Cl}(\text{Int}(B))) \backslash B \in I$, and
2. $B \backslash (\text{Int}(\text{Cl}(B)) \cup \text{Cl}(P)) \in I$.

In addition, we say that a set B is a b_I^*-closed set if B^C is b_I^*-open.

Consider the ideal space $(\{q, r, s\}, \{\varnothing, \{q\}, \{r\}, \{q, r\}, \{q, r, s\}\}, \{\varnothing, \{r\}\})$. Then $B = \{r, s\}$ is a b^*-open with respect to the ideal $I = \{\varnothing, \{r\}\}$. To see this, we let P be the open set $\{r\}$ and S be the closed set $\{r, s\}$. Then $\text{Int}(S) \cup \text{cl}(\text{int}(\{r, s\})) \backslash \{r, s\} = \text{int}(\{r, s\}) \cup \text{cl}(\{r\}) \backslash \{r, s\} = \{r\} \cup \{r, s\} \backslash \{r, s\} = \{r, s\} \backslash \{r, s\} = \varnothing \in I$. Also, $\text{Int}(\text{cl}(\{r, s\}) \cup \text{cl}(P) \backslash \{r, s\} = \text{int}(\{r, s\}) \cup \text{cl}(\{r\}) \backslash \{r, s\} = \{r\} \cup \{r, s\} \backslash \{r, s\} = \{r, s\} \backslash \{r, s\} = \varnothing \in I$. This shows that $B = \{r, s\}$ is a b_I^*-open.

The succeeding sections presents the rudimentary properties of b_I^{**}-hyperconnected spaces and b_I^*-separated spaces.

II. Results

This section presents the results of this study.

a) Preliminary Result: The following Lemmas were established in [4]. They will used in the proofs of some of the succeeding statements. In particular, Lemma 2.1 is used in Theorem 2.13 and Remark 2.15, while Lemma 2.2 is used in Lemma 2.19.

Lemma 2.1. *[4] Let (X, τ, I) be an ideal topological space. Then every b^*-open set is a b_I^*-open set.*

Lemma 2.2. *[4] Let (X, τ, I) be an ideal topological space with $I = \{\varnothing\}$. Then A is a b^*-open set if and only if A is a b_I^*-open set.*

b) b_I^{**}-Hyperconnected Ideal Topological Spaces: The concept $*$-hyperconnectedness was introduced by Ekici et al. [8], and the concept $I*$-hyperconnectedness was introduced by Abd El-Monsef et al. [12]. These insights motivated us to create the concept called b_I^{**}-hyperconnectedness. One may see [15] to gain more insights on these ideas.

Definition 2.3. *Let (X, τ) be a topological space and I be an ideal on X. A function $()^*(I, \tau) : P(X) \to P(X)$ given by $A^*(I, \tau) = \{x \in X : A \cup U \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ is called a local of A with respect to τ and I.*

Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, and $I = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ (note that τ is a topology on X and I is an ideal on X). Then, $\varnothing^* = \varnothing$, $\{a\}^* = \{c\}$, $\{b\}^* = \{c\}$, $\{c\}^* = X$, $\{a, b\}^* = \{c\}$, $\{a, c\}^* = X$, $\{b, c\}^* = X$ and $X^* = X$.

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4. M. P. Baldado. b^*_J sets and b^*_J-compact ideal spaces. European Journal of Pure & Applied Mathematics, 16(3), 2023.

Definition 2.4. *Let (X, τ) be a topological space and I be an ideal on X. The Kuratowski closure operator $Cl()^*(I, \tau) : P(X) \to P(X)$ for the topology $\tau^*(I, \tau)$ is given by $Cl(A)^*(I, \tau) = A \cup A^*$.*

Consider the ideal space in the previous example. We have, $Cl(\varnothing)^* = \varnothing \cup \varnothing^* = \varnothing \cup \varnothing = \varnothing$, $Cl(\{a\})^* = \{a\} \cup \{a\}^* = \{a\} \cup \{c\} = \{a, c\}$, $Cl(\{b\})^* = \{b\} \cup \{b\}^* = \{b\} \cup \{c\} = \{b, c\}$, $Cl(\{c\})^* = \{c\} \cup \{c\}^* = \{c\} \cup X = X$, $Cl(\{a, b\})^* = \{a, b\} \cup \{a, b\}^* = \{a, b\} \cup \{c\} = X$, $Cl(\{a, c\})^* = \{a, c\} \cup \{a, c\}^* = \{a, c\} \cup X = X$, $Cl(\{b, c\})^* = \{b, c\} \cup \{b, c\}^* = \{b, c\} \cup X = X$, and $Cl(X)^* = X \cup X^* = X \cup X = X$.

Definition 2.5. *Let (X, τ) be a topological space and I be an ideal on X. The Kuratowski interior operator $Int()^*(I, \tau) : P(X) \to P(X)$ for the topology $\tau^*(I, \tau)$ is given by $Int(A)^*(I, \tau) = X - Cl(X - A)^*$.*

Definition 2.6. *An ideal space (X, τ, I) is called $*$-hyperconnected [8] if $cl(A)^* = X$ for all non-empty open set $A \subseteq X$.*

Definition 2.7. *An ideal space (X, τ, I) is called I^*-hyperconnected [2] if $X - cl(A)^* \in I$ for all non-empty open set $A \subseteq X$.*

Definition 2.8. *An ideal topological space (X, τ, I) is said to be b_I^{**}-hyperconnected space if $X - cl(A)^* \in I$ for every non-empty b_I^*-open subset A of X.*

The next theorem says that the family of all b_I^{**}-hyperconnected space contains all $I*$-hyperconnected space.

Theorem 2.9. *Let (X, τ, I) be an ideal topological space. If X is $I*$-hyperconnected, then it is b_I^{**}-hyperconnected also.*

Proof. Let X be $I*$-hyperconnected, and A be a non-empty open set. Because X is I_*-hyperconnected, we have $X - cl(A)^* \in I$ for all non-empty open set $A \subseteq X$. And, because an open set is also a b_I^*-open set, we have $X - cl(A)^* \in I$ for all non-empty b_I^*-open set $A \subseteq X$. Hence, X is b_I^{**}-hyperconnected.

The next lemma is clear.

Lemma 2.10. *Let (X, τ) be a topological space. Then the intersection of any family of ideals on X is an ideal on X.*

Theorem 2.11 is taken from [2]. It says that when I is the minimal ideal, then the notions $*$-hyperconnected and I^*-hyperconnected are equivalent.

Theorem 2.11 *[2] Let (X, τ) be a clopen ideal topological space with $I = \{\varnothing\}$. Then, X is $*$-hyperconnected if and only if it is I^*-hyperconnected.*

The next remark is clear.

Remark 2.12. *If (X, τ) is a clopen topological space (a space in which every open set is also closed), then A is open if and only if A is b^*-open.*

Theorem 2.13 says that in a clopen space, with respect to the minimal ideal I, the notions b_I^{**}-hyperconnected and I^*-hyperconnected are equivalent.

Theorem 2.13 *Let (X, τ, I) be a clopen ideal topological space with $I = \{\varnothing\}$. Then, X is I^*-hyperconnected if and only if it is b_I^{**}-hyperconnected.*

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2. M. E. Abd El-Monsef, A. A. Nasef, A. E. Radwan, F. A. Ibrahem, and R. B. Esmaeel. Some properties of semi-open sets with respect to an ideal.

Proof. Suppose that X is I^*-hyperconnected. Let A is a non-empty element of τ. Then $X - cl^*(A) \in I$. By Remark 2.12 and Lemma 2.2, every open set is precisely b_I^*-open. Thus, $X - cl^*(A) \in I$ for all b_I^*-open set A $(\neq \varnothing)$. Therefore, X is b_I^{**}-hyperconnected also. Conversely, suppose that X is b_I^{**}-hyperconnected. Let A be a non-empty b_I^*-open set. Then $X - cl^*(A) \in I$. By Remark 2.12 and Lemma 2.2, b_I^*-open set is precisely open. Thus, $X - cl(A)^* \in I$ for all open set A $(\neq \varnothing)$. Therefore, X is I^*-hyperconnected also. □

Notes

Corollary 2.14 says that in a clopen ideal topological space, relative to the minimal ideal I, the notions b_I^{**}-hyperconnected, I^*-hyperconnected, and $*$-hyperconnected are equivalent.

Corollary 2.14. Let (X, τ, I) be a clopen ideal topological space with $I = \{\varnothing\}$. Then the following statements are equivalent.

i. X is $*$-hyperconnected.

ii. X is I^*-hyperconnected.

iii. X is b_I^{**}-hyperconnected.

Theorem 2.15 may be an important property.

Remark 2.15. If an ideal topological space $(X, \tau, \{\varnothing\})$ is a $b_I^*{*}$-hyperconnected space, then $X - cl^*(A) \in I$ for every non-empty b^*-open subset A of X.

To see this, let A is a non-empty b-open set. Then by Lemma 2.2 A is b_I^*-open. Since X is b_I^{**}-hyperconnected, $X - cl^*(A) \in I$.

Theorem 2.16 is a characterization of b_I^{**}-hyperconnected space.

Theorem 2.16. Let (X, τ, I) be an ideal topological space. Then the following statements are equivalent.

i. X is a b_I^{**}-hyperconnected space.

ii. $int(A)^* \in I$ for all b_I^*-closed proper subset A of X.

Proof. $(i) \Rightarrow (ii)$ Let B be b_I^*-closed. Then $X - B$ is b_I^*-open. Since $B \neq X$, $X - B \neq \varnothing$. Hence, by assumption we have $\text{int}(B)^* = X - \text{cl}(X - B)^* \in I$.
$(ii) \Rightarrow (i)$ Let A $(\neq X)$ be a non-empty b_I^*-open set. Then $X - A$ is a non-empty b_I^*-open set. Hence, by assumption, we have $X - \text{cl}(A)^* = X - \text{cl}(X - (X - A))^* = \text{int}(X - A)^* \in I$. Thus, X is b_I^{**}-hyperconnected. □

c) b_I^*-Separated Ideal Topological Spaces: In this section, we present the concepts b_I^*-separated sets and b_I^*-connected sets. We also present some of their important properties.

Definition 2.17. Let (X, τ, I) be an ideal topological space and A be a subset of X. The b^*-closure of A, denoted by $cl_{b^*}(A)$, is the smallest b^*-closed set that contains A. The b_I^*-closure of A, denoted by $cl_{b_I^*}(A)$, is the smallest b_I^*-closed set that contains A.

Next, we define b_I^*-separated sets, b_I^*-connected sets, and b_I^*-connected spaces.

Definition 2.18. Let (X, τ, I) be an ideal topological space. A pair of subsets, say A and B, of X is said to be b_I^*-separated if $cl_{b_I^*}(A) \cap B = \varnothing = A \cap cl_{b_I^*}(B)$. A subset A of X is said to be b_I^*-connected if it cannot be expressed as a union of two b_I^*-separated sets. The topological space X is said to be b_I^*-connected if it is b_I^*-connected as a subset.

Lemma 2.19 says that every b_I^*-connected space is connected. Recall, a space is connected if it cannot be written as a union of two non-empty open sets.

Lemma 2.19. Let (X, τ) be a τ_ς-space (a topological space in which every element is b^*-open also) and I be an ideal in X. If X is b_I^*-connected, then it is connected.

Notes

Proof. Suppose that to the contrary X is not connected. Let A and B be non-empty disjoint elements of τ with $X = A \cup B$. By Lemma 2.1, A and B are b_I^*-open sets also. Because $A = B^C$ and $B = A^C$, A and B are also b_I^*-closed. And so, $A = \text{cl}_{b_I^*}(A)$ and $B = \text{cl}_{b_I^*}(B)$. Thus, $\text{cl}_{b_I^*}(A) \cap B = A \cap B = \varnothing$ and $A \cap \text{cl}_{b_I^*}(B) = A \cap B = \varnothing$. This implies that X is b_I^*-separated, that is X is not b_I^*-connected, a contradiction. □

Remark 2.20. Let (X, τ) be a topology and I be an ideal in X. If $Y \subseteq X$, then $I_Y = \{Y \cap A : A \in I\}$ is an ideal in the relative topology (Y, τ_Y).

To see this, for the first property, let $B \in I_Y$ and $A \subseteq B$. Then $A \subseteq B \subseteq Y$. Now, if $A \in I_Y$, then there exist $C \in I$ such that $Y \cap C = A$. Note that $A \subseteq B \subseteq C$. Hence, $A, B \in I$. Thus, $A = Y \cap A \in I_Y$. Next, for the second, let $D, E \in I_Y$. Then $D \subseteq Y$ and $E \subseteq Y$. if $D \in I_Y$, then there exist $F \in I$ such that $Y \cap F = D$. Similarly, if $E \in I_Y$, then there exist $G \in I$ such that $Y \cap G = E$. Since I is an ideal, $F \cup G \in I$. Now, because $D \cup E \subseteq F \cup G$, $D \cup E \in I$. Thus, $D \cup E = (D \cup E) \cap Y \in I_Y$.

The next statement, Theorem 2.24, presents a way to construct b_I^*-open sets in a subspace.

Theorem 2.21. Let (X, τ, I) be an ideal topological space, and $Y \subseteq X$. If A is a b_I^*-open subset of X, then $A \cap Y$ is a $b_{I_Y}^*$-open set in Y.

Proof. Let A be a b_I^*-open set in X (τ, I). Then there exists an open set O with $O \subseteq \text{int}(A)$, and a closed set F with $\text{cl}(A) \subseteq F$ such that $\text{int}(F) \cup \text{cl}(\text{int}(A)) \backslash A \in I$, and $A \backslash \text{int}(\text{cl}(A)) \cup \text{cl}(O) \in I$. Let $O' = O \cap Y$, and $F' = F \cap Y$. Then O' is open in Y, and F' is closed in Y. Also, $\text{int}(F) \cup \text{cl}(\text{int}(A)) \backslash A \supseteq (\text{int}(F) \cup \text{cl}(\text{int}(A))) \cap Y \backslash (A \cap Y) \supseteq (\text{int}(F)) \cap Y \cup \text{cl}(\text{int}(A))) \cap Y \backslash (A \cap Y) \supseteq \text{int}(F \cap Y) \cup \text{cl}(\text{int}(A \cap Y)) \backslash (A \cap Y) \supseteq \text{int}(F') \cup \text{cl}(\text{int}(A \cap Y)) \backslash (A \cap Y)$, and $A \backslash \text{int}(\text{cl}(A)) \cup \text{cl}(O) \supseteq (A \cap Y) \backslash (\text{int}(\text{cl}(A)) \cup \text{cl}(O)) \cap Y \supseteq (A \cap Y) \backslash \text{int}(\text{cl}(A)) \cap Y \cup \text{cl}(O) \cap Y \supseteq (A \cap Y) \backslash \text{intcl}(A \cap Y) \cup \text{cl}(O \cap Y) \supseteq (A \cap Y) \backslash \text{intcl}(A \cap Y) \cup \text{cl}(O')$. Hence, by heredity $\text{int}(F') \cup \text{cl}(\text{int}(A \cap Y)) \backslash (A \cap Y) \in I$, and $(A \cap Y) \backslash \text{intcl}(A \cap Y) \cup \text{cl}(O') \in I$. This shows that $A \cap Y$ is a $b_{I_Y}^*$-open set in Y. □

Corollary 2.22. Let (X, τ, I) be an ideal topological space and $Y \subseteq X$. If A is a b_I^*-closed set in X (τ, I), then $A \cap Y$ is a $b_{I_Y}^*$-closed set in Y.

Proof. If A is b_I^*-closed, then A^C is b_I^*-open. By Theorem 2.21, $A^C \cap Y$ is $b_{I_Y}^*$-open. Hence, $A \cap Y = (A^C \cap Y)^C$ is $b_{I_Y}^*$-closed in Y. □

Remark 2.23. Let (X, τ, I) be an ideal topological space and $Y \subseteq X$. Then $I_Y = \{A \cap Y : A \in I\}$ is a subset of I.

The next statement, Theorem 2.24, say something about the closure of a set in the subspace.

Theorem 2.24. Let (X, τ, I) be an ideal topological space, and Y be an open subset of X. If $A \subseteq X$, then $cl_{b_{I_Y}^*}(A \cap Y) \subseteq cl_{b_I^*}(A) \cap Y$.

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Proof. Since $\mathrm{cl}_{b_I^*}(A)$ is a b_I^*-closed set in X, by Corollary 2.22 $\mathrm{cl}_{b_I^*}(A) \cap Y$ is a $b_{I_Y}^*$-closed set in Y. Hence, $\mathrm{cl}_{b_{I_Y}^*}(A \cap Y) \subseteq \mathrm{cl}_{b_{I_Y}^*}(\mathrm{cl}_{b_I^*}(A) \cap Y) = \mathrm{cl}_{b_I^*}(A) \cap Y$. □

The next statement, Theorem 2.25, says that if two sets, say A and B, are separated in the mother space, then $A \cap Y$ and $B \cap Y$ are also separated in the subspace.

Theorem 2.25. Let (X, τ, I) be an ideal topological space, and Y be a subset of X. If A and B are b_I^*-separated in X, then $A \cap Y$ and $B \cap Y$ are $b_{I_Y}^*$-separated in Y.

Notes

Proof. If A and B are b_I^*-separated in X, then $\mathrm{cl}_{b_I^*} A \cap B = \varnothing = A \cap \mathrm{cl}_{b_I^*} B$. Thus, by Theorem 2.24 $\varnothing = \varnothing \cap Y = (\mathrm{cl}_{b_I^*} A \cap B) \cap Y = ((\mathrm{cl}_{b_I^*} A) \cap Y) \cap (B \cap Y) \supseteq \mathrm{cl}_{b_{I_Y}^*}(A \cap Y) \cap (B \cap Y)$ and $\varnothing = \varnothing \cap Y = (A \cap \mathrm{cl}_{b_I^*} B) \cap Y = (A \cap Y) \cap ((\mathrm{cl}_{b_I^*} B) \cap Y) \supseteq (A \cap Y) \cap \mathrm{cl}_{b_{I_Y}^*}(B \cap Y)$. Thus, $A \cap Y$ and $B \cap Y$ are $b_{I_Y}^*$-separated. □

The next statement, Remark 2.26, says that the non-empty components of a space that makes it b_I^*-separated are b_I^*-open.

Remark 2.26. Let (X, τ, I) be a b_I^*-separated ideal topological space. If $X = A \cup B$ with $A \neq \varnothing$, $B \neq \varnothing$ such that $\mathrm{cl}_{b_I^*} A \cap B = \varnothing = A \cap \mathrm{cl}_{b_I^*} B$, then A and B are b_I^*-open.

To see this, we have $A^C = \mathrm{cl}_{b_I^*}(B)$ and $B^C = \mathrm{cl}_{b_I^*}(A)$. Hence, A^C and B^C are b_I^*-closed. Thus, A and B are b_I^*-open.

Recall, a pair of subsets, say A and B, of X is said to be b_I^*-separated if $\mathrm{cl}_{b_I^*}(A) \cap B = \varnothing = A \cap \mathrm{cl}_{b_I^*}(B)$. A subset A of X is said to be b_I^*-connected if it cannot be expressed as a union of two b_I^*-separated sets. A topological space X is said to be b_I^*-connected if it is b_I^*-connected as a subset.

The next statement, Theorem 2.27, says that two b_I^*-separated set cannot contain portions of a connected set.

Theorem 2.27. Let (X, τ, I) be a b_I^*-separated ideal topological space, and A be a b_I^*-connected set. If $A \subseteq H \cup G$ with H and G are b_I^*-separated sets, then either $A \subseteq H$ or $A \subseteq G$.

Proof. Suppose that to the contrary, $A = (A \cap H) \cup (A \cap G)$ with $A \cap H \neq \varnothing$ and $A \cap G \neq \varnothing$. Since H and G are b_I^*-separated sets, $\mathrm{cl}_{b_I^*}(A \cap H) \cap (A \cap G) \subseteq \mathrm{cl}_{b_I^*} H \cap G = \varnothing$ and $(A \cap H) \cap \mathrm{cl}_{b_I^*}(A \cap G) \subseteq H \cap \mathrm{cl}_{b_I^*} G = \varnothing$. Thus, $\mathrm{cl}_{b_I^*}(A \cap H) \cap (A \cap G) = \varnothing = (A \cap H) \cap \mathrm{cl}_{b^*}(A \cap G)$. Therefore, A can be expressed as a union of two b_I^*-separated sets $A \cap H$ and $A \cap G$. This is a contradiction. □

The next statement, Theorem 2.28, says that subsets of each of two b_I^*-separated sets are also separated.

Theorem 2.28. Let (X, τ, I) be an ideal topological space, and, A and B be b_I^*-separated sets. If $C \subseteq A$ $(C \neq \varnothing)$ and $D \subseteq B$ $(D \neq \varnothing)$, then C and D are also b_I^*-separated.

Proof. Suppose that A and B are b_I^*-separated. Then $\mathrm{cl}_{b_I^*} A \cap B = \varnothing = A \cap \mathrm{cl}_{b_I^*} B$. Thus, $\mathrm{cl}_{b_I^*} C \cap D \subseteq \mathrm{cl}_{b_I^*} A \cap B = \varnothing$ and $C \cap \mathrm{cl}_{b_I^*} D = A \cap \mathrm{cl}_{b_I^*} B = \varnothing$. Hence, $\mathrm{cl}_{b_I^*} C \cap D = \varnothing = C \cap \mathrm{cl}_{b_I^*} D$. Therefore, C and D is b_I^*-separated. □

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